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ASYMPTOTIC PROPERTIES OF THE GROWTH CURVE MODEL WITH COVARIANCE COMPONENTS

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Abstract. We consider a multivariate regression (growth curve) model of the form $Y = XBZ + \varepsilon$, $\mathrm{E}\,\varepsilon = 0$, $\mathrm{var}(\mathrm{vec}\,\varepsilon) = W \otimes \Sigma$, where $W = \sum_{i=1}^{k} \theta_i V_i$ and θ_i's are unknown scalar covariance components. In the case of replicated observations, we derive the explicit form of the locally best estimators of the covariance components under normality and asymptotic confidence ellipsoids for certain linear functions of the first order parameters $\{B_{ij}\}$ estimating simultaneously the first and the second order parameters.

Keywords: Replicated growth curve model, covariance components, multivariate regression, asymptotic confidence region

MSC 2000: 62F12; secondary 62J12

The Growth Curve Model (GCM) with covariance components is a multivariate regression model defined by the relations:

$$Y = XBZ + \varepsilon \tag{1}$$

$$\mathrm{E}\,\varepsilon = 0, \qquad \mathrm{var}\,(\mathrm{vec}\ \varepsilon) = \sum_{i=1}^{k} \theta_i V_i \otimes \Sigma = W \otimes \Sigma.$$

Here Y is an $n \times p$ matrix of p-dimensional observations, X, Z, Σ, $V_1, \ldots, V_k$ known matrices (of the types $n \times m$, $r \times p$, $n \times n$ and $p \times p$, respectively), B an $m \times r$ matrix of the first order parameters and $\theta_1, \ldots, \theta_k$ the second order (scalar) parameters. The sign $\otimes$ denotes the Kronecker product. In this paper, we do not consider other possible structures of $\mathrm{var}(\mathrm{vec}\,\varepsilon)$.

The uniformly best estimators of unknown parameters B and θ exist only in very special cases. On the other hand, locally best estimators (which are commonly used) depend on a priori chosen parameter value, which may be misleading. That is why we want to weaken the dependence of the estimator on the a priori value. A possible way of doing that is to study asymptotic behavior of the local estimators in the replicated GCM.

The replicated GCM (with s replications) is described as follows:

$$Y_j = XBZ + \varepsilon_j, \quad j = 1, \dots, s \tag{2}$$
$$\mathrm{E}\,\varepsilon_j = 0, \qquad \mathrm{var}\,(\mathrm{vec}\ \varepsilon_j) = \sum_{i=1}^{k} \theta_i V_i \otimes \Sigma$$

and $Y_1, \dots, Y_s$ are stochastically independent random variables. Moreover, we suppose throughout the paper that Y_j's are normally distributed.

Remark: a) The fact that e.g. the $n \times p$ matrix Y_j is normally distributed with mean XBZ and variance of $\mathrm{vec}\, Y_j$ equal to $W \otimes \Sigma$, is written in the following way:

$$Y_j \sim \mathcal{N}_{n\times p}(XBZ, W, \Sigma).$$

b) If G is a $p \times p$ matrix, we denote the $k \times k$ matrix whose (i,j)-th element is $\mathrm{Tr}\,(GV_iGV_j)$ by S_G.

c) $P_X^T \left(= X(X'TX)^- X'T\right)$ is the projection matrix onto the column space $\mathcal{M}(X)$ of a matrix X, $M_X^T \left(= I - P_X^T\right)$ is the projection matrix onto the orthogonal complement $\mathcal{O}(X)$ to $\mathcal{M}(X)$.

Formally, we can write our model also as the multivariate model

$$Y = \mathbf{1}_s \otimes XBZ + \varepsilon,$$

where

$$Y = (Y_1', \dots, Y_s')', \ \mathbf{1}_s = (\underbrace{1, \dots, 1}_{s})'$$

with

$$\mathrm{E}\,\varepsilon = 0 \qquad \text{and} \qquad \mathrm{var}\,(\mathrm{vec}\ \varepsilon) = W \otimes I_s \otimes \Sigma, \tag{3}$$

or as the one-dimensional model

$$\mathrm{vec}\ Y = (\mathbf{1} \otimes Z' \otimes X)\,\mathrm{vec}\ B + \mathrm{vec}\ \varepsilon,$$

where

$$\mathrm{vec}\ Y = \left((\mathrm{vec}\ Y_1)', \dots, (\mathrm{vec}\ Y_s)'\right)'$$

and

$$\text{(4)} \qquad \operatorname{var}(\operatorname{vec}\varepsilon) = \sum_{i=1}^{k} \theta_i (I_s \otimes V_i \otimes \Sigma).$$

That is why we can apply in it the known results from the above mentioned models.

First, we need to know the locally best estimator.

Theorem 1. *Let* $Y_j \sim \mathscr{N}_{n\times p}\Big(XBZ, \sum\limits_{i=1}^{k} \theta_i V_i, \Sigma\Big)$, $j = 1,\dots,s$, *be independent random variables. The locally at the point* $\theta = \theta_0$ *minimum variance unbiased invariant estimator (LMVUIE) of the function* $\gamma = f'\theta$ *is*

$$\widehat{\gamma} = \operatorname{Tr}[AW_\lambda],$$

where

$$W_\lambda = \sum_{i=1}^{k} \lambda_i V_i, \quad W_0 = \sum_{i=1}^{k} \theta_{oi} V_i,$$

$$\text{(5)} \qquad A = \sum_{l=1}^{s} W_0^{-1}(Y_l - \overline{Y})'\Sigma^{-1}(Y_l - \overline{Y})W_0^{-1}$$
$$+ s \cdot W_0^{-1}\overline{Y}'\Sigma^{-1}M_X^{\Sigma^{-1}}\overline{Y}W_0^{-1} + s \cdot W_0^{-1}M_{Z'}^{W_0^{-1}}\overline{Y}'\Sigma^{-1}P_X^{\Sigma^{-1}}\overline{Y}W_0^{-1}M_{Z'}^{W_0^{-1}},$$

λ *is an arbitrary solution of the system*

$$\Big[(sn - r(X))S_{W_0^{-1}} + r(X)S_{W_0^{-1}M_{Z'}^{W_0^{-1}}}\Big]\lambda = f$$

and $r(X)$ *is the rank of the matrix* X. *The estimator* $\widehat{\gamma}$ *does not depend on the choice of the solution.*

P r o o f. All statements can be obtained from Theorem 5.6.8 in Kubáček [7] for the one-dimensional replicated model. Calculation is rather tedious, but straightforward; we use only the well-known properties of the Kronecker product and the vec operator. □

The importance of the last two terms in the matrix A descends with an increasing number of replications s. It seems then that the estimator of W based on

$$S_1^* = \sum_{l=1}^{s} (Y_l - \overline{Y})'\Sigma^{-1}(Y_l - \overline{Y})$$

could be a good estimator independent of the a priori value of θ. In fact, as is shown in the next theorem, we have two options: either to use S_1^* or $S_1^* + S_2^*$, where $S_2^* = s \cdot \overline{Y}'\Sigma^{-1}M_X^{\Sigma^{-1}}\overline{Y}$.

Theorem 2. *Random variables*

$$W^* = \frac{1}{sn - r(X)}\Bigl(\sum_{l=1}^{s}(Y_l - \overline{Y})'\Sigma^{-1}(Y_l - \overline{Y}) + s \cdot \overline{Y}'\Sigma^{-1}M_X^{\Sigma^{-1}}\overline{Y}\Bigr) \tag{6}$$

and $P_X^{\Sigma^{-1}}\overline{Y}$ *are independent, and*

$$W^* \sim \mathscr{W}_p\Bigl(sn - r(X), \frac{1}{sn - r(X)}W\Bigr).$$

In particular,

$$\mathrm{E}\, W^* = W$$

and

$$\operatorname{var}(\operatorname{vec} W^*) = (I + K_p)(W \otimes W).$$

R e m a r k s. a) $\mathscr{W}_p(.,.)$ denotes the Wishart distribution. For the definition of the commutation matrix K_p, see e.g. Magnus and Neudecker [8].

b) It can be shown that the estimator W^* is in this (normal) case optimal.

P r o o f. We shall show that $S_1^* \sim \mathscr{W}_p(n(s-1), W)$ and it is independent of $\overline{Y}$, $S_2^* \sim \mathscr{W}_p(n - r(X), W)$ and it is independent of $P_X^{\Sigma^{-1}}\overline{Y}$ and S_1^* and S_2^* are mutually independent.

Let Q be an orthogonal matrix of order s such that its first column is $(s^{-1/2}, \ldots, s^{-1/2})'$. According to the properties of orthogonal matrices, all other columns have zero sums. Let us define $T_i = \Sigma^{-1/2}Y_i$ and $U_i = \sum_{j=1}^{s} q_{ji}T_i$. Then it is easy to show that

$$S_1^* = \sum_{i=1}^{s}(T_i - \overline{T})'(T_i - \overline{T}) = \sum_{i=1}^{s} U_i'U_i - U_1'U_1 = \sum_{i=2}^{s} U_i'U_i.$$

Because $\operatorname{vec} Y_i \sim \mathscr{N}_{np}((Z' \otimes X)\operatorname{vec} B, W \otimes \Sigma)$, we have

$$\operatorname{vec} U_i \sim \mathscr{N}_{np}\Bigl(\sum_{j=1}^{s} q_{ji}(Z' \otimes \Sigma^{-1/2}X)\operatorname{vec} B, \sum_{j=1}^{s} q_{ji}^2(W \otimes I)\Bigr)$$

and

$$\operatorname{cov}(\operatorname{vec} U_i, \operatorname{vec} U_j) = \delta_{ij}W \otimes I.$$

It is clear now that all U_i's are mutually independent and (because $\sum_{j=1}^{s} q_{ji}^2 = \|q_{\cdot i}\|^2 = 1\ \forall i$) $U_i \sim \mathscr{N}_{n\times p}(0, W, I)$ for $i = 2, \dots, s$. It follows that the rows in each matrix U_i $(i = 2, \dots, s)$ are independent identically distributed random vectors with the variance matrix W. Then $U_i' U_i \sim \mathscr{W}_p(n, W)$ for $i = 2, \dots, s$ and $S_1^* = \sum_{i=2}^{s} U_i' U_i \sim \mathscr{W}_p\,(n(s-1), W)$.

It is easy to show that $\operatorname{cov}(\operatorname{vec} \overline{Y}, \operatorname{vec}(T_i - \overline{T})) = 0\ \forall i = 1, \dots, s$. This implies that $\overline{Y}$ and $S_1^* = \sum_{i=1}^{s} (T_i - \overline{T})'(T_i - \overline{T})$ are independent random variables.

Trivially, $\sqrt{s}\,\overline{Y} \sim \mathscr{N}_{n\times p}(\sqrt{s} XBZ, W, \Sigma)$. Then, according to Rao [9],

$$S_2^* = s(\Sigma^{-1/2}\overline{Y})'\Sigma^{-1/2} M_X^{\Sigma^{-1}} \Sigma^{1/2}(\Sigma^{-1/2}\overline{Y}) \sim \mathscr{W}_p(n - r(X), W),$$

because $\Sigma^{-1/2} M_X^{\Sigma^{-1}} \Sigma^{1/2}$ is an idempotent matrix of the rank $n - r(X)$. S_2^* and $P_X^{\Sigma^{-1}} \overline{Y}$ are independent because $\left(P_X^{\Sigma^{-1}} \Sigma^{1/2}\right)\left(\Sigma^{-1/2} M_X^{\Sigma^{-1}} \Sigma^{1/2}\right) = 0$. Independence of S_1^* and S_2^* follows from the fact that $S_2^* = s \cdot U_1' \Sigma^{-1/2} M_X^{\Sigma^{-1}} \Sigma^{1/2} U_1$ and all U_i's are independent. Then $S^* = S_1^* + S_2^* \sim \mathscr{W}_p(sn - r(X), W)$. All properties of W^* now follow from the known properties of the Wishart distribution. □

We need the estimate of W^* as a basis for further estimation of the second order parameters θ.

Theorem 3. *Let $Y_i \sim \mathscr{N}_{n\times p}\,(XBZ, W, \Sigma)$, $i = 1, \dots, s$, be independent identically distributed random variables and let $W = \sum_{i=1}^{k} \theta_i V_i$, where $\theta \in \Theta \subset \mathbb{R}^k$, Θ is an open bounded set in$\mathbb{R}^k$, W is regular $\forall \theta \in \Theta$, Σ is a regular matrix and matrices $V_1, \dots, V_k$ are linearly independent. Then LMVUE of the vector θ at the point θ_0 based on the matrix W^* exists and is of the form*

$$\widehat{\theta}(\theta_0) = S^{-1}_{W_0^{-1}} \eta(\theta_0), \tag{7}$$

where the vector η has elements $\eta_i = \operatorname{Tr}\left(W^* W_0^{-1} V_i W_0^{-1}\right)$. Moreover,

$$\operatorname*{var}_{\theta}\ \widehat{\theta}(\theta_0) = \frac{2}{sn - r(X)} S^{-1}_{W_0^{-1}} S_{W_0^{-1} W W_0^{-1}} S^{-1}_{W_0^{-1}}.$$

P r o o f. According to Kubáček [6], Theorem 3.2, its consequence and Remark 3.3, it is sufficient to take the first two terms from Theorem 1 and to construct the unbiased estimator of θ on this basis. The assumptions of our theorem guarantee the fulfilment of all requirements of the quoted theorems. Then, let us consider the estimator

$$\widehat{\gamma} = \operatorname{Tr}\left(S^* W_0^{-1} W_\lambda W_0^{-1}\right) = \sum_{i=1}^{k} \lambda_i \operatorname{Tr}\left(S^* W_0^{-1} V_i W_0^{-1}\right).$$

According to Theorem 2, we have

$$\begin{aligned} \operatorname*{E}_{\theta} \widehat{\gamma} &= \sum_{i=1}^{k} \lambda_i \operatorname{Tr}\left(W_0^{-1} V_i W_0^{-1} \operatorname{E} S^*\right) = (sn - r(X)) \sum_{i=1}^{k} \lambda_i \operatorname{Tr}\left(W_0^{-1} V_i W_0^{-1} W\right) \\ &= (sn - r(X)) \sum_{i=1}^{k} \sum_{j=1}^{k} \lambda_i \theta_j \operatorname{Tr}\left(W_0^{-1} V_i W_0^{-1} V_j\right) = (sn - r(X)) \theta' \mathrm{S}_{W_0^{-1}} \lambda. \end{aligned}$$

This estimator is unbiased iff

$$\operatorname*{E}_{\theta} \widehat{\gamma} = \theta' f \quad \forall \theta \quad \Leftrightarrow \quad f = (sn - r(X)) \mathrm{S}_{W_0^{-1}} \lambda.$$

Independence of the matrices V_i guarantees that $\mathrm{S}_{W_0^{-1}}$ is regular and this implies that all θ_i are estimable. Then, using $f = e_i$, we have

$$\widehat{\theta}_i(\theta_0) = \lambda'_{(i)} (sn - r(X)) \eta(\theta_0) = e_i' \mathrm{S}_{W_0^{-1}}^{-1} \eta(\theta_0),$$

which proves the first assertion.

Because $\operatorname{var}(\operatorname{vec} W^*) = \frac{1}{sn - r(X)} (I + K_p)(W \otimes W)$, we conclude that

$$\begin{aligned} \operatorname{cov}(\eta_i, \eta_j) &= \operatorname{cov}\left(\operatorname{Tr}\left(W^* W_0^{-1} V_i W_0^{-1}\right), \operatorname{Tr}\left(W^* W_0^{-1} V_j W_0^{-1}\right)\right) \\ &= \frac{2}{sn - r(x)} \operatorname{Tr}\left(W_0^{-1} V_i W_0^{-1} W W_0^{-1} V_j W_0^{-1} W\right) = \frac{2}{sn - r(x)} \left\{\mathrm{S}_{W_0^{-1} W W_0^{-1}}\right\}_{ij}. \end{aligned}$$

This proves the second assertion. □

Now we are able to define new estimates

(8) $$\widehat{W} = \sum_{i=1}^{k} \widehat{\theta}_i(\theta_0) V_i$$

(9) $$\widehat{\theta}^* = \widehat{\theta}\left(\widehat{\theta}(\theta_0)\right) = \mathrm{S}_{\widehat{W}^{-1}}^{-1} \eta\left(\widehat{\theta}(\theta_0)\right)$$

(10) $$\widehat{W}_* = \sum_{i=1}^{k} \widehat{\theta}_i^*(\theta_0) V_i$$

(11) $$\widehat{\widehat{B}} = (X' \Sigma^{-1} X)^- X' \Sigma^{-1} \overline{Y} \widehat{W}_*^{-1} Z' (Z \widehat{W}_*^{-1} Z')^-,$$

where the vector $\eta\left(\widehat{\theta}(\theta_0)\right)$ has elements $\eta_i = \operatorname{Tr}\left(W^* \widehat{W}^{-1} V_i \widehat{W}^{-1}\right)$.

We will derive their properties in the following theorems. Naturally, we will concentrate on the estimator $\widehat{\widehat{B}}$.

Lemma 4. *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of estimators of a parameter $\theta \in \Theta \subset \mathbb{R}^k$ such that*

$$\sqrt{n}(T_n - \theta) \underset{n\to\infty}{\overset{\mathscr{D}}{\longrightarrow}} \mathscr{N}_k(0, \Sigma(\theta)).$$

Let Θ be an open set in $\mathbb{R}^k$, Σ be a continuous function of the parameter θ, and let $\Sigma(\theta)$ be regular $\forall \theta \in \Theta$. Further, let a function $g : \Theta \to \mathbb{R}^m$ have continuous partial derivatives $\dfrac{\partial g_i}{\partial \theta_j}$ $\forall i, j$ and let the matrix $\dfrac{\partial g}{\partial \theta'}\Sigma(\theta)\dfrac{\partial g'}{\partial \theta}$ be regular $\forall \theta \in \Theta$. Then

$$\sqrt{n}\Big(\frac{\partial g}{\partial T_n'}\Sigma(T_n)\frac{\partial g'}{\partial T_n}\Big)^{-1/2}\big(g(T_n) - g(\theta)\big) \underset{n\to\infty}{\overset{\mathscr{D}}{\longrightarrow}} \mathscr{N}_m(0, I).$$

P r o o f. See Rao [9], section 6a.2. □

Lemma 5. *Under the assumptions of Theorem 3 we have*

$$\widehat{W} \underset{s\to\infty}{\overset{\mathscr{P}}{\longrightarrow}} W$$

and

$$\sqrt{s}\Big(\frac{n}{2}S_{\widehat{W}^{-1}}\Big)^{1/2}\left(\widehat{\theta}^* - \theta\right) \underset{s\to\infty}{\overset{\mathscr{D}}{\longrightarrow}} \mathscr{N}_k(0, I).$$

P r o o f. The first assertion is trivial, because the estimator $\widehat{\theta}(\theta_0)$ is unbiased and $\mathrm{var}_\theta\ \widehat{\theta}(\theta_0) \underset{s\to\infty}{\longrightarrow} 0$. This implies, together with the fact that $\mathrm{S}_{W^{-1}}$ is a continuous function of the variance components, that

$$\Big(\frac{n}{2}\mathrm{S}_{\widehat{W}^{-1}}\Big)^{1/2} - \Big(\frac{n}{2}\mathrm{S}_{W^{-1}}\Big)^{1/2} \underset{s\to\infty}{\overset{\mathscr{P}}{\longrightarrow}} 0$$

and

$$\widehat{\theta}^* - \widehat{\theta}(\theta) = \mathrm{S}^{-1}_{W_0^{-1}}\eta\left(\widehat{\theta}(\theta_0)\right) - \mathrm{S}^{-1}_{W_0^{-1}}\eta(\theta) \underset{s\to\infty}{\overset{\mathscr{P}}{\longrightarrow}} 0.$$

According to Theorem 2, W^* is the sum of independent identically distributed random variables; that is why the sequence $\left\{\sqrt{s}(W^* - W)\right\}_{s=p+1}^{\infty}$ is asymptotically normally distributed. The estimator $\widehat{\theta}(\theta)$ is a function of W^*; then, according to Lemma 4, also the sequence $\left\{\sqrt{s}(\widehat{\theta}(\theta) - \theta)\right\}_{s=p+1}^{\infty}$ is asymptotically normal and, with respect to Theorem 3, we have

$$\sqrt{s}\left(\frac{n}{2}\mathrm{S}_{W^{-1}}\right)^{1/2}(\widehat{\theta}(\theta) - \theta) \underset{s\to\infty}{\overset{\mathscr{D}}{\longrightarrow}} \mathscr{N}_k(0, I).$$

Since
$$\sqrt{s}\left(\frac{n}{2}\mathrm{S}_{\widehat{W}^{-1}}\right)^{1/2}(\widehat{\theta}^{*}-\theta)=\sqrt{s}\left(\frac{n}{2}\mathrm{S}_{W^{-1}}\right)^{1/2}(\widehat{\theta}(\theta)-\theta)+\varepsilon_s$$
where
$$\begin{aligned}\varepsilon_s=&\sqrt{s}\left(\frac{n}{2}\mathrm{S}_{W^{-1}}\right)^{1/2}(\widehat{\theta}^{*}-\widehat{\theta}(\theta_0))\\&+\sqrt{s}\left[\left(\frac{n}{2}\mathrm{S}_{\widehat{W}^{-1}}\right)^{1/2}-\left(\frac{n}{2}\mathrm{S}_{W^{-1}}\right)^{1/2}\right](\widehat{\theta}(\theta_0)-\theta)\\&+\sqrt{s}\left[\left(\frac{n}{2}\mathrm{S}_{\widehat{W}^{-1}}\right)^{1/2}-\left(\frac{n}{2}\mathrm{S}_{W^{-1}}\right)^{1/2}\right](\widehat{\theta}^{*}-\widehat{\theta}(\theta_0))\xrightarrow[s\to\infty]{\mathscr{P}}0,\end{aligned}$$
the sequences
$$\left\{\sqrt{s}\left(\frac{n}{2}\mathrm{S}_{\widehat{W}^{-1}}\right)^{1/2}(\widehat{\theta}^{*}-\theta)\right\}_{s=p+1}^{\infty}$$
and
$$\left\{\sqrt{s}\left(\frac{n}{2}\mathrm{S}_{W^{-1}}\right)^{1/2}(\widehat{\theta}(\theta)-\theta)\right\}_{s=p+1}^{\infty}$$
have the same asymptotic distribution. □

We can consider two types of linear functions of B:

1) $\alpha_1 = CBD$, where C and D are of the types $u\times m$ and $r\times v$,
2) $\alpha_2 = \mathrm{Tr}(F'B)$, where F is of the type $m\times r$.

The first type is a matrix function; all elements of the matrix are functions of the type $c'Bd=\mathrm{Tr}(Bdc')$. This implies that the second class of functions is larger. On the other hand, structures of the first type are very natural in our model. That is why we will investigate both cases. The following theorems show that estimators of these parametric functions are asymptotically normal and determine an asymptotic confidence ellipsoid for them.

Theorem 6. *Let $\alpha_1 = CBD$ be an estimable function. Then, under the assumptions of Theorem 3, if moreover the matrices Z' and D' have full column rank and the matrix $C(X'\Sigma^{-1}X)^{+}C'$ is regular, we have*
$$\sqrt{s}\,\widehat{\Gamma}^{-1/2}\,\mathrm{vec}\left(C\widehat{\widehat{B}}D-CBD\right)\xrightarrow[s\to\infty]{\mathscr{D}}\mathscr{N}_{uv}(0,I)$$
and
$$s\left[\mathrm{vec}\,(C\widehat{\widehat{B}}D-CBD)\right]'\widehat{\Gamma}^{-1}\left[\mathrm{vec}\,(C\widehat{\widehat{B}}D-CBD)\right]\xrightarrow[s\to\infty]{\mathscr{D}}\chi^2_{uv},$$

where

$$\widehat{\Gamma} = D'(Z\widehat{W}_*^{-1}Z')^{-1}D \otimes C(X'\Sigma^{-1}X)^+C' + \frac{2}{n} FVS^{-1}_{\widehat{W}_*^{-1}} V'F', \tag{12}$$

$$F = D'(Z\widehat{W}_*^{-1}Z')^{-1}Z\widehat{W}_*^{-1} \otimes C(X'\Sigma^{-1}X)^+X'\Sigma^{-1}\widehat{r}\widehat{W}_*^{-1}M_{Z'}^{\widehat{W}_*^{-1}}, \tag{13}$$

$$V = (\operatorname{vec} V_1, \dots, \operatorname{vec} V_k) \quad \textit{and} \quad \widehat{r} = \overline{Y} - X\widehat{\widehat{B}}Z.$$

P r o o f. One can easily find that estimability of α_1 is equivalent to the existence of matrices $M_{u\times n}$ and $N_{p\times v}$ such that $MX = C$ and $ZN = D$. That means $\alpha_1 = MXBZN$. Again, we want to make use of Lemma 4. Let us consider the function $g\left(\operatorname{vec} MP_X^{\Sigma^{-1}}\overline{Y}, \widehat{\theta}^*\right) = \operatorname{vec} C\widehat{\widehat{B}}D = \left(N'P_{Z'}^{\widehat{W}_*^{-1}} \otimes I\right) \operatorname{vec} MP_X^{\Sigma^{-1}}\overline{Y}$. Random variables W^* and $P_X^{\Sigma^{-1}}\overline{Y}$ are, according to Theorem 2, independent; this implies that $\widehat{\theta}^*$ and $MP_X^{\Sigma^{-1}}\overline{Y}$ are also independent. It can be easily seen that

$$MP_X^{\Sigma^{-1}}\overline{Y} \sim \mathscr{N}_{n\times p}\left(MP_X^{\Sigma^{-1}}XBZ, \frac{1}{s}\,W,\, \Omega\right),$$

where $\Omega = C(X'\Sigma^{-1}X)^+C'$. Now, with respect to Lemma 5, we can state that

$$\sqrt{s}\begin{pmatrix} \operatorname{vec}(MP_X^{\Sigma^{-1}}\overline{Y} - MP_X^{\Sigma^{-1}}XBZ) \\ \widehat{\theta}^* - \theta \end{pmatrix} \xrightarrow[s\to\infty]{\mathscr{D}} \mathscr{N}_{np+k}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} W\otimes\Omega & 0 \\ 0 & \frac{2}{n}\mathrm{S}^{-1}_{W^{-1}} \end{pmatrix}\right).$$

Let us denote $\operatorname{vec} MP_X^{\Sigma^{-1}}\overline{Y} = y$. Then

$$\frac{\partial g}{\partial(y,\widehat{\theta}^*)'}\begin{pmatrix} \widehat{W}_*\otimes\Omega & 0 \\ 0 & \frac{2}{n}\mathrm{S}^{-1}_{\widehat{W}_*^{-1}} \end{pmatrix}\frac{\partial g'}{\partial(y,\widehat{\theta}^*)} = \frac{\partial g}{\partial y'}\left(\widehat{W}_*\otimes\Omega\right)\frac{\partial g'}{\partial y} + \frac{\partial g}{\partial\widehat{\theta}^{*\prime}}\left(\frac{2}{n}\mathrm{S}^{-1}_{\widehat{W}_*^{-1}}\right)\frac{\partial g'}{\partial\widehat{\theta}^*},$$

$$\frac{\partial g}{\partial y'} = N'P_{Z'}^{\widehat{W}_*^{-1}} \otimes I,$$

$$\frac{\partial g}{\partial\widehat{\theta}^{*\prime}} = -\left(N'P_{Z'}^{\widehat{W}_*^{-1}} \otimes MP_X^{\Sigma^{-1}}\widehat{r}\widehat{W}_*^{-1}M_{Z'}^{\widehat{W}_*^{-1}}\right)(\operatorname{vec} V_1, \dots, \operatorname{vec} V_k)$$

(we have exchanged $\overline{Y}$ for $\widehat{r}$, because $X\widehat{\widehat{B}}Z\left(M_{Z'}^{\widehat{W}_*^{-1}}\right)' = 0$). Taking into account the properties of the projection matrices and the relations between the matrices M, N and C, D, respectively, we are led to the relation

$$N'P_{Z'}^{\widehat{W}_*^{-1}}\widehat{W}_*\left(P_{Z'}^{\widehat{W}_*^{-1}}\right)'N\otimes\Omega = D'(Z\widehat{W}_*^{-1}Z')^{-1}D \otimes C(X'\Sigma^{-1}X)^+C'.$$

This proves the first assertion. The second is a direct consequence of the first and Sverdrup's theorem. □

Theorem 7. *Let $\alpha_2 = \mathrm{Tr}(F'B) = \mathrm{Tr}(U'XBZ)$ be an estimable function, let the matrix Z' have full rank and let the matrix $U'P_X^{\Sigma^{-1}}\Sigma U$ be regular. Then, under the assumptions of Theorem 3,*

$$\sqrt{\frac{s}{\widehat{\xi}}}\left(\mathrm{Tr}(F'\widehat{\widehat{B}}) - \mathrm{Tr}(F'B)\right) \xrightarrow[s\to\infty]{\mathscr{D}} \mathscr{N}(0,1)$$

and

$$\frac{s}{\widehat{\xi}}\left(\mathrm{Tr}(F'\widehat{\widehat{B}}) - \mathrm{Tr}(F'B)\right)^2 \xrightarrow[s\to\infty]{\mathscr{D}} \chi_1^2$$

holds, where

$$\widehat{\xi} = \mathrm{Tr}\left(F'(X'\Sigma^{-1}X)^+F(ZW_*^{-1}Z')^{-1}\right) + \frac{2}{n}\tau' S^{-1}_{\widehat{W}_*^{-1}}\tau \tag{14}$$

and the vector τ has elements

$$\tau_i = \mathrm{Tr}\left(F'(X'\Sigma^{-1}X)^+X'\Sigma^{-1}\widehat{r}\widehat{W}_*^{-1}M_{Z'}^{\widehat{W}_*^{-1}}V_i\widehat{W}_*^{-1}Z'(Z\widehat{W}_*^{-1}Z')^{-1}\right). \tag{15}$$

P r o o f. It is analogous to that of Theorem 6. Again it is clear that estimability of α_2 is equivalent to the existence of a matrix U such that $F = X'UZ'$. Let us consider the function

$$g\left(\mathrm{vec}\ U'P_X^{\Sigma^{-1}}\overline{Y}, \widehat{\theta}^*\right) = \mathrm{Tr}(U'X\widehat{\widehat{B}}Z) = \left(\mathrm{vec}\ P_{Z'}^{\widehat{W}_*^{-1}}\right)'\left(\mathrm{vec}\ U'P_X^{\Sigma^{-1}}\overline{Y}\right).$$

Then $\mathrm{var}\left(\mathrm{vec}\ U'P_X^{\Sigma^{-1}}\overline{Y}\right) = \frac{1}{s}W\otimes U'P_X^{\Sigma^{-1}}\Sigma U$. Denoting $u = \mathrm{vec}\ U'P_X^{\Sigma^{-1}}\overline{Y}$ we get

$$\begin{aligned}\frac{\partial g}{\partial u'}\left(\widehat{W}_*\otimes U'P_X^{\Sigma^{-1}}\Sigma U\right)\frac{\partial g'}{\partial u} &= \left(\mathrm{vec}\ P_{Z'}^{\widehat{W}_*^{-1}}\right)'\left(\widehat{W}_*\otimes U'P_X^{\Sigma^{-1}}\Sigma U\right)\left(\mathrm{vec}\ P_{Z'}^{\widehat{W}_*^{-1}}\right)\\ &= \mathrm{Tr}\left[U'P_X^{\Sigma^{-1}}\Sigma U\widehat{W}_*\left(P_{Z'}^{\widehat{W}_*^{-1}}\right)'\right] = \mathrm{Tr}\left[F'(X'\Sigma^{-1}X)^+F(Z\widehat{W}_*^{-1}Z')^{-1}\right]\end{aligned}$$

and

$$\begin{aligned}\frac{\partial g}{\partial\widehat{\theta}_i^*} &= -\left(\mathrm{vec}\ P_{Z'}^{\widehat{W}_*^{-1}}V_i\widehat{W}_*^{-1}M_{Z'}^{\widehat{W}_*^{-1}}\right)'\left(\mathrm{vec}\ U'P_X^{\Sigma^{-1}}\overline{Y}\right)\\ &= -\mathrm{Tr}\left[U'P_X^{\Sigma^{-1}}\overline{Y}\widehat{W}_*^{-1}M_{Z'}^{\widehat{W}_*^{-1}}V_i\left(P_{Z'}^{\widehat{W}_*^{-1}}\right)'\right]\\ &= -\mathrm{Tr}\left[F'(X'\Sigma^{-1}X)^+X'\Sigma^{-1}\widehat{r}\widehat{W}_*^{-1}M_{Z'}^{\widehat{W}_*^{-1}}V_i\widehat{W}_*^{-1}Z'(Z\widehat{W}_*^{-1}Z')^{-1}\right] = -\tau_i\end{aligned}$$

(we have exchanged $\overline{Y}$ for $\widehat{r}$ again). The first assertion is a consequence of Lemma 4. The second is again an immediate consequence of the first one. □

R e m a r k s. a) We could write $\overline{Y}$ instead of $\widehat{r}$ in both Theorems 6 and 7.

b) We could leave the assumption of regularity of the matrix $U'P_X^{\Sigma^{-1}}\Sigma U$, if we started from the less effective estimator S_1^* of the matrix W instead of S^*; we could then take the function g as a function of vec Y which has a regular variance matrix and is independent of θ^*. The assertion of Theorem 7 would be still valid—only convergence would be slower.

If we knew the matrix W, then the standard least squares estimator of B would be $\widehat{B} = (X'\Sigma^{-1}X)^{-}X'\Sigma^{-1}\overline{Y}W^{-1}Z'(ZW^{-1}Z')^{-}$. The variance of the estimators of the parametric functions α_1 and α_2 would consist only of the first terms (with W instead of $\widehat{W_*}$) of the formulas in Theorems 6 and 7. These theorems thus show how much the variance of the estimators of α_1 and α_2 is affected by estimating the variance matrix. Naturally, this correction is only an asymptotic one. Numerical studies in the one-dimensional case show that, especially when the number of replications is small, the real variance of the estimator is rather larger than the asymptotic one (see Kubáček [6]).

The following two theorems show the situation in two important special cases: when the matrix W is completely unknown (including its structure) and when the matrix W is diagonal. Proofs are analogous to those of Theorems 6 and 7 and that is why we omit them. We consider only the function α_1.

Theorem 8. *Let $\alpha_1 = CBD$ be an estimable function, let the matrices Z' and D' have full column rank and let the matrix $C(X'\Sigma^{-1}X)^{+}C'$ be regular. Further, let the matrix W be unknown and* $\operatorname{vech} W \in \Theta \subset \mathbb{R}^{p(p+1)/2}$*, where Θ is an open bounded set such that W is regular* $\forall \operatorname{vech} W \in \Theta$*. Then*

$$\sqrt{s}\,\widetilde{\Gamma}^{-1/2}\operatorname{vec}(C\widetilde{B}D - CBD) \xrightarrow[s\to\infty]{\mathscr{D}} \mathscr{N}_{uv}(0, I)$$

and

$$s\left[\operatorname{vec}(C\widetilde{B}D - CBD)\right]'\widetilde{\Gamma}^{-1}\left[\operatorname{vec}(C\widetilde{B}D - CBD)\right] \xrightarrow[s\to\infty]{\mathscr{D}} \chi^2_{uv},$$

where

$$\widetilde{B} = (X'\Sigma^{-1}X)^{-}X'\Sigma^{-1}\overline{Y}\,(W^*)^{-1}Z'\left(Z\,(W^*)^{-1}Z'\right)^{-}, \tag{16}$$

$$\widetilde{\Gamma} = D'\left(Z\,(W^*)^{-1}Z'\right)^{-1}D \otimes C(X'\Sigma^{-1}X)^{+}C' + \frac{1}{n}\,[(cC)F]Q[(cR)F'], \tag{17}$$

$$F = D'\left(Z\,(W^*)^{-1}Z'\right)^{-1}Z\,(W^*)^{-1} \otimes C(X'\Sigma^{-1}X)^{+}X'\Sigma^{-1}\widehat{r}\,(W^*)^{-1}M_{Z'}^{(W^*)^{-1}}, \tag{18}$$

$$\{Q\}_{ij,kl} = w^*_{ik}w^*_{jl} + w^*_{jk}w^*_{il} \text{ for } i \leqslant j, k \leqslant l \quad \text{and} \quad \widehat{r} = \overline{Y} - X\widetilde{B}Z.$$

Remark: vech is the vec-half operator and (cC), (cR) are the operations of appropriate columns and rows summation. For exact definitions see Kubáček [5].

In the case of W being diagonal, using a "diagonal estimator" of W would be more appropriate than using W^* as the starting estimate. This is the only difference of the next theorem from the other ones. It is clear from Theorem 2 that

$$\operatorname{var} \sqrt{s}\, w_{ii}^* \xrightarrow[s\to\infty]{\mathscr{D}} \frac{2}{n}\, w_{ii}^2.$$

Theorem 9. *Under the assumptions of Theorem 6, if W is a diagonal matrix then*

$$\sqrt{s}\,\widehat{\Gamma}_\Delta^{-1/2} \operatorname{vec}(C\widehat{B}_\Delta D - CBD) \xrightarrow[s\to\infty]{\mathscr{D}} \mathscr{N}_{uv}(0, I)$$

and

$$s\left[\operatorname{vec}(C\widehat{B}_\Delta D - CBD)\right]' \widehat{\Gamma}^{-1}\left[\operatorname{vec}(C\widehat{B}_\Delta D - CBD)\right] \xrightarrow[s\to\infty]{\mathscr{D}} \chi^2_{uv},$$

where

$$\Delta = \operatorname{Diag}(W^*), \qquad \widehat{r} = \overline{Y} - X\widehat{B}_\Delta Z,$$

(19) $$\widehat{B}_\Delta = (X'\Sigma^{-1}X)^- X'\Sigma^{-1}\overline{Y}\Delta^{-1}Z'(Z\Delta^{-1}Z')^-,$$

(20) $$\widehat{\Gamma}_\Delta = D'(Z\Delta^{-1}Z')^{-1}D \otimes C(X'\Sigma^{-1}X)^+C' + \frac{2}{n}\sum_{i=1}^{p} \delta_{ii}^2 \left[D'F_iF_i'D \otimes CG_iG_i'C'\right],$$

(21) $$F_i = D'(Z\Delta^{-1}Z')^{-1}Z\Delta^{-1}e_i,$$

(22) $$G_i = C(X'\Sigma^{-1}X)^+X'\Sigma^{-1}\widehat{r}\Delta^{-1}M_{Z'}^{\Delta^{-1}}e_i,$$

or

(23) $$\widehat{\Gamma}_\Delta = \left[D'(Z\Delta^{-1}Z')^{-1}D \otimes C(X'\Sigma^{-1}X)^+C'\right]\Big[Z\Delta^{-1}Z' \otimes X'\Sigma^{-1}X + \frac{2}{n}\sum_{i=1}^{p} \delta_{ii}^2 Z\Delta^{-1}e_ie_i'\Delta^{-1}Z' \otimes X'\Sigma^{-1}\widehat{r}\Delta^{-1}M_{Z'}^{\Delta^{-1}}e_ie_i'\Delta^{-1}M_{Z'}^{\Delta^{-1}}\widehat{r}'\Sigma^{-1}X\Big] \times\left[(Z\Delta^{-1}Z')^{-1}D \otimes (X'\Sigma^{-1}X)^+C'\right].$$

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