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APPROXIMATIONS AND ERROR BOUNDS FOR COMPUTING THE INVERSE MAPPING

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Abstract. In this paper we propose a procedure to construct approximations of the inverse of a class of $\mathcal{C}^m$ differentiable mappings. First of all we determine in terms of the data a neighbourhood where the inverse mapping is well defined. Then it is proved that the theoretical inverse can be expressed in terms of the solution of a differential equation depending on parameters. Finally, using one-step matrix methods we construct approximate inverse mappings of a prescribed accuracy.

Keywords: approximations, inverse mapping, error bounds

MSC 2000: 26B10, 34G20, 65D30, 15A24

1. Introduction

The aim of this paper is to propose a constructive method to provide continuous approximate functions and error bounds of the local inverse of a class of differentiable mappings acting between finite-dimensional Banach spaces. More precisely, we consider mappings $f\colon\ \Omega \subset E \to E$, which are $\mathcal{C}^m$ continuously differentiable in an open set Ω containing a disk centered at the origin of the finite-dimensional Banach space E, satisfying the conditions

$$f(0) = 0, \text{ and } \quad Df(0) \quad \text{is an isomorphism.} \tag{1.1}$$

The paper is organized as follows. In Section 2 we determine, in terms of the data, a neighbourhood where a mapping f of a clas $\mathcal{C}^m$ in Ω and satisfying (1.1) is invertible and its inverse mapping is expressed in terms of the solution of a differential initial

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value problem depending on parameters. In Section 3, using one-step matrix methods for the numerical solution of initial value matrix differential problems, we construct an approximate inverse mapping whose error in the predetermined neighbourhood is uniformly upper bounded by a prescribed admissible error ε.

If A is a matrix in $\mathbb{C}^{r\times q}$ we denote by $\|A\|$ its operator norm which may be computed by the square root of the maximum of the set

$$\big\{|z|:\ z \text{ eigenvalue of } A^H A\big\}$$

where A^H denotes the conjugate transpose of A, see [13, p. 21]. If f is a differentiable mapping $f\colon E \to E$, where E is a finite-dimensional Banach space, we denote by $\|Df(x)\|$ the supremum of the set

$$\{\|(Df(x))(v)\|;\ v \in E,\ \|v\| \leqslant 1\}.$$

The open disk of radius $r > 0$ centered at the origin of E is denoted by U_r and the corresponding closed disk is denoted by D_r. The set of all continuous linear mappings $u\colon E \to E$, endowed with the operator norm $\|u\| = \sup\{\|u(x)\|;\ \|x\| \leqslant 1\}$ is a Banach space denoted by $\mathcal{L}(E,E)$.

If A is a matrix in $\mathbb{C}^{p\times q}$, then it follows from [5, p. 14] that

$$\max|a_{ij}| \leqslant \|A\| \leqslant \sqrt{pq}|a_{ij}| \tag{1.2}$$

2. The inverse mapping as the solution of a differential equation depending on parameters

We begin this section with a lemma which determines the neigbourhood where the differential equation satisfied by the inverse mapping is well stated.

Lemma 2.1. *Let E be a finite dimensional Banach space, let Ω be an open set in E containing a closed disk D_r of radius $r > 0$ centered at the origin of E, and let $f\colon \Omega \to E$ be a differentiable mapping of class $\mathcal{C}^2$ such that $Df(0)$ is an isomorphism.*

(i) *Let $M_0 = \sup\big\{\|D^2 f(y)\|;\ \|y\| \leqslant r\big\}$ and let $r' > 0$ be defined by*

$$r' = min\Big\{r, \big[2\|(Df(0))^{-1}\|M_0\big]^{-1}\Big\}.$$

Then for $x \in U_{r'}$, $Df(x)$ is an isomorphism and

$$\sup\big\{\|(Df(z))^{-1}\|;\ \|z\| < r^{-1}\big\} \leqslant 2\|(Df(0))^{-1}\|. \tag{2.1}$$

(ii) *Let us consider the mapping* $H\colon U_{r'} \longrightarrow \mathcal{L}(E,E)$ *defined by* $H(x) = (Df(x))^{-1}$*. Then*

$$\sup\{\|H(x)\|;\ \|x\| < r'\} \leqslant 4M_0\|(Df(0))^{-1}\|^2 \tag{2.2}$$

where M_0 *is defined in* (i).

Let $\gamma > 0$ *and let* $G\colon U_r \times U_\gamma \to E$ *be defined by*

$$G(x,v) = (Df(x))^{-1}(v). \tag{2.3}$$

Then G *satisfies the Lipschitz condition*

$$\big\|G(x,v) - G(y,v)\big\| \leqslant 4\gamma M_0\big\|\big(Df(0)\big)^{-1}\big\|^2\|x-y\|;\ x,y \in U_{r'},\ v \in U_\gamma. \tag{2.4}$$

Proof. (i) From the Mean Value Theorem [2, p. 158], if $x \in U_r$ one gets

$$\|Df(x) - Df(0)\| \leqslant \|x\|M_0.$$

From the definition of r', if $\|x\| < r'$ it follows that

$$\|Df(x) - Df(0)\| < \big\|(Df(0))^{-1}\big\|^{-1}.$$

The perturbation lemma [3, p. 584] and the last inequality imply the invertibility of $Df(x)$. □

From the Banach lemma, the Mean Value Theorem and the definition of r' it follows that

$$\begin{aligned}
\big\|(Df(x))^{-1} - (Df(0))^{-1}\big\| &\leqslant \|(Df(x))^{-1}\|\|(Df(0))^{-1}\|\|Df(x) - Df(0)\| \\
&< \|(Df(x))^{-1}\|\|(Df(0))^{-1}\|r'M_0, \\
\big|\|(Df(x))^{-1}\| - \|(Df(0))^{-1}\|\big| &< \|(Df(x))^{-1}\|\|(Df(0))^{-1}\|r'M_0, \\
\|(Df(x))^{-1}\|\big(1 - r'M_0\big\|(Df(0))^{-1}\big\|\big) &< \|(Df(0))^{-1}\|, \\
\|(Df(x))^{-1}\| &< \big(1 - r'M_0\big\|(Df(0))^{-1}\big\|\big)^{-1}\|(Df(0))^{-1}\| \\
&< 2\|(Df(0))^{-1}\|.
\end{aligned}$$

Thus (2.1) is proved.

(ii) Note that $H(x) = (g_2 \circ g_1)(x)$, where $g_1\colon U_{r'} \longrightarrow \mathcal{L}(E,E)$ and $g_2\colon \mathcal{L}(E,E) \to \mathcal{L}(E,E)$ are defined by $g_1(x) = Df(x)$ and $g_2(y) = y^{-1}$, respectively. From Theorem 8.2.1 of [2, p. 149] it follows that $H'(x) = g_2'(g_1(x)) \cdot g_1'(x)$. Hence, from Theorem 8.3.2 of [2, p. 151] it follows that

$$\|g_2'(g_1(x))\| = \|g_2'(Df(x))\| \leqslant \big\|(Df(x))^{-1}\big\|^2. \tag{2.5}$$

Taking into account that $g_1'(x) = D^2 f(x)$, from (2.5) we obtain that

$$\|H'(x)\| \leqslant \left\|(Df(x))^{-1}\right\|^2 \|D^2 f(x)\|. \tag{2.6}$$

From (2.6) and (2.1) one gets (2.2).

Note that $G(x, v) = (Df(x))^{-1}(v) = H(x)(v)$. From the Mean Value Theorem and (2.2) it follows that

$$\begin{aligned} \|G(x,v) - G(x,y)\| &= \|H(x)(v) - H(y)(v)\| = \|[H(x) - H(y)](v)\| \\ &\leqslant \|H(x) - H(y)\|\|v\| \leqslant \|H(x) - H(y)\|\gamma \\ &\leqslant \sup\{\|H'(z)\|;\ \|z\| < r'\}\,\gamma\|x - y\|. \end{aligned}$$

From (2.6) and (2.1) one gets (2.4).

Theorem 2.1. *Let E be a finite-dimensional Banach space, let Ω be an open set in E containing a closed disk D_r of radius $r > 0$ centered at the origin of E. Let $f\colon \Omega \to E$ be a differentiable mapping of class $\mathcal{C}^m$, $m \geqslant 2$, such that $f(0) = 0$ and $Df(0)$ is an isomorphism.*

(i) *Let $r' > 0$ be defined by Lemma 2.1 and let us consider the differential system depending on parameters*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = G(x, v);\quad x(0, v) = 0,\ \|v\| < \gamma \tag{2.7}$$

where $G(x, v)$ is defined by (2.3) and $\gamma > 0$. Let δ be defined by

$$\delta = 2\|(Df(0))^{-1}\|r'\left[1 + 2r'M_0\gamma\|(Df(0))^{-1}\|\right]. \tag{2.8}$$

Then the system (2.7) has only one solution in the interval $]-\delta, \delta[$.

(ii) *Let δ be defined by (2.8) and let $\varepsilon = \gamma\delta/2$. Then the function $f\colon U_{r'} \to U_\varepsilon$ admits an inverse mapping $g\colon U_\varepsilon \to U_{r'}$, defined by*

$$g(v) = X(\delta/2, 2v/\delta),\quad v \in U_\varepsilon$$

where $x(t, v)$ is the solution of (2.7).

Proof. (i) From Lemma 2.1, $Df(x)$ is an isomorphism for $x \in U_{r'}$, and thus problem (2.7) is well stated. From Theorem 10.7.1, 10.7.3 and 10.7.4 of [2], for every $x \in U_\gamma$ there exists a unique solution $t \to x(t, v)$ of problem (2.7) of class $\mathcal{C}^{m-1}$ defined in $]-\delta, \delta[$ such that

$$\mathrm{d}f(x(t, v))\frac{\partial x(t, v)}{\partial t} = v.$$

Consequently, $\frac{\mathrm{d}f(x(t,v))}{\mathrm{d}t} = v$, and then $f(x(t,v)) = tv + \Phi(v)$. Taking t=0, we conclude that $\Phi = 0$ and

$$f(x(t,v)) = tv. \tag{2.9}$$

(ii) Let δ be defined by (2.7) and let $g(v) = x(\delta/2, 2v/\delta)$ for $v \in U_\varepsilon$ with $\varepsilon = \gamma\delta/2$. From (2.9) it follows that

$$(f \circ g)(v) = f\left(x(\delta/2, 2v/\delta)\right) = v$$

and $f \circ g = \mathrm{Id}$. Thus, g is a right inverse mapping of f of class $\mathcal{C}^{m-1}$, where $g\colon U_\varepsilon \to U_r$. Otherwise, as $(Df)^{-1}$ is of class $\mathcal{C}^{m-1}$, the identity $Dg = (Df)^{-1} \circ g$ shows that g is of class $\mathcal{C}^m$ indeed. Furthermore, the equality $f \circ g = \mathrm{Id}$, implies that $Df(0) \circ Dg(0) = \mathrm{Id}$. Since $Df(0)$ is an isomorphism, we conclude that $Dg(0)$ is also an isomorphism. Thus we can apply the first part of the proof to the mapping g, obtaining a right inverse h for it, i.e., $g \circ h = \mathrm{Id}$, in an appropiate neigbourhood of the origin. From the equations $f \circ g$, $g \circ h = \mathrm{Id}$, it follows that $f = h$ and the result is proved. □

The following example shows the utility of the above theorem in the case where the inverse mapping is known.

E x a m p l e 2.1. Let $f\colon \Re^n \to \Re^n$ be the function $f(x) = Ax$, where $x \in \Re^n$ and A is an invertible square matrix of order n. Since $f'(x) = A$, the corresponding differential system (2.7) takes the form

$$\frac{\mathrm{d}x(t,v)}{\mathrm{d}t} = A^{-1}(v), \quad x(0,v) = 0.$$

Integrating one gets $x(t,v) = A^{-1}vt$, and the solution x(t,v) is defined on the whole real line. Taking any values of r and of the corresponding δ given by Theorem 2.1, the inverse mapping is

$$g(v) = x\left(\frac{\delta}{2}, \frac{2v}{\delta}\right) = A^{-1}v$$

3. Approximate inverse mappings and error bounds

For the sake of clarity of presentation we summarize some results about the numerical solution of initial value matrix problems recently given in Section 2 of [9]. Let us consider the problem

$$Y'(t) = f(t, Y(t)), \quad Y(0) = Y_0 \in \mathbb{C}^{r\times q}, \quad 0 \leqslant t \leqslant b \tag{3.1}$$

where $f\colon\ [0,b] \times \mathbb{C}^{r\times q} \longrightarrow \mathbb{C}^{r\times q}$ is bounded, continuous and satisfies the Lipschitz condition

$$\|f(t,P) - f(t,Q)\| \leqslant L\|P - Q\|, \tag{3.2}$$

which guarantees the existence of a unique continuously differentiable matrix function $Y(t)$, a solution of (3.1), [4, p. 99].

A one-step matrix method is a relationship of the form

$$Y_{n+1} - Y_n = h\left\{B_1 f_{n+1} + B_0 f_n\right\}, \quad n \geqslant 0 \tag{3.3}$$

where B_0, B_1 are matrices in $\mathbb{C}^{r\times r}$ and Y_n, $f_n = f(t_n, Y_n) \in \mathbb{C}^{r\times q}$, $t_n = nh \in [0,b]$, $h > 0$ and

$$B_0 + B_1 = I. \tag{3.4}$$

Let C_s be the matrix in $\mathbb{C}^{r\times r}$ defined by

$$C_0 = 0;\quad C_1 = I - (B_0 + B_1) = 0;\ \ldots;\ C_s = \frac{I}{s!} - \frac{B_1}{(s-1)!};\quad s = 2, 3, \ldots \tag{3.5}$$

The method (3.3)–(3.4) is said to be of order p, if in (3.5) we have $C_0 = C_1 = \ldots = C_p = 0$ and $C_{p+1} \neq 0$.

Theorem 3.1. ([9]) *Let us consider a one-step matrix method of the type* (3.3)–(3.4) *of order* $p \geqslant 1$, *and let* h, Γ^* *be positive constants defined by*

$$h < \left(L\|B_1\|\right)^{-1}, \quad \Gamma^* = \left(1 - hL\|B_1\|\right)^{-1}, \tag{3.6}$$

where L *is the Lipschitz constant given by* (3.2). *If* G *and* D *are given by*

$$G = \|C_{p+1}\|, \quad D \geqslant \max\left\{\|Y^{p+1}(t)\|;\ \ 0 \leqslant t \leqslant b\right\}, \tag{3.7}$$

where $Y(t)$ is the theoretical solution of (3.1), *then the discretization error,* $e_n = Y(t_n) - Y_n$, *is upper bounded by the inequality*

$$\|e_n\| \leqslant \Gamma^* h^p G D t_n \exp(\Gamma^* L B^* t_n), \quad n \geqslant 0. \tag{3.8}$$

Example 3.1. Let us consider the one-step matrix method

$$Y_{n+1} - Y_n = h f_n\,;\ n \geqslant 0,\ Y_0 = \Omega \tag{3.9}$$

where $A_0 = I$, $B_0 = I$, $B_1 = 0$. From (3.5) it follows that $C_0 = C_1 = 0$ and $C_2 = I/2$ and thus the method (3.9) is of order $p = 1$. In accordance with the notation of Theorem 3.1, we have $C = \|C_2\| = 1/2$, $\Gamma^* = 1$,

$$D_2 \geqslant \left\{\|Y^{(2)}(t)\|\,;\ t \in [0,b]\right\} \tag{3.10}$$

and the discretization error e_n verifies

$$\|e_n\| \leqslant \frac{h t_n D_2}{2} \exp(L t_n),\ n \geqslant 0,\ t_n = nh \tag{3.11}$$

From a practical point of view it is important to obtain the constant D_2 in terms of the data because the theoretical solution $Y(t)$ of problem (3.1) is not known. For the sake of clarity of presentation we recall the concept and some properties of the Kronecker product of matrices. If $A = [a_{ij}] \in \mathbb{C}^{m\times n}$ and $B = [b_{ij}] \in \mathbb{C}^{r\times s}$, then the Kronecker product of A and B, denoted by $A \otimes B$, is the block matrix defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \vdots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}.$$

The column vector operator acting on the matrix $A \in \mathbb{C}^{m\times n}$ is defined by

$$\operatorname{vec}(A) = \begin{bmatrix} A_{.1} \\ \vdots \\ A_{.n} \end{bmatrix}, \quad A_{.k} = \begin{bmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{bmatrix}.$$

If $A \in \mathbb{C}^{m\times n}$, $Y \in \mathbb{C}^{n\times r}$ and $B \in \mathbb{C}^{r\times s}$, then [6, p. 25], implies that

$$\operatorname{vec}(AYB) = (B^T \otimes A) \operatorname{vec} Y$$

where B^T is the transpose matrix of B. If $Y = [y_{ij}] \in \mathbb{C}^{p\times q}$ and $X = [x_{rs}] \in \mathbb{C}^{m\times n}$, then [6, p. 62 and p. 81], yields

$$\frac{\partial Y}{\partial x_{rs}} = \begin{bmatrix} \frac{\partial y_{11}}{\partial x_{rs}} & \cdots & \frac{\partial y_{1q}}{\partial x_{rs}} \\ \vdots & \vdots & \vdots \\ \frac{\partial y_{p1}}{\partial x_{rs}} & \cdots & \frac{\partial y_{pq}}{\partial x_{rs}} \end{bmatrix}; \quad \frac{\partial Y}{\partial X} = \begin{bmatrix} \frac{\partial Y}{\partial x_{11}} & \cdots & \frac{\partial Y}{\partial x_{1n}} \\ \vdots & \vdots & \vdots \\ \frac{\partial Y}{\partial x_{m1}} & \cdots & \frac{\partial Y}{\partial x_{mn}} \end{bmatrix}.$$

The chain rule for the derivative of a matrix $Z = Y(X)$ with respect to a matrix X, with $X \in \mathbb{C}^{m\times n}$, $Y \in \mathbb{C}^{n\times r}$, $Z \in \mathbb{C}^{p\times q}$, takes the form [6, p. 88], [14]

$$\frac{\partial Z}{\partial X} = \left[\frac{\partial [\operatorname{vec} Y]^T}{\partial X} \otimes I_p\right]\left[I_n \otimes \frac{\partial Z}{\partial \operatorname{vec} Y}\right]. \tag{3.12}$$

If we consider the theoretical solution $x(t,v)$ of problem (2.7) in an interval $[0,\delta^*]$ where $\delta^* < \delta$ and δ is defined by (2.8), then by virtue of (3.12) the second derivative of the solution $x(t,v)$ of problem (2.7) takes the form

$$\frac{\mathrm{d}^2 x(t,v)}{\mathrm{d}t^2} = \left(\left[\operatorname{vec}(Df(x(t,v)))^{-1}(v)\right]^T \otimes I_n\right) \frac{\partial \left(Df(x(t,v))\right)^{-1}(v)}{\partial \operatorname{vec} X} \tag{3.13}$$

where n is the dimension of the Banach space E. Taking into account that [11, p. 439] yields

$$\|A \otimes B\| = \|A\| \|B\| \tag{3.14}$$

we obtain from Lemma 2.1, Theorem 2.1 and (3.12), (3.13), (3.14), that

$$\sup\left\{\left\|\frac{\mathrm{d}^2 x(t,v)}{\mathrm{d}t^2}\right\|;\ 0 \leqslant t \leqslant \delta^* < \delta,\ \|v\| < \gamma\right\} \leqslant 8\left\|(Df(0))^{-1}\right\|^3 M_0\gamma^2. \tag{3.15}$$

Thus for the problem (2.7), the constant D_2 appearing in (3.11) takes the form

$$D_2 = 8\gamma^2\left\|(Df(0))^{-1}\right\|^3 M_0. \tag{3.16}$$

The following result summarizes the procedure for constructing approximate inverse mappings and its proof is a direct consequence of Lemma 2.1, Theorem 2.1 and (3.11), (3.16).

Theorem 3.2. *Let $\gamma > 0$ and let E be a finite-dimensional Banach space, let Ω be an open set in E containing a closed disk D_r of radius $r > 0$ centered at the origin of E. Let $f\colon \Omega \to E$ be a differentiable mapping of class $\mathcal{C}^m$, $m \geqslant 2$, such that $f(0) = 0$ and $Df(0)$ is an isomorphism. Let $\delta^* < \delta$ where δ is defined by (2.8),* *let* $L = 4\gamma M_0 \|(Df(0))^{-1}\|^2$ *and* $D_2 = 8\gamma^2 \left\|(Df(0))^{-1}\right\|^3 M_0$. *Let us consider the one step method*

$$Y_{n+1} - Y_n = h\left[Df(Y_n)\right]^{-1}(v);\ Y_0 = 0,\ 0 \leqslant n \leqslant N-1 \tag{3.17}$$

where $\gamma > 0$, $h > 0$, $v \in E$, $\|v\| < \gamma\delta^*/2$ *and* $N = [\delta^*/h] = 2p$ *is an even integer.*

Let us define the approximate inverse mapping $\hat{g}(\cdot, h, \gamma)\colon U_{\gamma\delta^*/2} \to U_{r'}$ *by the expression*

$$\hat{g}(v, h, \gamma) = Y_{N/2}(2v/\delta^*) \tag{3.18}$$

where r' *is given by Lemma* 2.1.

The error of $\hat{g}$ *with respect to the theoretical inverse mapping* f^{-1} *of f is upper bounded by the inequality*

$$\|f^{-1}(v) - \hat{g}(v, h, \gamma)\| \leqslant h\gamma\|(Df(0))^{-1}\|K(\gamma)\exp[K(\gamma)];\ \|v\| < \gamma\delta^*/2 \tag{3.19}$$

where

$$K(\gamma) = 4\gamma M_0\|(Df(0))^{-1}\|^3(1 + 2r'M_0\gamma)\|(Df(0))^{-1}\|)r'. \tag{3.20}$$

Given an admissible error $\varepsilon > 0$, *taking* $h < \varepsilon\left[\gamma K(\gamma)\|(Df(0))^{-1}\|\exp(K(\gamma))\right]^{-1}$, *the corresponding approximate mapping* $\hat{g}(., h, \gamma)$ *satisfies*

$$\|f^{-1}(v) - \hat{g}(v, h, \gamma)\| < \varepsilon,\ v \in U_{\gamma\delta^*/2}. \tag{3.21}$$

R e m a r k 3.1. Note that by Theorem 3.2, for a given admissible error $\varepsilon > 0$, the error bound (3.19) as well as the domain of the inverse f^{-1} and of the approximate inverse $\hat{g}(., h, \gamma)$ depend on the parameter γ. So, the required size of h and the domain of the inverse change if the parameter γ changes. This fact is illustrated by the next example.

E x a m p l e 3.2. Let us consider the mapping $f\colon \Re^{2\times 2} \to \Re^{2\times 2}$ defined by

$$f(X) = X^2 + X. \tag{3.22}$$

By Theorem 8.14 of [2, p. 148] it is easy to show that

$$\begin{gathered} Df(X)\colon \Re^{2\times 2} \to \Re^{2\times 2}, \\ (Df(X))(V) = XV + VX + V,\ V \in \Re^{2\times 2},\ X \in \Re^{2\times 2}, \end{gathered} \tag{3.23}$$

It is clear that $f(0)=0$ and $(Df(0))(V)=V$. Thus $Df(0)$ is an isomorphism, and the condition (1.1) is satisfied. Let us take $r=1$, then in accordance with the notation of Lemma 2.1, Theorem 2.1 and Theorem 3.2 we have $(Df(0))^{-1}(V)=V$ and

$$\|(Df(0))^{-1}\|=1,\ M_0=3,\ r'=1/6,\ L=4\|(Df(0))^{-1}\|^2M_0=12\gamma$$
$$\delta=\frac{1}{3}(1+\gamma),\ D_2=24\gamma^2,\ K(\gamma)=2\gamma(1+\gamma).$$

Note that to compute the constant δ appearing in (2.8) we need the expression of $(Df(0))^{-1}$ which, by Lemma 2.1, is well defined for $\|X\|<1/6$. From (3.22), if X, T are matrices in $\Re^{2\times 2}$ and $\|X\|<1/6$, it follows that

$$(Df(X))(T)=TX+XT+T=(X+I)T+TX$$
$$(Df(X))^{-1}((X+I)T+TX)=T$$

and in view of linearity

$$(X+I)(Df(X))^{-1}(T)+(Df(X))^{-1}(T)X=T.$$

If we denote $A=(Df(X))^{-1}(T)$, then it follows that A satisfies the Sylvester matrix equation

$$(X+I)A+AX=T. \tag{3.24}$$

Taking into account [8] and [1] or [12], we can write

$$\begin{aligned}A&=(Df(X))^{-1}(T)\\&=[(1+\operatorname{tr}X)T+TX-XT]\left[(1+\operatorname{tr}X+|X|)I+(2+\operatorname{tr}X)X+X^2\right]^{-1}\end{aligned} \tag{3.25}$$

where $\operatorname{tr}X$ denotes the trace of X and $|X|$ denotes the determinant of X. The one-step method (3.9) takes the form

$$X_{n+1}=h\sum_{j=0}^{n}(Df(X_j))^{-1}(V),\ \|V\|<\delta^*\gamma/2. \tag{3.26}$$

Taking $N=10$, Table 1 shows the results obtained for different values of the parameter γ where $E(\gamma,h)=2h(1+\gamma)\gamma^2\exp(2\gamma(1+\gamma))$. The approximate inverse mapping for different values of h is given by $\hat{g}(V, h, \gamma) = \frac{2}{\delta} X_5$.

γ	δ	h	Error bound E(γ, h)
0.1	0.366666	0.036666	1.0051685×10^{-3}
0.2	0.400000	0.040000	6.205725×10^{-3}
0.3	0.433333	0.043333	2.212012×10^{-2}
0.4	0.466666	0.046666	6.407588×10^{-2}
0.5	0.500000	0.050000	1.680633×10^{-1}

Table 3.1. Example 3.2

Using Mathematica, [15], for $h = 0'0400$, $\delta = 0'4$ and $V = \begin{bmatrix} 0.01 & 0.01 \\ 0 & 0.01 \end{bmatrix}$ from (3.25) one gets the value of the approximate inverse at V,

$$\hat{g}(V,\ 0'04,\ 0'2) = \frac{2}{0'4} X_5 = 5 \times 10^{-3} \begin{bmatrix} 1.99681 & 1.99363 \\ 0 & 1.99681 \end{bmatrix}.$$

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