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ON A NEW KIND OF 2-PERIODIC TRIGONOMETRIC INTERPOLATION

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Summary. It is well-known that the interpolation theory plays an important role in many fields of computer vision, especially in surface reconstruction. In this paper, we introduce a new kind of 2-period interpolation of functions with period 2π. We find out the necessary and sufficient conditions for regularity of this new interpolation problem. Moreover, a closed form expression for the interpolation polynomial is given. Our interpolation is of practical significance. Our results provide the theoretical basis for using our interpolation in practical problems.

Keywords: Interpolation, trigonometric polynomial, regularity, computer vision

AMS classification: 41A05

1. INTRODUCTION

It is well-known that the interpolation theory plays an important role in many fields of computer vision, especially in surface reconstruction. Recently, many works have been devoted to studying the Hermite interpolation and Birkhoff interpolation. Here we only mention a very small part of them, for example, see [1–4]. As we know, these interpolations are only adequate for smooth functions. In many practical issues of computer vision, we do not know if the interpolated function is differentiable or we only know the values of the interpolated function at nodes, so we can not use these interpolations. From computational view, divided difference is a natural candidate for replacing derivative because divided difference is the discretization of the derivative of functions. In this paper, we will replace the conditions of derivatives for 2-periodic interpolation by those of differences at the nodes. We only consider the interpolation of 2π-periodic functions at the nodes $x_k = x_{k,n} = k\pi/n$ ($k = 0, 1, \ldots, 2n-1$).

For $f \in C_{2\pi}$ and $0 < h < \pi/n$, we define

$$\delta f(x) = \delta^1 f(x) = f(x+h) - f(x-h),$$
$$\delta^m f(x) = \delta(\delta^{m-1} f(x)) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(x + (m-2k)h), \quad m \geqslant 2.$$

We shall say that $t(x) \in \mathcal{T}_n$, if $t(x)$ is a polynomial of the form

$$t_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

If

$$t_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx) + a_n \cos(nx + \varepsilon\pi/2),$$
$$(\varepsilon = 0 \text{ or } 1),$$

we shall say that $t_n(x) \in \mathcal{T}_{n,\varepsilon}$ $(\varepsilon = 0$ or $1)$.

Let $p(t) = p_e(t) + p_o(t)$ be a real algebraic polynomial, where $p_e(t)$ is even and $p_o(t)$ is odd, and $p_e(0) = 0$. Our problems are

P_1: For any two given sets of complex numbers $\{\alpha_k\}_0^{n-1}$ and $\{\beta_k\}_0^{n-1}$, if $p(2h) \neq 0$, decide whether or not there exists a unique trigonometric polynomial $t_n(x) \in \mathcal{T}_{n,\varepsilon}$ $(\varepsilon = 0$ or $1)$ satisfying the conditions

$$t_n(x_{2k}) = \alpha_k, \quad (p(\delta)t_n)(x_{2k+1})/p(2h) = \beta_k, \quad (k = 0, 1, \ldots, n-1).$$

We call the interpolation problem satisfying the above conditions a 2-*periodic interpolation.*

P_2: If the answer to Problem P_1 is affirmative, then usually, we say the interpolation problem is regular. Find necessary and sufficient conditions on n such that Problem P_1 is regular.

P_3: Find the fundamental polynomials of the interpolation when Problem P_1 is regular.

2. Lemmas

In order to prove our main results we need the following lemmas.

Lemma 2.1. *If*

$$K_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx), \tag{1}$$

then

(i) $K_n(x_{2k}) = \delta_{0,k}$ $(k = 0, 1, \ldots, n-1)$ *if and only if*

$$\begin{cases} \frac{1}{2}a_0 + a_n = \frac{1}{n} \\ a_k + a_{n-k} = \frac{2}{n} \\ b_k - b_{n-k} = 0 \\ k = 1, \ldots, n-1; \end{cases}$$

(ii) $K_n(x_{2k}) = 0$ $(k = 0, 1, \ldots, n-1)$ *if and only if*

$$\begin{cases} \frac{1}{2}a_0 + a_n = 0, \\ a_k + a_{n-k} = 0, \\ b_k - b_{n-k} = 0, \\ k = 1, \ldots, n-1. \end{cases}$$

P r o o f. It is an easy consequence of the following two known identities.

$$\frac{\sin \frac{nx}{2}}{n \sin \frac{x}{2}} = \frac{1}{n}\left[1 + 2\sum_{j=1}^{(n-1)/2} \cos jx\right], \qquad n \text{ odd},$$

and

$$\frac{\cos \frac{x}{2} \sin \frac{nx}{2}}{n \sin \frac{x}{2}} = \frac{1}{n}\left[1 + 2\sum_{j=1}^{(n-2)/2} \cos jx + \cos \frac{nx}{2}\right], \qquad n \text{ even}.$$

□

The above two identities come from formula (2.15) in [3].

Lemma 2.2. *If* $K_n(x)$ *is given by* (1), *then*

(i) $K_n(x_{2k+1}) = \delta_{0,k}$ $(k = 0, 1, \ldots, n-1)$ *if and only if*

$$\begin{cases} \frac{1}{2}a_0 - a_n = \frac{1}{n} \\ a_k - a_{n-k} = \frac{2}{n}\cos jx_1 \\ b_k + b_{n-k} = \frac{2}{n}\sin jx_1 \\ k = 1, \ldots, n-1; \end{cases}$$

(ii) $K_n(x_{2k+1}) = 0$ $(k = 0, 1, \ldots, n-1)$ *if and only if*

$$\begin{cases} \frac{1}{2}a_0 - a_n = 0, \\ a_k - a_{n-k} = 0, \\ b_k + b_{n-k} = 0, \\ k = 1, \ldots, n-1. \end{cases}$$

The proof consists in applying Lemma 2.1 after the observation that

$$\begin{aligned} K_n(x_{2k+1}) &= K_n\Big(x_{2k} + \frac{\pi}{n}\Big) \\ &= \frac{a_0}{2} - a_n + \sum_{j=1}^{n-1}(P_j \cos jx_{2k} + Q_j \sin jx_{2k}) \\ &\quad (k = 0, 1, \ldots, n-1), \end{aligned}$$

where

$$P_j = a_j \cos jx_1 + b_j \sin jx_1,$$

and

$$Q_j = b_j \cos jx_1 - a_j \sin jx_1.$$

Lemma 2.3. *For* $s = 1, 2, \ldots$, *we have*

$$\begin{aligned} \delta^{2s} \cos jx &= (\mathrm{i}2 \sin jh)^{2s} \cos jx, \\ \delta^{2s} \sin jx &= (\mathrm{i}2 \sin jh)^{2s} \sin jx, \\ \delta^{2s+1} \cos jx &= \mathrm{i}(\mathrm{i}2 \sin jh)^{2s+1} \sin jx, \\ \delta^{2s+1} \sin jx &= -\mathrm{i}(\mathrm{i}2 \sin jh)^{2s+1} \cos jx. \end{aligned}$$

Proof. It is easy to prove it by induction. □

Lemma 2.4. *Let $p(t) = p_e(t) + p_o(t)$ be a real algebraic polynomial, where $p_e(t)$ is even and $p_o(t)$ is odd. We have*

$$p(\delta)\cos jx = p_e(\mathrm{i}2\sin jh)\cos jx + \mathrm{i}p_o(\mathrm{i}2\sin jh)\sin jx,$$
$$p(\delta)\sin jx = p_e(\mathrm{i}2\sin jh)\sin jx - \mathrm{i}p_o(\mathrm{i}2\sin jh)\cos jx.$$

Proof. From Lemma 2.3, we have

$$\begin{aligned}
p_e(\delta)\cos jx &= p_e(\mathrm{i}2\sin jh)\cos jx,\\
p_o(\delta)\cos jx &= \mathrm{i}p_o(\mathrm{i}2\sin jh)\sin jx,\\
p_e(\delta)\sin jx &= p_e(\mathrm{i}2\sin jh)\sin jx,\\
p_o(\delta)\sin jx &= -\mathrm{i}p_o(\mathrm{i}2\sin jh)\cos jx.
\end{aligned}$$

This implies Lemma 2.4. □

3. Main results and their proofs

Theorem 3.1. (i) *For $\varepsilon = 0$, Problem P_1 is regular if and only if*

$$p_e(\mathrm{i}2\sin jh) \neq 0, \quad \Delta_j := A_j^2 + B_j^2 \neq 0, \tag{2}$$

where

$$\begin{aligned}
A_j &= p_e(\mathrm{i}2\sin jh) + p_e(\mathrm{i}2\sin(n-j)h),\\
B_j &= \mathrm{i}[p_o(\mathrm{i}2\sin(n-j)h) - p_o(\mathrm{i}2\sin jh)],
\end{aligned}$$

$j = 0, 1, 2, \ldots, n-1.$

(ii) *For $\varepsilon = 1$, Problem P_1 is regular if and only if*

$$p_o(\mathrm{i}2\sin jh) \neq 0, \quad \Delta_j \neq 0, \tag{3}$$

$j = 0, 1, 2, \ldots, n-1.$

Proof. We only prove part (i) because the proof of part (ii) is similar. Let $Q_n(x) \in \mathcal{T}_{n,0}$ have the form

$$Q_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n-1}(a_k\cos kx + b_k\sin kx) + a_n\cos nx,$$

and let

$$\begin{cases} Q_n(x_{2k}) = 0 \\ (p(\delta)Q_n)(x_{2k+1}) = 0 \\ k = 0, 1, \ldots, n-1. \end{cases}$$

Taking $K_n(x) = Q_n(x)$ in part (2) of Lemma 2.1, we have

$$\begin{cases} \frac{1}{2}a_0 + a_n = 0 \\ a_j + a_{n-j} = 0 \\ b_j - b_{n-j} = 0 \\ j = 1, \ldots, n-1. \end{cases} \tag{4}$$

Applying Lemma 2.4 and $p_e(0) = 0$, we have

$$\begin{aligned} (p(\delta)Q_n)(x) = \sum_{j=1}^{n-1} & \{[a_j p_e(\mathrm{i}2\sin jh) - \mathrm{i}b_j p_o(\mathrm{i}2\sin jh)]\cos jx \\ & + [b_j p_e(\mathrm{i}2\sin jh) + \mathrm{i}a_j p_o(\mathrm{i}2\sin jh)]\sin jx\} \\ & + a_n p_e(\mathrm{i}2\sin nh)\cos nx + a_n \mathrm{i} p_0(\mathrm{i}2\sin nh)\sin nx. \end{aligned}$$

Taking $K_n(x) = (p(\delta)Q_n)(x)$ in part (2) of Lemma 2.2, we have

$$\begin{cases} -a_n p_e(\mathrm{i}2\sin nh) = 0 \\ a_j p_e(\mathrm{i}2\sin jh) - \mathrm{i}b_j p_o(\mathrm{i}2\sin jh) \\ \qquad -a_{n-j} p_e(\mathrm{i}2\sin(n-j)h) + \mathrm{i}b_{n-j} p_o(\mathrm{i}2\sin(n-j)h) = 0 \\ b_j p_e(\mathrm{i}2\sin jh) + \mathrm{i}a_j p_o(\mathrm{i}2\sin jh) \\ \qquad +b_{n-j} p_e(\mathrm{i}2\sin(n-j)h) + \mathrm{i}a_{n-j} p_o(\mathrm{i}2\sin(n-j)h) = 0, \\ j = 1, 2, \ldots, n-1. \end{cases} \tag{5}$$

From the first equations (4) and (5) we have $a_0 = a_n = 0$ if and only if $p_e(\mathrm{i}2\sin nh) \neq 0$.

From the other equations of (4) and (5) we have

$$\begin{cases} a_j[p_e(\mathrm{i}2\sin jh) + p_e(\mathrm{i}2\sin(n-j)h)] \\ \qquad +\mathrm{i}b_j[-p_o(\mathrm{i}2\sin jh) + p_o(\mathrm{i}2\sin(n-j)h)] = 0 \\ \mathrm{i}a_j[p_o(\mathrm{i}2\sin jh) - p_o(\mathrm{i}2\sin(n-j)h)] \\ \qquad +b_j[p_e(\mathrm{i}2\sin jh) + p_e(\mathrm{i}2\sin(n-j)h)] = 0, \\ j = 1, 2, \ldots, n-1. \end{cases}$$

That is,

$$\begin{cases} A_j a_j + B_j b_j = 0 \\ -B_j a_j + A_j b_j = 0 \\ j = 1, 2, \ldots, n-1. \end{cases}$$

Hence $a_j = b_j = 0$ $(j = 1, 2, \ldots, n-1)$ if and only if $\Delta_j = A_j^2 + B_j^2 \neq 0$ $(j = 1, 2, \ldots, n-1)$. This completes the proof of Theorem 3.1. □

If we denote the fundamental polynomials of $(0, p(\delta))$ interpolation by $r_{j,\varepsilon}(x)$ and $\varrho_{j,\varepsilon}(x)$, respectively, then it is clear that

$$r_{j,\varepsilon}(x) = r_{0,\varepsilon}(x - x_j), \quad \varrho_{j,\varepsilon}(x) = \varrho_{0,\varepsilon}(x - x_j), j = 1, 2, \ldots, n-1.$$

We come to give the explicit forms of the fundamental polynomials $r_{0,\varepsilon}(x)$ and $\varrho_{0,\varepsilon}(x)$. They are determined by the conditions

$$r_{0,\varepsilon}(x_{2k}) = \delta_{0,k}, \quad (p(\delta) r_{0,\varepsilon})(x_{2k+1}) = 0, \tag{6}$$

$$\varrho_{0,\varepsilon}(x_{2k}) = 0, \quad (p(\delta)\varrho_{0,\varepsilon})(x_{2k+1}) = \delta_{0,k}, \tag{7}$$

$k = 1, 2, \ldots, n-1$.

Theorem 3.2. *Let*

$$\begin{cases} C_{1,j} = A_j p_e(\mathrm{i}2\sin(n-j)h) + \mathrm{i}B_j p_o(\mathrm{i}2\sin(n-j)h) \\ D_{1,j} = B_j p_e(\mathrm{i}2\sin(n-j)h) - A_j p_o(\mathrm{i}2\sin(n-j)h)] \\ C_{2,j} = A_j \cos jx_1 - B_j \cos jx_1 \\ D_{2,j} = B_j \cos jx_1 + A_j \cos jx_1, \\ j = 1, 2, \ldots, n-1, \end{cases}$$

where A_j and B_j are the same as in Theorem 3.1. (i) *If $\varepsilon = 0$ and* (2) *hold, then*

$$\begin{cases} r_{0,0} = \dfrac{1}{n} + \dfrac{2}{n}\displaystyle\sum_{k=1}^{n-1} \dfrac{C_{1,k}\cos kx + D_{1,k}\sin kx}{\Delta_k}, \\ \varrho_{0,0} = \dfrac{\cos nx}{np_e(\mathrm{i}2\sin nh)} + \dfrac{2}{n}\displaystyle\sum_{k=1}^{n-1} \dfrac{C_{2,k}\cos kx + D_{2,k}\sin kx}{\Delta_k}. \end{cases} \tag{8}$$

(ii) *If $\varepsilon = 1$ and* (3) *hold, then*

$$\begin{cases} r_{0,1} = \dfrac{1}{n} + \dfrac{2}{n}\displaystyle\sum_{k=1}^{n-1} \dfrac{C_{1,k}\cos kx + D_{1,k}\sin kx}{\Delta_k}, \\ \varrho_{0,1} = \dfrac{\mathrm{i}\sin nx}{np_o(\mathrm{i}2\sin nh)} + \dfrac{2}{n}\displaystyle\sum_{k=1}^{n-1} \dfrac{C_{2,k}\cos kx + D_{2,k}\sin kx}{\Delta_k}. \end{cases} \tag{9}$$

Proof. We only prove part (ii) because the proof of part (i) is similar. For $\varepsilon = 1$, we denote

$$Q_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n-1}(a_k \cos kx + b_k \sin kx) + b_n \sin nx. \tag{10}$$

Then

$$\begin{aligned}(p(\delta)Q_n)(x) = \sum_{j=1}^{n-1} &\{[a_j p_e(\mathrm{i}2\sin jh) - \mathrm{i}b_j p_o(\mathrm{i}2\sin jh)]\cos jx \\ &+[b_j p_e(\mathrm{i}2\sin jh) + \mathrm{i}a_j p_o(\mathrm{i}2\sin jh)]\sin jx\} \\ &+ b_n p_e(\mathrm{i}2\sin nh)\sin nx - \mathrm{i}b_n p_o(\mathrm{i}2\sin nh)\cos nx.\end{aligned}$$

If $Q_n(x)$ satisfies conditions (6), then taking $K_n(x) = Q_n(x)$ in the part (i) of Lemma 2.1 and $K_n(x) = (p(\delta)Q_n)(x)$ in part (ii) of Lemma 2.2 we have

$$\begin{cases} \frac{1}{2}a_0 = \frac{1}{n} \\ a_j + a_{n-j} = \frac{2}{n} \\ b_j - b_{n-j} = 0 \\ j = 1, \ldots, n-1, \end{cases} \tag{11}$$

and

$$\begin{cases} \mathrm{i}b_n p_o(\mathrm{i}2\sin nh) = 0 \\ a_j p_e(\mathrm{i}2\sin jh) - \mathrm{i}b_j p_o(\mathrm{i}2\sin jh) \\ \qquad -a_{n-j}p_e(\mathrm{i}2\sin(n-j)h) + \mathrm{i}b_{n-j}p_o(\mathrm{i}2\sin(n-j)h) = 0 \\ b_j p_e(\mathrm{i}2\sin jh) + \mathrm{i}a_j p_o(\mathrm{i}2\sin jh) \\ \qquad +b_{n-j}p_e(\mathrm{i}2\sin(n-j)h) + \mathrm{i}a_{n-j}p_o(\mathrm{i}2\sin(n-j)h) = 0, \\ j = 1, 2, \ldots, n-1. \end{cases} \tag{12}$$

From (11) and (12) we have $a_0 = 2/n$, $b_n = 0$, and

$$\begin{cases} a_j p_e(\mathrm{i}2\sin jh) - (\frac{2}{n} - a_j)p_e(\mathrm{i}2\sin(n-j)h) + B_j b_j = 0 \\ \mathrm{i}a_j p_o(\mathrm{i}2\sin jh) + \mathrm{i}(\frac{2}{n} - a_j)p_o(\mathrm{i}2\sin(n-j)h) + A_j b_j = 0, \\ j = 1, 2, \ldots, n-1. \end{cases}$$

That is,

$$\begin{cases} A_j a_j + B_j b_j = \frac{2}{n}p_e(\mathrm{i}2\sin(n-j)h) \\ -B_j a_j + A_j b_j = -\frac{\mathrm{i}2}{n}p_o(\mathrm{i}2\sin(n-j)h), \\ j = 1, 2, \ldots, n-1. \end{cases} \tag{13}$$

From (13) we have

$$\begin{cases} a_j = \frac{2}{n\Delta_j}(A_j p_e(\mathrm{i}2\sin(n-j)h) + B_j \mathrm{i} p_o(\mathrm{i}2\sin(n-j)h)) \\ b_j = \frac{2}{n\Delta_j}(B_j p_e(\mathrm{i}2\sin(n-j)h) - A_j \mathrm{i} p_o(\mathrm{i}2\sin(n-j)h)), \\ j = 1, 2, \dots, n-1. \end{cases}$$

This implies the first equality of (9).

If $Q_n(x)$ satisfies conditions (7), then taking $K_n(x) = Q_n(x)$ in part (ii) of Lemma 2.1 and $K_n(x) = (p(\delta)Q_n)(x)$ in part (i) of Lemma 2.2 we have

$$\begin{cases} \frac{1}{2}a_0 = 0 \\ a_j + a_{n-j} = 0 \\ b_j - b_{n-j} = 0 \\ j = 1, \dots, n-1, \end{cases} \tag{14}$$

and

$$\begin{cases} \mathrm{i}b_n p_o(\mathrm{i}2\sin nh) = \frac{1}{n} \\ a_j p_e(\mathrm{i}2\sin jh) - \mathrm{i}b_j p_o(\mathrm{i}2\sin jh) \\ \quad -a_{n-j} p_e(\mathrm{i}2\sin(n-j)h) + \mathrm{i}b_{n-j} p_o(\mathrm{i}2\sin(n-j)h) = \frac{2}{n}\cos jx_1 \\ b_j p_e(\mathrm{i}2\sin jh) + \mathrm{i}a_j p_o(\mathrm{i}2\sin jh) \\ \quad b_{n-j} p_e(\mathrm{i}2\sin(n-j)h) + \mathrm{i}a_{n-j} p_o(\mathrm{i}2\sin(n-j)h) = \frac{2}{n}\sin jx_1, \\ j = 1, 2, \dots, n-1. \end{cases} \tag{15}$$

From (14) and (15) we have $a_0 = 0$, $b_n = -\mathrm{i}/(np_o(\mathrm{i}2\sin nh))$, and

$$\begin{cases} A_j a_j + B_j b_j = \frac{2}{n}\cos jx_1 \\ -B_j a_j + A_j b_j = \frac{2}{n}\sin jx_1, \\ j = 1, 2, \dots, n-1. \end{cases} \tag{16}$$

From (16) we have

$$\begin{cases} a_j = \dfrac{2(A_j \cos jx_1 - B_j \cos jx_1)}{n\Delta_j} \\ b_j = \dfrac{2(B_j \cos jx_1 + A_j \cos jx_1)}{n\Delta_j}, \\ j = 1, 2, \dots, n-1. \end{cases}$$

This implies the second equality of (9). □

4. Conclusion

In this paper, we have replaced the conditions of derivatives for Hermite interpolation and Birkhoff interpolation by those of differences at the nodes. First, we have introduced a new kind of 2-period interpolation of functions with period 2π. Second, we found the necessary and sufficient conditions for regularity of this new interpolation problem. Moreover, a closed form expression for the interpolation polynomial was given. Our interpolation is of practical significance. Our results provide the theoretical basis for using our interpolation in practical ploblems. In future, we will try to extend our idea to the case of algebraic polynomial interpolation and of multivariate interpolation.

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