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Zdeněk Skalák

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A CONTINUITY PROPERTY FOR THE INVERSE OF MAÑÉ'S PROJECTION

Zdeněk Skalák, Praha

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Abstract. Let X be a compact subset of a separable Hilbert space H with finite fractal dimension $d_F(X)$, and P_0 an orthogonal projection in H of rank greater than or equal to $2d_F(X)+1$. For every $\delta > 0$, there exists an orthogonal projection P in H of the same rank as P_0, which is injective when restricted to X and such that $\|P - P_0\| < \delta$. This result follows from Mañé's paper. Thus the inverse $(P|_X)^{-1}$ of the restricted mapping $P|_X\colon X \to PX$ is well defined. It is natural to ask whether there exists a universal modulus of continuity for the inverse of Mañé's projection $(P|_X)^{-1}$. It is known that when H is finite dimensional then $(P|_X)^{-1}$ is Hölder continuous. In this paper we shall prove that if X is a global attractor of an infinite dimensional dissipative evolutionary equation then in some cases (e.g. two-dimensional Navier-Stokes equations with homogeneous Dirichlet boundary conditions) $\|x-y\| \cdot \ln\ln \frac{1}{\gamma\|Px-Py\|} \leqslant 1$ for every $x, y \in X$ such that $\|Px - Py\| \leqslant \frac{1}{\gamma e^{e}}$, where γ is a positive constant.

Keywords: dissipative evolutionary equations, Navier-Stokes equations, attractors, Mañé's projection, fractal dimension

MSC 2000: 35Q10, 76D05, 76F99

Introduction

A large class of infinite dimensional dissipative evolution equations has a global attractor (see [26]) which describes the long-time behaviour of the solutions of these equations. The attractor is usually a subset of a suitable infinite dimensional Hilbert space H. It is supposed to have a complex geometric structure (see [6, 17]), perhaps a fractal one, which reflects the complicated dynamics of the corresponding system.

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It was proved in many cases of the infinite dimensional dissipative evolution equations that the global attractor is a subset of the graph of a mapping φ defined from a finite dimensional subspace of H to H. If φ is sufficiently smooth then the dynamics on the attractor can be described by a system of finite dimensional ordinary differential equations. This fact has often been used in numerical computations (see [21, 23]). The existence of the (smooth) mapping φ gives us also better understanding of the geometrical structure of the attractor.

In those cases of the evolution dissipative equations where the existence of inertial manifolds was proved (see [4, 5, 10, 15, 16]), the mapping φ is Lipschitz and is defined on the finite dimensional space which is spanned by the first n eigenvectors of the linear part of the corresponding equation. In these cases, the dynamics on the global attractor is described by the finite dimensional system of ordinary differential equations, referred to as an inertial form. This leads to considerable simplification in the study of the dynamics. The existence of inertial manifolds was proved for the systems satisfying the spectral gap condition in the spectrum of the linear part of the equation, for example for the modified Navier-Stokes equations and some Reaction-Diffusion equations (see [15]) or the Kuramoto-Shivasinski equation (see [14]). A different method of showing the existence of inertial manifolds was used in [20] for the two-dimensional Navier-Stokes equations with periodic boundary conditions or in [27] for the Navier-Stokes equations on a two-dimensional sphere.

However, for some dissipative evolution equations the existence of inertial manifolds has not been proved yet. Here, in many cases, only the existence of approximate inertial manifolds was established (see [7, 11, 12, 13, 19, 25, 28]), which means that the global attractor is not a subset of a smooth finite dimensional manifold but only a subset of its open neighborhood. Fortunately, it was shown in some cases of these systems (e.g. the two-dimensional Navier-Stokes equations with homogeneous Dirichlet boundary conditions) that the fractal dimension of their global attractor is finite (see [2, 3, 9, 18]). Therefore, as was proved in [22], any orthogonal projection in the underlying Hilbert space H of sufficiently high finite rank can be perturbed so that its restriction to the global attractor is injective. It means that in the case of these systems the global attractor is a subset of the graph of a continuous mapping defined on a finite dimensional subspace of H. It was proved in [1] that the mapping is Hölder continuous as long as the underlying Hilbert space H is finite dimensional. It is natural to ask whether the same theorem also holds for the infinite dimensional space H. In this paper we show the following weaker result:

Theorem R. *Let us consider a system of infinite dimensional dissipative evolution equations which has a global attractor X of finite fractal dimension in the phase space H (the precise definition of the system together with the necessary conditions*

[A1], [A2] and [A3] will be stated in Section 1). Let P_0 be an orthogonal projection in H of sufficiently high finite rank N. Then for any $\delta > 0$ there exists an orthogonal projection P in H of rank N and a positive number $\gamma = \gamma(P_0, \delta)$ such that

(1) $\|P - P_0\| \leqslant \delta$,

(2) $\|x - y\| \cdot \ln\ln \frac{1}{\gamma\|Px-Py\|} \leqslant 1$ *for all* $x, y \in X$ *such that* $\|Px - Py\| \leqslant \frac{1}{\gamma \mathrm{e}^{\mathrm{e}}}$.

1. Preliminaries, notation and statement of Main Theorem

Assume we are given an infinite dimensional Hilbert space H with a scalar product $(\cdot,\cdot)$ and a norm $\|\cdot\|$. In this paper we consider a class of nonlinear evolutionary equations of the type

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au + R(u) = f, \tag{1.1}$$

where A is a linear unbounded self-adjoint positive operator in H with domain $D(A)$ dense in H. We assume that A^{-1} is compact and, accordingly, H has an orthonormal basis $\{w_j\}_{j=1}^{\infty}$ consisting of eigenfunctions of the operator A with the corresponding eigenvalues λ_j (i.e., $Aw_j = \lambda_j w_j$, $j = 1, 2, \ldots$) satisfying $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$, $\lambda_j \to +\infty$ as $j \to +\infty$. R is an arbitrary (nonlinear) operator from $D(A)$ into H and $f \in H$.

For $n = 1, 2, \ldots$ we denote by P_n the orthogonal projection of H onto $\mathrm{Span}\{w_1, w_2, \ldots, w_n\}$.

In the paper, we use the concept of the fractal dimension (for more details see e.g. [1] or [26]):

Definition 1.1. Let Y be a compact subset of a Hilbert space H and $\varepsilon > 0$. Let $N_\varepsilon(Y)$ be the minimum number of ε-balls necessary to cover Y. The fractal dimension of Y, denoted by $d_F(Y)$, is then defined as

$$d_F(Y) = \limsup_{\varepsilon \to 0_+} \frac{\ln N_\varepsilon(Y)}{\ln(1/\varepsilon)}.$$

Moreover, let the following assumptions be satisfied:

[A1] For every $u_0 \in H$ there is a unique global solution $u = u(t)$ to (1.1) such that

$$u(0) = u_0 \tag{1.2}$$

and $u \in C(\langle 0, \infty), H)$.

[A2] There exists a global attractor $X \subset H$ for (1.1). The fractal dimension of X, denoted by $d_F(X)$, is finite.

[A3] There exist a real number $a \in (0,1)$ and a natural number M such that for all natural numbers $n \geqslant M$ and for all $x, y \in X$ we have: $\|x-y\| \geqslant a^n \Longrightarrow \|P_n x - P_n y\| \geqslant \frac{1}{2}\|x-y\|$.

R e m a r k 1.2. The assumptions [A1] and [A2] have been proved for a large class of dissipative systems defined in (1.1), see e.g. [26]. As far as [A3] is concerned, let us now point out that this condition holds for global attractors of some dissipative systems defined in (1.1). This follows from the theory of inertial and approximate inertial manifolds developed in [4, 5, 7, 10, 11, 12, 13, 15, 16, 19, 25, 28], where it has been shown that the global attractors are included in a thin neighbourhood of these manifolds. In the article, we do not work directly with the systems (1.1) (and therefore we do not specify the terms of (1.1) in detail), but we consider the global attractors of those particular systems (1.1) where [A3] can be proved and use only this key condition. For example, the following proposition has been proved in the case of the two-dimensional Navier-Stokes equations:

Proposition. *Let X be the global attractor of two-dimensional Navier-Stokes equations with periodic or homogeneous Dirichlet boundary conditions. Then there exist a natural number M and a real number $a \in (0,1)$ such that for all natural numbers $n \geqslant M$ there exists a Lipschitz mapping (approximate inertial manifold) $\Sigma_n \colon P_n H \longrightarrow (I-P_n)H$ with a Lipschitz constant 1 such that*

$$\operatorname{dist}(x, \Sigma_n) \leqslant a^n, \qquad \forall x \in X.$$

P r o o f. The proof follows immediately from Theorem 3.1 in [13]. $\square$

Clearly, [A3] is now a simple consequence of the proposition.

Let us also note that [A3] is easily verifiable for those dissipative systems where the existence of Lipschitz inertial manifolds was proved (see [4, 5, 10, 15, 16]).

Let us denote $A_X = \{x-y;\ x, y \in X\}$ and $S_X = \sup\{\|u\|;\ u \in A_X\}$. Obviously, one has $d_F(A_X) \leqslant 2d_F(X) < \infty$. Let P_0 be an orthogonal projection in H of rank $N > d_F(A_X)$ and $\delta > 0$. Let D and θ be numbers such that $d_F(A_X) < D < N$ and $0 < \theta < 1 - D/N$. Since $D > d_F(A_X)$, there exists a number $c \geqslant 1$ such that $N_\varepsilon(A_X) \leqslant c(1/\varepsilon)^D$ for every $\varepsilon \in (0,1)$, where $N_\varepsilon(A_X)$ denotes the minimum number of ε-balls necessary to cover A_X.

The following theorem is the main result of this paper.

Theorem 1.3. *Let $X \in H$ be the global attractor of the system defined in* (1.1) *and let the assumptions* [*A*1], [*A*2] *and* [*A*3] *be satisfied. Assume we are given the operator P_0 and the numbers N, δ, D, θ and c as in the preceding paragraph. Then there exists an orthogonal projection P in H of rank N such that $\|P - P_0\| \leqslant \delta$ and $\frac{\|u\|}{f(\|Pu\|)} \leqslant 1$ for every $u \in A_X$. The real function f is defined on the interval* $(0,1)$ *as*

$$f(x) = \begin{cases} \dfrac{1}{\ln\ln\frac{1}{\gamma x}} & \text{for every } x \in \Big(0, \dfrac{1}{\gamma \mathrm{e}^{\mathrm{e}}}\Big\rangle, \\ 1 & \text{for every } x \in \Big(\dfrac{1}{\gamma \mathrm{e}^{\mathrm{e}}}, 1\Big) \end{cases}$$

and $\gamma = \gamma(a, c, N, D, P_0, \delta, \theta, S_X)$ is a positive constant.

2. Statement of Auxiliary Lemmas

In this section we formulate three lemmas which will be used in the proof of Theorem 1.3.

Let us denote by $G(N, M)$ the set consisting of all N dimensional subspaces of the Euclidean space $\mathbb{R}^M$. There exists a nontrivial measure m defined on $G(N, M)$ which is invariant under the action of the orthogonal group. Let us note that all open subsets of $G(N, M)$ are m-measurable and their measure is greater than zero. For a subspace S of $\mathbb{R}^M$, we denote by $S^\perp$ the orthogonal complement of S in $\mathbb{R}^M$. For $0 \neq a \in \mathbb{R}^M$ and $\varrho > 0$, let us define the set

$$W(a, \varrho) = \{S \in G(N, M);\; S^\perp \cap B(a, 2\varrho) \neq \emptyset\}.$$

For every natural number k let O_k denote the surface of the k dimensional unique sphere of $\mathbb{R}^{k+1}$ and let $O(M, N) = O_{M-1} O_{M-2} \dots O_{M-N}$ for $1 \leqslant N < M$. Let us note that the definitions and the propositions of this section are also mentioned in [1] and intensively studied in [24].

Lemma 2.1. *If $0 \neq a \in \mathbb{R}^M$ and $\varrho > 0$ then for $W(a, \varrho) \subset G(N, M)$,*

$$m(W(a, \varrho)) \leqslant 4^N M^N O(M, N) \Big(\frac{\varrho}{\|a\|}\Big)^N. \tag{2.1}$$

Lemma 2.2. *Let $P_0 \in G(N, M)$ and $\delta \in (0, 1)$. Then*

$$m(\{P \in G(N, M);\; \|P - P_0\| \leqslant \delta\}) > \varepsilon^M \delta^{MN} O(M, N), \tag{2.2}$$

where $\varepsilon = 1/(4N)^N$.

Lemma 2.3. *Let P_0 be an orthogonal projection in H of rank N and $\delta > 0$. Then there exists an orthogonal projection P in H of rank N and a natural number $M = M(P_0, \delta)$ such that $\|P_0 - P\| \leqslant \delta$ and $PH \subset \operatorname{Span}\{w_1, w_2, \dots, w_M\}$.*

Remark 2.4. For the methods to prove Lemma 2.1 and Lemma 2.2, see [24]. The proof of Lemma 2.3 is elementary.

3. Proof of Main Theorem

From now on, we suppose that all assumptions of Theorem 1.3 hold. Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of natural numbers and $0 = M_0 < 4 \leqslant M_1 < M_2 < \dots$, $M_1 > N$ and $M_1 \geqslant M$, where M is the number from the assumption [A3]. Using Lemma 2.3 we can assume, without loss of generality, that for $M_1 = M_1(\delta, P_0)$ sufficiently large, $P_0 H \subset \operatorname{Span}\{w_1, w_2, \dots, w_{M_1}\}$. We can further assume that $S_X \leqslant 1$ and $\delta \in (0, \frac{1}{2})$.

In the proof of the following lemma we use a method which was developed in [1].

Lemma 3.1. *Let k be a natural number or zero, let $\widetilde{P}$ be an orthogonal projection in $P_{M_{k+1}}(H)$ of rank N and $\beta > 0$. Then there exists an orthogonal projection P in $P_{M_{k+1}}(H)$ of rank N such that*

(1) $\frac{\|x\|}{\|Px\|^{\theta}} \leqslant \frac{2}{\eta a^{\theta}}, \qquad \forall x \in A_k \stackrel{\text{def}}{=} \{x \in A_X\,;\ a^{M_{k+1}} < \|x\| \leqslant a^{M_k}\}$ *and*

(2) $\|P - \widetilde{P}\| \leqslant \frac{1}{2}\beta$, *where*

$$\eta = \Big(\frac{\varepsilon^{M_{k+1}} \beta^{M_{k+1} N}}{B M_{k+1}^N G^{M_k} 2^{M_{k+1} N}}\Big)^{\theta/(N-D)}, \tag{3.1}$$

$$B = c 2^D 4^N H a^{(N-D)(\theta-1)/\theta},\ H = G/(1-G) \text{ and } G = a^{(N-D)/\theta - N}. \tag{3.2}$$

Proof. Let us consider the finite dimensional space $P_{M_{k+1}}(H)$. Then obviously, $P_{M_{k+1}}(A_k) \subset P_{M_{k+1}}(H)$, $d_F(P_{M_{k+1}}(A_k)) \leqslant d_F(A_k) \leqslant d_F(A_X) < D < N$ and $N_\varepsilon(P_{M_{k+1}}(A_k)) \leqslant c(1/\varepsilon)^D$ for every $\varepsilon \in (0,1)$. For all natural numbers n let $r_n = a^n$ and let us consider the sets $Y_k^n = \{y \in P_{M_{k+1}}(A_k);\ r_n < \|y\| \leqslant r_{n-1}\}$. Now we set $\varrho_n = (a^{\theta-1}\eta r_n)^{1/\theta}$ for every natural number n, where $\eta > 0$ is to be specified later. For each $n = 1, 2, \dots$ we choose a covering of $P_{M_{k+1}}(A_k)$ with $\frac{1}{2}\varrho_n$-balls consisting of at most $c(\frac{1}{2}\varrho_n)^{-D}$ balls. We disregard the balls not intersecting Y_k^n. Let $B(\alpha_j^{nk}, \frac{1}{2}\varrho_n)$, $j = 1, 2, \dots, m_n^k$, be the remaining balls, where $m_n^k \leqslant c(\frac{1}{2}\varrho_n)^{-D}$.

For each $j=1,2,\dots,m_n^k$ let us choose an arbitrary point $a_j^{nk}\in Y_k^n\cap B(\alpha_j^{nk},\frac{1}{2}\varrho_n)$. Then $B(\alpha_j^{nk},\frac{1}{2}\varrho_n)\subset B(a_j^{nk},\varrho_n)$ and therefore

$$Y_k^n\subset\bigcup_{j=1}^{m_n^k}B(\alpha_j^{nk},\frac{1}{2}\varrho_n)\subset\bigcup_{j=1}^{m_n^k}B(a_j^{nk},\varrho_n).$$

If a given projection P in $P_{M_{k+1}}(H)$ satisfies

$$\begin{aligned}(3.3)\qquad &(I-P)P_{M_{k+1}}(H)\cap B(a_j^{nk},2\varrho_n)=\emptyset\\ &\text{for every } n=1,2,\dots \text{ and } j=1,2,\dots,m_n^k,\end{aligned}$$

then it is easy to see that

$$\|y\|\leqslant\frac{1}{a^\theta\eta}\|Py\|^\theta,\qquad\forall y\in P_{M_{k+1}}(A_k).$$

This follows from the fact that if $y\in P_{M_{k+1}}(A_k)$ and $y\neq0$, then $y\in Y_k^n\subset\bigcup_{j=1}^{m_n^k}B(a_j^{nk},\varrho_n)$ for some natural number n, $\|y\|\leqslant r_{n-1}$, $\|Py\|=\operatorname{dist}(y,(\operatorname{Ker}P)^\perp)\geqslant\varrho_n$ (see (3.3)), and from the definition of r_{n-1} and ϱ_n.

Our goal is to perturb $\widetilde{P}$ just enough to realize (3.3). Since

$$P_{M_{k+1}}(A_k)\subset\bigcup_{n=M_k+1}^{\infty}Y_k^n\subset\bigcup_{n=M_k+1}^{\infty}\bigcup_{j=1}^{m_n^k}B(a_j^{nk},2\varrho_n),$$

we first proceed to estimate $\sum_{n=M_k+1}^{\infty}\sum_{j=1}^{m_n^k}m(W(a_j^{nk},\varrho_n))$. It follows from Lemma 2.1, the inequalities $m_n^k\leqslant c2^D\varrho_n^{-D}$, $r_n\leqslant|a_j^{nk}|$ and $\varrho_n=(a^{\theta-1}\eta r_n)^{1/\theta}$ that

$$\begin{aligned}(3.4)\quad \sum_{n=M_k+1}^{\infty}\sum_{j=1}^{m_n^k}m(W(a_j^{nk},\varrho_n))&\leqslant\sum_{n=M_k+1}^{\infty}m_n^k4^NM_{k+1}^NO(M_{k+1},N)\Big(\frac{\varrho_n}{r_n}\Big)^N\\ &\leqslant c2^D4^NM_{k+1}^NO(M_{k+1},N)\sum_{n=M_k+1}^{\infty}\varrho_n^{N-D}r_n^{-N}\\ &=c2^D4^Na^{(N-D)(\theta-1)/\theta}M_{k+1}^N\eta^{(N-D)/\theta}O(M_{k+1},N)\sum_{n=M_k+1}^{\infty}r_n^{(N-D)/\theta-N}\\ &=B\eta^{(N-D)/\theta}O(M_{k+1},N)M_{k+1}^NG^{M_k},\end{aligned}$$

where for B and G see (3.2). If we now set

$$\eta = \Big(\frac{\varepsilon^{M_{k+1}}\beta^{M_{k+1}N}}{BM_{k+1}^N G^{M_k} 2^{M_{k+1}N}}\Big)^{\theta/(N-D)},$$

then

$$B\eta^{(N-D)/\theta} O(M_{k+1}, N) M_{k+1}^N G^{M_k} \leqslant \varepsilon^{M_{k+1}} \Big(\frac{\beta}{2}\Big)^{M_{k+1}N} O(M_{k+1}, N)$$

and due to (3.3), (3.4) and Lemma 2.2 there exists an orthogonal projection P in $P_{M_{k+1}}(H)$ of rank N so that item (2) of Lemma 3.1 holds, i.e. $\|P - \widetilde{P}\| \leqslant \frac{1}{2}\beta$, and

$$\|P_{M_{k+1}}x\| \leqslant \frac{1}{\eta a^\theta}\|Px\|^\theta, \qquad \forall x \in A_k.$$

Since $\|x\| \geqslant a^{M_{k+1}}$ for every $x \in A_k$ it follows from the assumption [A3] that $\|P_{M_{k+1}}x\| \geqslant \frac{1}{2}\|x\|$ and therefore

$$\frac{\|x\|}{\|Px\|^\theta} \leqslant \frac{2}{\eta a^\theta}, \qquad \forall x \in A_k,$$

that is item (1) of Lemma 3.1 holds. □

R e m a r k 3.2. Let us show that B from (3.2) is greater than 1: Since $a, G \in (0,1)$ and $D > 0$, then $a^{-D} \geqslant 1 - G$ and therefore $a^{(N-D)(\theta-1)/\theta} a^{(N-D)/\theta - N} \geqslant 1 - G$. Since $c \geqslant 1$, it means according to (3.2) that $B > Ha^{(N-D)(\theta-1)/\theta} \geqslant 1$.

P r o o f o f T h e o r e m 1.3. For every $k = 0, 1, \ldots$, let us now define an orthogonal projection P^k in H of rank N and a positive number $\delta_k < 1$. The projection P from Theorem 1.3 will then be defined as $\lim\limits_{k\to\infty} P^k$. We put $P^0 = P_0$, $\delta_0 = \delta$ and use the principle of mathematical induction. Let us assume that for some $k \in \{0, 1, \ldots\}$ an orthogonal projection P^k in $P_{M_{k+1}}(H)$ of rank N and a positive number $\delta_k < 1$ have already been defined. Let us define P^{k+1} and δ_{k+1}. By virtue of Lemma 3.1, there exists an orthogonal projection P^{k+1} in $P_{M_{k+1}}(H) \subset P_{M_{k+2}}(H)$ of rank N such that

$$\|P^{k+1} - P^k\| \leqslant \frac{1}{2}\delta_k$$

and

$$\frac{\|x\|}{\|P^{k+1}x\|^\theta} \leqslant \frac{2}{\eta_{k+1}a^\theta}, \qquad \forall x \in A_k, \tag{3.5}$$

where

$$\eta_{k+1} = \Big(\frac{\varepsilon^{M_{k+1}}\delta_k^{M_{k+1}N}}{BM_{k+1}^N G^{M_k} 2^{M_{k+1}N}}\Big)^{\theta/(N-D)} \tag{3.6}$$

with constants B, H and G defined in (3.2). Let us set $c_{k+1} = \frac{2}{\eta_{k+1}a^\theta}$ and define

$$\delta_{k+1} = \frac{1}{2}\Big(\frac{a^{M_{k+1}}}{c_{k+1}}\Big)^{1/\theta} a^{-M_{k+1}}. \tag{3.7}$$

Let us show now that $\delta_{k+1} \leqslant \frac{1}{2}\delta_k$. It follows from the definitions of δ_{k+1}, c_{k+1} and η_{k+1} that

$$\begin{aligned} \delta_{k+1} &= \frac{1}{2}\Big(\frac{a^{M_{k+1}}}{c_{k+1}}\Big)^{1/\theta} a^{-M_{k+1}} \\ &= \frac{a}{2^{(1+\theta)/\theta}}\Big(\frac{\varepsilon^{M_{k+1}}\delta_k^{M_{k+1}N}}{BM_{k+1}^N G^{M_k} 2^{M_{k+1}N}}\Big)^{1/(N-D)} a^{M_{k+1}(1-\theta)/\theta} \\ &= \Big(\frac{a}{2^{(1+\theta)/\theta}B^{1/(N-D)}}\Big)\Big(\frac{\varepsilon^{M_{k+1}}}{M_{k+1}^N 2^{M_{k+1}N}}\Big)^{1/(N-D)} \\ &\quad\times \Big(\frac{a^{M_{k+1}(1-\theta)/\theta}}{G^{M_k/(N-D)}}\Big)\delta_k^{M_{k+1}N/(N-D)}. \end{aligned} \tag{3.8}$$

Using the definition of G in (3.2) one immediately has $\big(\frac{a^{M_{k+1}(1-\theta)/\theta}}{G^{M_k/(N-D)}}\big) \leqslant 1$. From here and since $B \geqslant 1$ and $\varepsilon \leqslant 1$, one obtains from (3.8) that $\delta_{k+1} \leqslant \frac{1}{2}\delta_k^{M_{k+1}N/(N-D)}$. Since $M_{k+1}N/(N-D) \geqslant 1$ and $\delta_k < 1$, we conclude that $\delta_{k+1} \leqslant \frac{1}{2}\delta_k < 1$.

Let Q be an orthogonal projection in H of rank N and $\|Q - P^{k+1}\| \leqslant \delta_{k+1}$ for $k = 0, 1, \dots$. Then for every $x \in A_k$

$$\begin{aligned} \|Q - P^{k+1}\| \cdot \|x\| &\leqslant \frac{1}{2}\Big(\frac{a^{M_{k+1}}}{c_{k+1}}\Big)^{1/\theta} a^{-M_{k+1}}\|x\| \\ &= \frac{1}{2}\Big(\frac{1}{c_{k+1}}\Big)^{1/\theta} a^{M_{k+1}(1/\theta-1)}\|x\| \\ &< \frac{1}{2}\Big(\frac{1}{c_{k+1}}\Big)^{1/\theta}\|x\|^{1/\theta-1}\|x\| = \frac{1}{2}\Big(\frac{\|x\|}{c_{k+1}}\Big)^{1/\theta} \end{aligned} \tag{3.9}$$

and since by virtue of the definition of c_{k+1} and (3.5) we have $\|P^{k+1}x\| \geqslant \big(\frac{\|x\|}{c_{k+1}}\big)^{1/\theta}$ for every $x \in A_k$, it follows from (3.9) that $\|Q - P^{k+1}\| \cdot \|x\| \leqslant \frac{1}{2}\|P^{k+1}x\|$ for every $x \in A_k$. Therefore we have $\|Qx\| \geqslant \|P^{k+1}x\| - \|Q - P^{k+1}\| \cdot \|x\| \geqslant \frac{1}{2}\|P^{k+1}x\|$ for every $x \in A_k$ and finally

$$\frac{\|x\|}{\|Qx\|^\theta} \leqslant \frac{\|x\|}{(\frac{1}{2}\|P^{k+1}x\|)^\theta} \leqslant 2^\theta c_{k+1}, \qquad \forall x \in A_k. \tag{3.10}$$

Let us sum up the properties of the sequences $\{\delta_k\}_{k=0}^{\infty}$ and $\{P^k\}_{k=0}^{\infty}$:

(1) $\delta_0 = \delta, \delta_k > 0$ and $\delta_{k+1} \leqslant \frac{1}{2}\delta_k, \qquad \forall k = 0, 1, \dots$

(2) $P^0 = P_0$ and P^k is an orthogonal projection in $P_{M_{k+1}}(H), \qquad \forall k = 0, 1, \dots$

(3) $\|P^{k+1} - P^k\| \leqslant \frac{1}{2}\delta_k, \qquad \forall k = 0, 1, \dots$

(4) $\frac{\|x\|}{\|P^{k+1}x\|^\theta} \leqslant c_{k+1}, \qquad \forall x \in A_k,$

where $c_{k+1} = \frac{2}{\eta_{k+1}a^\theta}$ and η_{k+1} was defined in (3.6). It follows easily from conditions (1) and (3) that $\lim\limits_{k\to\infty} P^k = P$, where P is an orthogonal projection in H of rank N. For $n = 0, 1, \dots$ one has

$$\|P - P^n\| \leqslant \sum_{k=0}^{\infty} \|P^{n+k} - P^{n+k+1}\| \leqslant \sum_{k=0}^{\infty} \frac{1}{2}\delta_{n+k} \leqslant \frac{1}{2}\sum_{k=0}^{\infty} \delta_n \frac{1}{2^k} = \delta_n \tag{3.11}$$

and due to (3.10)

$$\frac{\|x\|}{\|Px\|^\theta} \leqslant 2^\theta c_{k+1}, \qquad \forall x \in A_k, \forall k = 0, 1, \dots. \tag{3.12}$$

Using (3.6), (3.7) and the definition of c_{k+1} we shall now express c_{k+1} as a function of c_k:

$$\begin{aligned} c_{k+1} = \frac{2}{a^\theta \eta_{k+1}} &= \underbrace{\left(\frac{2}{a^\theta} B^{\theta/(N-D)}\right)}_{\geqslant 1} \underbrace{\left(G^{M_k\theta/(N-D)}\right)}_{\leqslant 1} \\ &\times \left(M_{k+1}^{N\theta/(N-D)}\right) \underbrace{\left(\frac{4^N}{\varepsilon}\right)^{M_{k+1}\theta/(N-D)}}_{\geqslant 1} \\ &\times \left(a^{(\theta-1)NM_kM_{k+1}/(N-D)}\right)\left(c_k^{M_{k+1}N/(N-D)}\right). \end{aligned} \tag{3.13}$$

Moreover, the maximum of the real function $x \longrightarrow x^{1/x}$ $\forall x > 0$ is $\mathrm{e}^{1/\mathrm{e}}$. Therefore $M_{k+1}^{1/M_{k+1}} \leqslant 16N$ and using the definition of ε from Lemma 2.2 one has $\left(M_{k+1}^{N\theta/(N-D)}\right) \leqslant \left(\frac{4^N}{\varepsilon}\right)^{M_{k+1}\theta/(N-D)}$. If we now denote

$$b = b(a, \theta, N, D, c) = \frac{2}{a^\theta} B^{\theta/(N-D)} \left(\frac{16^N}{\varepsilon^2}\right)^{\theta/(N-D)} a^{(\theta-1)N/(N-D)}$$

and

$$T = T(N, D) = \frac{N}{N-D},$$

it follows immediately from (3.13) that

$$c_{k+1} \leqslant b^{M_kM_{k+1}} c_k^{M_{k+1}T}, \qquad \forall k = 1, 2, \dots.$$

Let us define a sequence $\{\Gamma_k\}_{k=1}^{\infty}$ in the following way: $\Gamma_1 = \max(c_1, b^{M_1}, a^{-1})$ and $\Gamma_{k+1} = b^{M_k M_{k+1}} \Gamma_k^{M_{k+1} T}$, $\forall k = 1, 2, \ldots$. Obviously, $\Gamma_k \geqslant c_k$, $\forall k = 1, 2, \ldots$. Since $\Gamma_{k+1} = b^{M_k M_{k+1}} \Gamma_k^{M_{k+1} T} \geqslant b^{M_{k+1}}$, one has $\Gamma_{k+1} = \left(b^{M_k}\right)^{M_{k+1}} \Gamma_k^{M_{k+1} T} \leqslant \Gamma_k^{M_{k+1}} \Gamma_k^{M_{k+1} T} = \Gamma_k^{(T+1) M_{k+1}}$. Therefore one has $\Gamma_{k+1} \leqslant \Gamma_k^{(T+1) M_{k+1}}$, $\forall k = 1, 2, \ldots$ and subsequently

$$\Gamma_k \leqslant \Gamma_1^{M_2 M_3 \ldots M_k (T+1)^{k-1}}, \qquad \forall k = 2, 3, \ldots, \tag{3.14}$$

where $\Gamma_1 = \Gamma_1(a, c, N, D, \theta, \delta_0, M_1)$. Since $c_k \leqslant \Gamma_k$ for every natural k, it follows from (3.12) that

$$\frac{\|x\|}{\|Px\|^{\theta}} \leqslant 2^{\theta} \Gamma_{k+1}, \qquad \forall x \in A_k, \forall k = 0, 1, \ldots. \tag{3.15}$$

We shall now consider the real function f which was defined in Theorem 1.3, where we put

$$\gamma = \frac{2\Gamma_1^{1/\theta}}{a^{M_1/\theta} \mathrm{e}^{\mathrm{e}}}. \tag{3.16}$$

Clearly, f is a nondecreasing function on the interval $(0, 1)$ and $\lim\limits_{x \to 0_+} f(x) = 0$. Let $x \in A_k$. It follows from (3.15) and the definition of A_k that

$$\|Px\| \geqslant \frac{1}{2} \Big(\frac{\|x\|}{\Gamma_{k+1}} \Big)^{1/\theta} \geqslant \frac{1}{2} \frac{a^{M_{k+1}/\theta}}{\Gamma_{k+1}^{1/\theta}}.$$

If we have

$$f\Big(\frac{1}{2} \frac{a^{M_{k+1}/\theta}}{\Gamma_{k+1}^{1/\theta}} \Big) \geqslant a^{M_k} \tag{3.17}$$

then $f(\|Px\|) \geqslant a^{M_k} \geqslant \|x\|$. Since $A_X = \bigcup\limits_{k=0,1,\ldots} A_k$, one has

$$f(\|Px\|) \geqslant \|x\|, \qquad \forall x \in A_X. \tag{3.18}$$

Let us prove (3.17). By means of (3.16) and the definition of f, (3.17) is equivalent to

$$\ln \ln \Big(\frac{a^{M_1/\theta} \mathrm{e}^{\mathrm{e}}}{\Gamma_1^{1/\theta}} \frac{\Gamma_{k+1}^{1/\theta}}{a^{M_{k+1}/\theta}} \Big) \leqslant a^{-M_k}. \tag{3.19}$$

Obviously, (3.19) holds for $k = 0$. Let now $k > 0$. It is possible to choose M_1 so large that $\frac{a^{M_1/\theta} e^{e}}{\Gamma_1^{1/\theta}} \leqslant 1$. Since also $M_1 \geqslant 4$ and $\Gamma_1 \geqslant a^{-1}$, it is sufficient to prove (see also (3.14)) that $\ln\ln\bigl(\Gamma_1^{M_1 M_2 M_3 \dots M_{k+1}(T+1)^k/\theta}\bigr) \leqslant a^{-M_k}$ or

$$(3.20)\quad (\ln M_1 + \ln M_2 + \ldots + \ln M_{k+1}) + (k\ln(T+1)) + (\ln(1/\theta)) + (\ln\ln\Gamma_1) \leqslant a^{-M_k}.$$

It is elementary to show that (3.20) holds if $M_1 = M_1(a, N, D, \theta, c, \delta_0)$ is a sufficiently large number. Therefore (3.17) holds and this together with (3.11) concludes the proof of Theorem 1.3. □

Remark 3.3. By setting for example $D = \frac{1}{2}(N + d_F(A_X))$ and $\theta = 1 - D/2N$ in Theorem 1.3 we can have that $\gamma = \gamma(a, c, N, P_0, \delta, S_X)$.

References

[1] *A.Ben-Artzi, A.Eden, C.Foiaş and B.Nicolaenko*: Hölder continuity for the inverse of Mañé's projection. J. Math. Anal. Appl. *178* (1993), 22–29.

[2] *P.Constantin and C.Foiaş*: Global Lyapunov exponents, Kaplan-Yorke formulas and the dimension of the attractors for 2D Navier-Stokes equations. Comm. Pure Appl. Math. *38* (1985), 1–27.

[3] *P.Constantin, C.Foiaş, O.P.Manley and R.Temam*: Determining modes and fractal dimension of turbulent flows. J. Fluid. Mech. *150* (1985), 427–440.

[4] *P.Constantin, C.Foiaş, B.Nicolaenko and R.Temam*: Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations. Springer-Verlag, New-York, 1989.

[5] *P.Constantin, C.Foiaş, B.Nicolaenko and R.Temam*: Spectral barriers and inertial manifolds for dissipative partial differential equations. J. Dyn. Diff. Equ. *1* (1989), 45–73.

[6] *P.Constantin, C.Foiaş and R.Temam*: Attractors representing turbulent flows. Mem. Amer. Math. Soc. *53* (1985), 1–67.

[7] *A.Debussche and R.Temam*: Convergent families of approximate inertial manifolds. J. Math. Pures Appl. *73* (1994), 489–522.

[8] *A.Eden, C.Foiaş, B.Nicolaenko and Z.S.She*: Exponential attractors and their relevance to fluid dynamics systems. Phys. *D 63* (1993), 350–360.

[9] *A.Eden, C.Foiaş and R.Temam*: Local and global Lyapunov exponents. J.Dyn.Diff.Equ. *3* (1991), 133–177.

[10] *E.Fabes, M.Luskin and G.R.Sell*: Construction of inertial manifolds by elliptic regularization. J. Differential Equations *89* (1991), 355–387.

[11] *C.Foiaş, O.P.Manley and R.Temam*: Approximate inertial manifolds and effective viscosity in turbulent flows. Phys. Fluids *A 3* (1991), 898–911.

[12] *C.Foiaş, O.P.Manley and R.Temam*: Iterated approximate inertial manifolds for Navier-Stokes equations in 2-D. J. Math. Anal. Appl. *178* (1993), 567–583.

[13] *C.Foiaş, O.Manley and R.Temam*: Modelling of the interaction of small and large eddies in two dimensional turbulent flows. Math. Mod. Numer. Anal. *22* (1988), 93–118.

[14] *C.Foiaş, B.Nicolaenko, G.Sell and R.Temam*: Inertial manifolds for the Kuramoto Sivashinsky equation and an estimate of their lowest dimension. J. Math. Pures Appl. *67* (1988), 197–226.

[15] *C.Foiaş, G.R.Sell and R.Temam*: Inertial manifolds for nonlinear evolutionary equations. J. Differential Equations *73* (1988), 309–353.
[16] *C.Foiaş, G.R.Sell and E.S.Titi*: Exponential tracking and approximation of inertial manifolds for dissipative nonlinear equations. J. Dyn. Diff. Equ. *1* (1989), 199–244.
[17] *C.Foiaş and R.Temam*: Approximation of attractors by algebraic or analytic sets. SIAM J. Math. Anal. *25* (1994), 1269–1302.
[18] *J.M.Ghidaglia*: On the fractal dimension of attractors for viscous incompressible fluid flows. SIAM J. Math. Anal. *17* (1986), 1139–1157.
[19] *D.A.Jones and E.S.Titi*: A remark on quasi-stationary approximate inertial manifolds for the Navier-Stokes equations. SIAM J. Math. Anal. *25* (1994), 894–914.
[20] *M.Kwak*: Finite-dimensional inertial forms for the 2D Navier-Stokes equations. Indiana Univ. Math. J. *41* (1992), 925–981.
[21] *J.Laminie, F.Pascal and R.Temam*: Implementation and numerical analysis of the nonlinear Galerkin methods with finite elements discretization. Appl. Num. Math. *15* (1994), 219–246.
[22] *R.Mañé*: On the dimension of the compact invariant sets of certain non-linear maps. Lecture Notes in Math. 898 (1981). Springer-Verlag, New York, pp. 230–242.
[23] *M.Marion and R.Temam*: Nonlinear Galerkin methods. SIAM J. Numer. Anal. *26* (1989), 1139–1157.
[24] *L.A.Santaló*: Integral Geometry and Geometric Probability. Addison-Wesley, Reading, 1976.
[25] *R.Temam*: Induced trajectories and approximate inertial manifolds. Math. Mod. Numer. Anal. *23* (1989), 541–561.
[26] *R.Temam*: Infinite-Dimensional Dynamical Systems in Mechanics and Physics. Appl. Math. Sci. 68. Springer-Verlag, New-York, 1988.
[27] *R.Temam and S.Wang*: Inertial forms of Navier-Stokes equations on the sphere. J. Funct. Anal. *117* (1993), 215–241.
[28] *E.S.Titi*: On approximate inertial manifolds to the Navier-Stokes equations. J. Math. Anal. Appl. *149* (1990), 540–557.

Author's address: Institute of Hydrodynamics, The Academy of Sciences of the Czech Republic, Pod Paťankou 5/30, 166 12 Praha 6, Czech Republic.