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Applications of Mathematics, Vol. 40 (1995), No. 6, 411–431

Persistent URL: http://dml.cz/dmlcz/134305

40 (1995) APPLICATIONS OF MATHEMATICS No. 6, 411–431

PROPERLY RECORDED ESTIMATE AND CONFIDENCE REGIONS OBTAINED BY AN APPROXIMATE COVARIANCE OPERATOR IN A SPECIAL NONLINEAR MODEL

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(Received March 22, 1994)

Summary. The properly recorded standard deviation of the estimator and the properly recorded estimate are introduced. Bounds for the locally best linear unbiased estimator and estimate and also confidence regions for a linearly unbiasedly estimable linear functional of unknown parameters of the mean value are obtained in a special structure of nonlinear regression model. A sufficient condition for obtaining the properly recorded estimate in this model is also given.

Keywords: Properly recorded estimate, nonlinear regression, variances depending on the mean value parameters, confidence regions, bounds for the estimate

AMS classification: 62J05, 62F10

1. INTRODUCTION

Many situations in measurement can be represented by a model, where the result of observations $\mathbf{y}_{n,1}$ is a realization of a normally distributed random vector $\mathbf{Y}_{n,1}$ with mean value $\mathscr{E}(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta}$, where $\mathbf{X}$ is a known $n \times k$ design matrix and $\boldsymbol{\beta} \in \mathbb{R}^k$ is an unknown vector of parameters. A large class of measurement devices has its dispersion characteristic of the form $\sigma^2(a + b|s|)^2$, where s is the actual value of the measured quantity; σ^2, a and b are known positive constants (see e.g. [5], [2]). If we assume independent measurements, we obtain

$$(1) \qquad (\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}(\boldsymbol{\beta}))$$

as a model of measurement, where $\mathbf{\Sigma}(\boldsymbol{\beta})$ is the covariance matrix of the measurement which is of the form

$$\mathbf{\Sigma}(\boldsymbol{\beta}) = \sigma^2 \begin{pmatrix} (a+b|\mathbf{e}_1'\mathbf{X}\boldsymbol{\beta}|)^2 & 0 & \dots & 0 \\ 0 & (a+b|\mathbf{e}_2'\mathbf{X}\boldsymbol{\beta}|)^2 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & & \dots & (a+b|\mathbf{e}_n'\mathbf{X}\boldsymbol{\beta}|)^2 \end{pmatrix},$$

$\mathbf{e}'$ beeing the transpose of the i-th unity vector.

Let $\boldsymbol{\beta}^\circ \in \mathbb{R}^k$ be the actual value of the parameter $\boldsymbol{\beta}$.

Now we outline a typical situation occuring in practice:

We have an apriori information about the parameter $\boldsymbol{\beta}^\circ$ of the following form:

We know $\boldsymbol{\beta}_\circ$ and $\varrho_i \geqslant 0$, $i = 1, 2 \dots, n$, such that

$$\boldsymbol{\beta}^\circ \in \mathscr{B}_\circ = \{\gamma \in \mathbb{R}^k : (\mathbf{e}_i'\mathbf{X}\gamma - \mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ)^2 \leqslant \varrho_i^2,\ i = 1, 2, \dots, n\} \tag{2}$$

(or we have $\varepsilon \in \langle 0, 1\rangle$, $\boldsymbol{\beta}_\circ$ and ϱ_i, $i = 1, 2, \dots, n$ such that (2) is valid with probability at least $1 - \varepsilon$, i.e. $\mathrm{P}\{\boldsymbol{\beta}^\circ \in \mathscr{B}_\circ\} \geqslant 1 - \varepsilon$).

The problem considered is to get a properly recorded (see below) $\boldsymbol{\beta}^\circ$-LBLUe ($\boldsymbol{\beta}^\circ$-locally best linear unbiased estimate, see [3], [4]) in model (1) with information of the form (2) of the linear functional $\mathbf{f}'\boldsymbol{\beta}$ of the unknown parameter.

In Section 2 of the paper the properly recorded $\boldsymbol{\beta}^\circ$-LBLUe is defined.

In Section 4 bounds for this estimate are given.

In Section 5 bounds for the standard deviation of the $\boldsymbol{\beta}^\circ$-LBLUE ($\boldsymbol{\beta}^\circ$-locally best linear unbiased estimator) are given.

Sufficient conditions for obtaining the properly recorded standard deviation of the $\boldsymbol{\beta}^\circ$-LBLUE and $\boldsymbol{\beta}^\circ$-LBLUe are given in Section 6.

Even if we can not obtain a properly recorded standard deviation of the required estimator and a properly recorded estimate (realization of this estimator), we can obtain suitable bounds for the estimator and also confidence regions for the unknown parameters, which enable us to make very complete inferences about the the unknown values of the parameters.

In Section 7 we obtain bounds for the $\boldsymbol{\beta}^\circ$-LBLUE ($\boldsymbol{\beta}^\circ$-locally best linear unbiased estimator).

$(1 - \alpha)$ confidence regions for a linearly unbiasedly estimable linear functional $\mathbf{f}'\boldsymbol{\beta}$ of unknown parameter are given in Section 8 and Section 9 using two approaches.

These two results are compared in Section 10.

If we do not have an apriori information of the form (2), we can use Appendix to obtain a suitable one from measurement.

2. Some necessary definitions and examples

Definition 2.1. The standard deviation of the estimator is of *order* m if the first nonzero digit from the left in its decadic notation is of magnitude 10^m.

Definition 2.2. Let the standard deviation of the estimator be of order m. We say that this standard deviation is *properly recorded* if it is rounded to magnitude 10^{m-1}.

Definition 2.3. Let the standard deviation of the estimator be of order m. The realization of this estimator (the estimate) is *properly recorded* if it is rounded to magnitude 10^{m-1}.

Remark 2.4. According to Definition 2.2 and Definition 2.3 the task of obtaining the properly recorded standard deviation and also the properly recorded estimate is only a matter of rounding. Nonetheless, the conceptions of properly recorded standard deviation and properly recorded estimate seem to be very important in the field of experimental data (statistical) analysis. Practically it has no sense (in measurement) to consider a more precise standard deviation of the estimator than the properly recorded standard deviation and also a more precise estimate than the properly recorded one. (We hope to return to the conceptions of properly recorded standard deviation of an estimator and properly recorded estimate and investigate their probabilistic and statistical properties from the point of view of experimental data (statistical) evaluation methods in another paper).

Example 2.5. The measured vector $\mathbf{Y}_{10,1}$ is normally distributed with the mean value $\mathbf{x}\beta$ and covariance matrix $\mathbf{\Sigma}(\beta)$ (i.e. $\mathbf{Y} \sim N_{10}(\mathbf{x}\beta, \mathbf{\Sigma}(\beta))$, where

$$\mathbf{x}' = (1, 2, \dots, 10),$$
$$\mathbf{\Sigma}(\beta) = \begin{pmatrix} (0.5 + b|\beta|)^2 & 0 & \dots & 0 \\ 0 & (0.5 + b|2\beta|)^2 & & \\ \vdots & & \ddots & \\ 0 & & \dots & (0.5 + b|10\beta|)^2 \end{pmatrix},$$

$0.05 \leqslant \beta^\circ \leqslant 0.15$. (We independently measure points on the linear function passing through the origin with the slope β° knowing that $0.05 \leqslant \beta^\circ \leqslant 0.15$. The measuring device has dispersion $(0.5 + b|x\beta|)^2$.)

The γ-LBLUE (γ-locally best line r unbiased estimator) of β (see e.g. [3], [4]) is

$$(3) \qquad \hat{\beta}_\gamma = (\mathbf{x}'\mathbf{\Sigma}^{-1}(\gamma)\mathbf{x})^{-1}\mathbf{x}'\mathbf{\Sigma}^{-1}(\gamma)\mathbf{Y} = \left[\sum_{i=1}^{10} \frac{i^2}{(0.5 + ib\gamma)^2}\right]^{-1} \sum_{i=1}^{10} \frac{iY_i}{(0.5 + ib\gamma)^2}$$

and its standard deviation at γ is

$$\sigma_\gamma(\hat{\beta}_\gamma) = \sqrt{(\mathbf{x}'\boldsymbol{\Sigma}^1(\gamma)\mathbf{x})^{-1}} = \left[\sum_{i=1}^{10} \frac{i^2}{(0.5 + ib\gamma)^2}\right]^{-\frac{1}{2}}. \tag{4}$$

Let $b = 10^{-6}$ and let

$$\mathbf{y}' = (0.750719, 0.285096, 1.002530, 0.118773, 0.370847, \\ 0.706980, 0.111058, 0.990869, 0.361825, 1.530856)$$

be a realization of the random variable $\mathbf{Y}$.

If we take into account that $0.05 \leqslant \beta^\circ \leqslant 0.15$, we obtain from (3) that γ-LBLUe-s (for $\gamma = 0.05, 0.06, \ldots, 0.15$) are from the interval

$$\langle 0.099140392785, 0.099140393941 \rangle.$$

The standard deviations of these γ-LBLUe-s are (according to (4)) in the interval

$$\langle 0.025482379594, 0.025482419637 \rangle.$$

We see that the standard deviation of the β°-LBLUE is of order -2, the properly recorded standard deviation of the β°-LBLUE at β° is

$$\underline{\sigma} = 0.025$$

and the properly recorded β°-LBLUe is

$$\underline{\beta^\circ} = 0.099$$

(even if we do not know the true value of β°). It means we have obtained the properly recorded standard deviation of the β°-LBLUE (localized in the true but unknown parameter) and the properly recorded β°-LBLUe (where β° is again the true but unknown parameter).

Example 2.6. The measured vector $\mathbf{Y}_{10,1}$ is normally distributed with the mean value $\mathbf{x}\beta$ and covariance matrix $\boldsymbol{\Sigma}(\beta)$ (i.e. $\mathbf{Y} \sim N_{10}(\mathbf{x}\beta, \boldsymbol{\Sigma}(\beta))$, where

$$\mathbf{x}' = (1, 2, \ldots, 10),$$

$$\boldsymbol{\Sigma}(\beta) = \begin{pmatrix} (0.5 + b|\beta|)^2 & 0 & \ldots & 0 \\ 0 & (0.5 + b|2\beta|)^2 & & \\ \vdots & & \ddots & \\ 0 & & \ldots & (0.5 + b|10\beta|)^2 \end{pmatrix}.$$

To generate the measured data let the true value of β be $\beta^\circ = 1.000$.

The measured values $\mathbf{y}_{(b)}$ (for $b = 10^{-6}, 10^{-3}, 10^{-2}, 10^{-1}, 1$ and 10) are

$$\mathbf{y}'_{(10^{-6})} = (1.472332, 2.354972, 2.608851, 4.230250, 4.244317, 5.902735, 6.870571, 8.488452, 8.043104, 9.654645)$$

$$\mathbf{y}'_{(10^{-3})} = (-0.175442, 1.763118, 2.613595, 5.073383, 4.566302, 6.921577, 7.827280, 7.483188, 8.557624, 10.154784)$$

$$\mathbf{y}'_{(10^{-2})} = (1.750598, 1.238338, 3.105686, 4.839270, 5.043211, 5.349931, 6.937543, 7.459618, 9.243031, 10.458655)$$

$$\mathbf{y}'_{(10^{-1})} = (3.021137, 2.842871, 4.685383, 4.590931, 4.677528, 6.791524, 7.633321, 9.763429, 7.469816, 10.352657)$$

$$\mathbf{y}'_{(1)} = (-0.974661, -0.144880, -0.792570, 8.125993, 4.103920, 11.975135, 5.002477, 12.715016, 10.402275, 11.468850)$$

$$\mathbf{y}'_{(10)} = (0.237966, -9.910959, -43.953966, 6.956683, -87.212464, -23.509272, -142.373531, 26.479785, -79.344916, -35.998113).$$

We have an apriori information $0.95 \leqslant \beta^\circ \leqslant 1.15$. This information is of the form (2) because it is equivalent to the statement

$$\beta^\circ \in \mathscr{B}_\circ = \left\{\gamma \in \mathbb{R}^1 : (i\gamma - 1.05i)^2 \leqslant \left(\frac{i}{10}\right)^2, \quad i = 1, 2, \ldots, 10\right\}$$

(here $\beta_\circ = 1.05$). According to (3) and (4) the values of β°-LBLUe-s and the corresponding standard deviations are in the following intervals:

b	β° – LBLUe	σ_{β°
10^{-6}	$\langle 0.967542488, 0.967542508\rangle$	$\langle 0.025482739, 0.025482820\rangle$
10^{-3}	$\langle 1.010674, 1.010695\rangle$	$\langle 0.025862, 0.025942\rangle$
10^{-2}	$\langle 1.002538, 1.002617\rangle$	$\langle 0.029235, 0.030014\rangle$
10^{-1}	$\langle 1.107546, 1.115002\rangle$	$\langle 0.060597, 0.067606\rangle$
1	$\langle 0.933055, 0.949706\rangle$	$\langle 0.340921, 0.404891\rangle$
10	$\langle -6.896528, -6.887816\rangle$	$\langle 3.049, 3.682\rangle$

We see that for $b = 10^{-6}, 10^{-3}, 10^{-2}$ and 10 we have a properly recorded β°-LBLUe

b	$\underline{\beta^\circ}$
10^{-6}	0.968
10^{-3}	1.011
10^{-2}	1.003
10	−6.9

but we have none for the others. For $b = 10^{-6}$ and 10^{-3} we have also a properly recorded standard deviation of the β°-LBLUE

b	$\underline{\sigma}$
10^{-6}	0.025
10^{-3}	0.026

but we have none for the others. (Even if we do not know the true value of β°.)

Remark 2.7. From the previous examples it is seen that under the apriori information (2) it is possible in some cases to obtain in model (1) the properly recorded β°-LBLUe and the properly recorded standard deviation of the β°-LBLUE in the true (but unknown) value of β. Sufficient conditions for obtaining the properly recorded standard deviation of the β°-LBLUE and β°-LBLUe are given in Section 6.

3. Auxiliary results

Our investigations are based on a result of Cleveland in [1]. If $\boldsymbol{\beta}^\circ$ is the true value of the parameter $\boldsymbol{\beta}$, then the true covariance matrix of $\mathbf{Y}$ is $\boldsymbol{\Sigma}(\boldsymbol{\beta}^\circ)$. The $\boldsymbol{\beta}^\circ$-LBLUE of $\mathbf{X}\boldsymbol{\beta}$ is

$$\widehat{\mathbf{X}\boldsymbol{\beta}}(\mathbf{Y}) = \mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}^\circ)\mathbf{X})^{-}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}^\circ)\mathbf{Y}.$$

($\mathbf{A}^-$ denotes a g-inverse of the matrix $\mathbf{A}$.)

For an arbitrary but fixed positive definite (p.d.) matrix $\boldsymbol{\Sigma}^\star$ let us denote

$$\alpha_1 = \inf\{\alpha\colon \det(\boldsymbol{\Sigma}^\star\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}^\circ) - \alpha\mathbf{I}) = 0\}$$

$$\alpha_2 = \sup\{\alpha\colon \det(\boldsymbol{\Sigma}^\star\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}^\circ) - \alpha\mathbf{I}) = 0\}.$$

According to [1] for every realization $\mathbf{y}$ of $\mathbf{Y}$ and β°-LBLUe $\widehat{\mathbf{X}\boldsymbol{\beta}}(\mathbf{y})$ the inequalities

$$\begin{aligned}(5)\qquad \|\widehat{\mathbf{X}\boldsymbol{\beta}}(\mathbf{y}) - \widetilde{\mathbf{X}\boldsymbol{\beta}}(\mathbf{y})\|^2_{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}^\circ)} &\leqslant \frac{(\alpha_1-\alpha_2)^2}{4\alpha_1\alpha_2}\|\mathbf{y} - \widehat{\mathbf{X}\boldsymbol{\beta}}(\mathbf{y})\|^2_{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}^\circ)}\\ &\leqslant \frac{(\alpha_1-\alpha_2)^2}{4\alpha_1\alpha_2}\|\mathbf{y} - \widetilde{\mathbf{X}\boldsymbol{\beta}}(\mathbf{y})\|^2_{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}^\circ)}\end{aligned}$$

hold, where

$$\widetilde{\mathbf{X}\beta}(\mathbf{y}) = \mathbf{X}(\mathbf{X}'\mathbf{\Sigma}^{\star-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{\Sigma}^{\star-1}\mathbf{y} \tag{6}$$

and $\|\mathbf{z}\|^2_{\mathbf{\Sigma}^{-1}(\beta^\circ)} = \mathbf{z}'\mathbf{\Sigma}^{-1}(\beta^\circ)\mathbf{z}$.

Making use of (2) let us now construct the upper bound for $\frac{(\alpha_1-\alpha_2)^2}{4\alpha_1\alpha_2}$ in model (1). For $\beta \in \mathscr{B}_\circ$ we have

$$-\varrho_i + \mathbf{e}_i'\mathbf{X}\beta_\circ \leqslant \mathbf{e}_i'\mathbf{X}\beta \leqslant \varrho_i + \mathbf{e}_i'\mathbf{X}\beta_\circ, \qquad i = 1,2,\dots,n.$$

Thus for $i = 1,2,\dots,n$

(i) if $\mathbf{e}_i'\mathbf{X}\beta_\circ \geqslant 0$, $\mathbf{e}_i'\mathbf{X}\beta_\circ \geqslant \varrho_i$,

then

$$\max_{\beta\in\mathscr{B}_\circ} |\mathbf{e}_i'\mathbf{X}\beta| = \mathbf{e}_i'\mathbf{X}\beta_\circ + \varrho_i$$
$$\min_{\beta\in\mathscr{B}_\circ} |\mathbf{e}_i'\mathbf{X}\beta| = \mathbf{e}_i'\mathbf{X}\beta_\circ - \varrho_i$$

and

$$\frac{(a + b(\mathbf{e}_i'\mathbf{X}\beta_\circ - \varrho_i))^2}{(a + b(\mathbf{e}_i'\mathbf{X}\beta_\circ + \varrho_i))^2} \leqslant \frac{(a + b|\mathbf{e}_i'\mathbf{X}\beta|)^2}{(a + b|\mathbf{e}_i'\mathbf{X}\beta^\circ|)^2} \leqslant \frac{(a + b(\mathbf{e}_i'\mathbf{X}\beta_\circ + \varrho_i))^2}{(a + b(\mathbf{e}_i'\mathbf{X}\beta_\circ - \varrho_i))^2} \tag{7}$$

is valid for all $\beta \in \mathscr{B}_\circ$.

(ii) If $\mathbf{e}_i'\mathbf{X}\beta_\circ \geqslant 0$, $\mathbf{e}_i'\mathbf{X}\beta_\circ < \varrho_i$,

then

$$\max_{\beta\in\mathscr{B}_\circ} |\mathbf{e}_i'\mathbf{X}\beta| = \mathbf{e}_i'\mathbf{X}\beta_\circ + \varrho_i$$
$$\min_{\beta\in\mathscr{B}_\circ} |\mathbf{e}_i'\mathbf{X}\beta| = 0$$

and

$$\frac{a^2}{(a + b(\mathbf{e}_i'\mathbf{X}\beta_\circ + \varrho_i))^2} \leqslant \frac{(a + b|\mathbf{e}_i'\mathbf{X}\beta|)^2}{(a + b|\mathbf{e}_i'\mathbf{X}\beta^\circ|)^2} \leqslant \frac{(a + b(\mathbf{e}_i'\mathbf{X}\beta_\circ + \varrho_i))^2}{a^2} \tag{8}$$

is valid for all $\beta \in \mathscr{B}_\circ$.

(iii) If $\mathbf{e}_i'\mathbf{X}\beta_\circ < 0$, $-\mathbf{e}_i'\mathbf{X}\beta_\circ \leqslant \varrho_i$, then

$$\max_{\beta\in\mathscr{B}_\circ} |\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}| = -\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ + \varrho_i$$
$$\min_{\beta\in\mathscr{B}_\circ} |\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}| = 0$$

and

$$\text{(9)} \qquad \frac{a^2}{(a+b(-\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ+\varrho_i))^2} \leqslant \frac{(a+b|\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}|)^2}{(a+b|\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}^\circ|)^2} \leqslant \frac{(a+b(-\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ+\varrho_i))^2}{a^2}$$

is valid for all $\boldsymbol{\beta} \in \mathscr{B}_\circ$.

(iv) If $\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ < 0$, $-\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ > \varrho_i$,

then

$$\max_{\beta\in\mathscr{B}_\circ} |\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}| = -\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ + \varrho_i$$
$$\min_{\beta\in\mathscr{B}_\circ} |\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}| = -\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ - \varrho_i$$

and

$$\text{(10)} \qquad \frac{(a+b(-\mathbf{e}'\mathbf{X}\boldsymbol{\beta}_\circ-\varrho_i))^2}{(a+b(-\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ+\varrho_i))^2} \leqslant \frac{(a+b|\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}|)^2}{(a+b|\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}^\circ|)^2} \leqslant \frac{(a+b(-\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ+\varrho_i))^2}{(a+b(-\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ-\varrho_i))^2}$$

is true for all $\boldsymbol{\beta} \in \mathscr{B}_\circ$.

Thus for an arbitrary $\boldsymbol{\beta} \in \mathscr{B}_\circ$ we have

$$0 < \underline{\gamma_i} \leqslant \frac{(a+b|\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}|)^2}{(a+b|\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}^\circ|)^2} \leqslant \overline{\gamma_i} \qquad i=1,2,\ldots,n$$

(we obtain the bounds $\underline{\gamma_i}$ and $\overline{\gamma_i}$, $i=1,2,\ldots,n$ from (7), (8), (9) or (10) using the values $\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ$ and ϱ_i).

Also for every $\boldsymbol{\beta} \in \mathscr{B}_\circ$ we have

$$\begin{aligned}\underline{\gamma} &= \min\{\underline{\gamma_i}\colon i=1,2,\ldots,n\}\\ &\leqslant \alpha_1(\boldsymbol{\beta}) = \inf\{\alpha\colon \det(\boldsymbol{\Sigma}(\boldsymbol{\beta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}^\circ)-\alpha\mathbf{I})=0\}\\ &= \min\left\{\frac{(a+b|\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}|)^2}{(a+b|\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}^\circ|)^2}\colon i=1,2,\ldots,n\right\}\end{aligned}$$

and

$$\begin{aligned}\overline{\gamma} &= \max\{\overline{\gamma_i}\colon i=1,2,\ldots,n\}\\ &\geqslant \alpha_2(\boldsymbol{\beta}) = \sup\{\alpha\colon \det(\boldsymbol{\Sigma}(\boldsymbol{\beta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}^\circ)-\alpha\mathbf{I})=0\}\\ &= \max\left\{\frac{(a+b|\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}|)^2}{(a+b|\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}^\circ|)^2}\colon i=1,2,\ldots,n\right\}.\end{aligned}$$

So for every $\beta \in \mathscr{B}_\circ$

$$(11)\qquad \frac{(\alpha_1(\beta)-\alpha_2(\beta))^2}{4\alpha_1(\beta)\alpha_2(\beta)} \leqslant \frac{(\underline{\gamma}-\overline{\gamma})^2}{4\underline{\gamma}\overline{\gamma}} = \gamma.$$

Example 3.1. What are the bounds $\underline{\gamma}$, $\overline{\gamma}$ and γ in Example 2.6?

As

$$\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ = 1.05i \geqslant 0$$

and

$$\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ = 1.05i \geqslant \varrho_i = \frac{i}{10}$$

for $i = 1, 2, \ldots, 10$, we have

$$\underline{\gamma_i} = \frac{(0.5 + b(1.05i - \frac{i}{10}))^2}{(0.5 + b(1.05i + \frac{i}{10}))^2} = \frac{(0.5 + 0.95bi)^2}{(0.5 + 1.15bi)^2},$$

$$\overline{\gamma_i} = \frac{(0.5 + 1.15bi)^2}{(0.5 + 0.95bi)^2}.$$

So

$$\underline{\gamma} = \min_i \frac{(0.5 + 0.95bi)^2}{(0.5 + 1.15bi)^2} = \left(\frac{0.5 + 9.5b}{0.5 + 11.5b}\right)^2,$$

$$\overline{\gamma} = \max_i \frac{(0.5 + 1.15bi)^2}{(0.5 + 0.95bi)^2} = \left(\frac{0.5 + 11.5b}{0.5 + 9.5b}\right)^2$$

and

$$\gamma = \frac{1}{4}\left[\left(\frac{0.5 + 9.5b}{0.5 + 11.5b}\right)^2 - \left(\frac{0.5 + 11.5b}{0.5 + 9.5b}\right)^2\right]^2.$$

For $b = 10^{-6}, 10^{-3}, 10^{-2}, 10^{-1}, 1$ and 10 we obtain the corresponding γ values:

b	γ
10^{-6}	6.4×10^{-11}
10^{-3}	0.000061396
10^{-2}	0.004378458
10^{-1}	0.068282629
1	0.138963272
10	0.15172681

Let us return to model (1). Because of $\beta \in \mathscr{B}_\circ$ we consider all possible covariance matrices $\{\Sigma(\beta)\colon \beta \in \mathscr{B}_\circ\}$ and the corresponding set of estimators

$$\mathscr{T} = \{\widetilde{\mathbf{X}\beta}(\mathbf{Y}) = \mathbf{X}(\mathbf{X}'\Sigma^{-1}(\beta)\mathbf{X})^{-}\mathbf{X}'\Sigma^{-1}(\beta)\mathbf{Y}\colon \beta \in \mathscr{B}_\circ\}.$$

According to (5) and (11) we obtain that for every realization $\mathbf{y}$ of $\mathbf{Y}$ and every $\widetilde{\mathbf{X}\beta} \in \mathscr{T}$

$$\text{(12)} \qquad \begin{aligned} \|\widehat{\mathbf{X}\beta}(\mathbf{y}) - \widetilde{\mathbf{X}\beta}(\mathbf{y})\|^2_{\Sigma^{-1}(\beta^\circ)} &\leqslant \gamma\|\mathbf{y} - \widehat{\mathbf{X}\beta}(\mathbf{y})\|^2_{\Sigma^{-1}(\beta^\circ)} \\ &\leqslant \gamma\|\mathbf{y} - \widetilde{\mathbf{X}\beta}(\mathbf{y})\|^2_{\Sigma^{-1}(\beta^\circ)}. \end{aligned}$$

For $\mathbf{f} \in \mu(\mathbf{X}') = \{\mathbf{X}'\mathbf{u}\colon \mathbf{u} \in \mathbb{R}^n\}$ (i.e. $\mathbf{f} = \mathbf{X}'\mathbf{u}_\mathbf{f}$) and $\widetilde{\mathbf{X}\beta}(\mathbf{Y}) \in \mathscr{T}$ we denote

$$\mathbf{u}_\mathbf{f}'\widetilde{\mathbf{X}\beta}(\mathbf{Y}) = \widetilde{\mathbf{f}'\beta}(\mathbf{Y})$$

and

$$\mathscr{T}_\mathbf{f} = \{\mathbf{u}_\mathbf{f}'\widetilde{\mathbf{X}\beta}(\mathbf{Y})\colon \widetilde{\mathbf{X}\beta}(\mathbf{Y}) \in \mathscr{T}\}.$$

4. Bounds for the β°-LBLUE

Let $\widetilde{\mathbf{X}\beta}(\mathbf{Y}) \in \mathscr{T}$. According to (12) for every realization $\mathbf{y}$ of $\mathbf{Y}$ we have

$$\text{(13)} \qquad \begin{aligned} &(\widehat{\mathbf{X}\beta}(\mathbf{y}) - \widetilde{\mathbf{X}\beta}(\mathbf{y}))'\Sigma^{-1}(\beta^\circ)(\widehat{\mathbf{X}\beta}(\mathbf{y}) - \widetilde{\mathbf{X}\beta}(\mathbf{y})) \\ &\leqslant \gamma(\mathbf{y} - \widetilde{\mathbf{X}\beta}(\mathbf{y}))'\Sigma^{-1}(\beta^\circ)(\mathbf{y} - \widetilde{\mathbf{X}\beta}(\mathbf{y})). \end{aligned}$$

If we denote

$$\Sigma(\min) = \sigma^2 \begin{pmatrix} \min\limits_{\beta\in\mathscr{B}_\circ}(a + b|\mathbf{e}_1'\mathbf{X}\beta|)^2 & 0 & \dots & 0 \\ 0 & \min\limits_{\beta\in\mathscr{B}_\circ}(a + b|\mathbf{e}_2'\mathbf{X}\beta|)^2 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & & \dots & \min\limits_{\beta\in\mathscr{B}_\circ}(a + b|\mathbf{e}_n'\mathbf{X}\beta|)^2 \end{pmatrix},$$

$$\Sigma(\max) = \sigma^2 \begin{pmatrix} \max\limits_{\beta\in\mathscr{B}_\circ}(a + b|\mathbf{e}_1'\mathbf{X}\beta|)^2 & 0 & \dots & 0 \\ 0 & \max\limits_{\beta\in\mathscr{B}_\circ}(a + b|\mathbf{e}_2'\mathbf{X}\beta|)^2 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & & \dots & \max\limits_{\beta\in\mathscr{B}_\circ}(a + b|\mathbf{e}_n'\mathbf{X}\beta|)^2 \end{pmatrix}$$

and

$$(\mathbf{y} - \widetilde{\mathbf{X}\beta}(\mathbf{y}))'\boldsymbol{\Sigma}^{-1}(\min)(\mathbf{y} - \widetilde{\mathbf{X}\beta}(\mathbf{y})) = U(\mathbf{y}),$$

then (13) implies

$$(\widehat{\mathbf{X}\beta}(\mathbf{y}) - \widetilde{\mathbf{X}\beta}(\mathbf{y}))'[\gamma U(\mathbf{y})\boldsymbol{\Sigma}(\max)]^{-1}(\widehat{\mathbf{X}\beta}(\mathbf{y}) - \widetilde{\mathbf{X}\beta}(\mathbf{y})) \leqslant 1. \tag{14}$$

Further, let us denote

$$\mathbf{D} = [\gamma U(\mathbf{y})\boldsymbol{\Sigma}(\max)]^{-1}.$$

From (14) we obtain that for every $\mathbf{f} \in \mu(\mathbf{X}')$, every $\widetilde{\mathbf{f}'\beta}(\mathbf{Y}) \in \mathscr{T}_\mathbf{f}$ and every realization $\mathbf{y}$ of $\mathbf{Y}$

$$\begin{aligned}(\widehat{\mathbf{f}'\beta}(\mathbf{y}) - \widetilde{\mathbf{f}'\beta}(\mathbf{y}))^2 &= (\mathbf{u}'\mathbf{X}'\mathbf{D}\widehat{\mathbf{X}\beta}(\mathbf{y}) - \mathbf{u}'\mathbf{X}'\mathbf{D}\widetilde{\mathbf{X}\beta}(\mathbf{y}))^2 \\ &\leqslant \mathbf{u}'\mathbf{X}'\mathbf{D}\mathbf{X}\mathbf{u} = \mathbf{f}'(\mathbf{X}'\mathbf{D}\mathbf{X})^-\mathbf{f}.\end{aligned}$$

It means that for every linearly unbiasedly estimable linear functional $\mathbf{f}'\beta$, every $\widetilde{\mathbf{f}'\beta}(\mathbf{Y}) \in \mathscr{T}_\mathbf{f}$ and every realization $\mathbf{y}$ of $\mathbf{Y}$ the following inequalities hold:

$$\widetilde{\mathbf{f}'\beta}(\mathbf{y}) - \sqrt{\mathbf{f}'(\mathbf{X}'\mathbf{D}\mathbf{X})^-\mathbf{f}} \leqslant \widehat{\mathbf{f}'\beta}(\mathbf{y}) \leqslant \widetilde{\mathbf{f}'\beta}(\mathbf{y}) + \sqrt{\mathbf{f}'(\mathbf{X}'\mathbf{D}\mathbf{X})^-\mathbf{f}}. \tag{15}$$

Remark 4.1. $U(\mathbf{y})$ and $\mathbf{D}$ depend on the choice of β, but we can choose an arbitrary $\beta \in \mathscr{B}_o$, i.e. an arbitrary $\widetilde{\mathbf{X}\beta}(\mathbf{Y}) \in \mathscr{T}$. For every realization $\mathbf{y}$ of $\mathbf{Y}$ and the corresponding $U(\mathbf{y})$, $\mathbf{D}$ relation (15) gives bounds for the β°-LBLUe of $\mathbf{f}'\beta$ (even if we do not know the true value β°).

Example 4.2. Using results of Example 3.1 we obtain for model (1) given in Example 2.6

$$\boldsymbol{\Sigma}(\min) = \begin{pmatrix} (0.5+0.95b)^2 & 0 & \dots & 0 \\ 0 & (0.5+1.90b)^2 & \dots & 0 \\ \vdots & & \ddots & 0 \\ 0 & & \dots & (0.5+9.5b)^2 \end{pmatrix},$$

$$\boldsymbol{\Sigma}(\max) = \begin{pmatrix} (0.5+1.15b)^2 & 0 & \dots & 0 \\ 0 & (0.5+2.30b)^2 & \dots & 0 \\ \vdots & & \ddots & 0 \\ 0 & & \dots & (0.5+11.5b)^2 \end{pmatrix}$$

and the values $U(\mathbf{y})$ (for various b)

b	$U(\mathbf{y})$
10^{-6}	8.080997
10^{-3}	19.299078
10^{-2}	9.816710
10^{-1}	19.190809
1	6.254972
10	6.571786

(we have chosen $\beta = \beta_{\circ} = 1.05$). According to (15) we obtain the following bounds for the β°-LBLUe (for various b)

b	β° – LBLUe
10^{-6}	$\langle 0.9675419; 0.9675430\rangle$
10^{-3}	$\langle 1.00979; 1.01157\rangle$
10^{-2}	$\langle 0.99635; 1.00879\rangle$
10^{-1}	$\langle 1.03396; 1.18874\rangle$
1	$\langle 0.56334; 1.31831\rangle$
10	$\langle -10.5687; -3.2148\rangle$

5. Bounds for the standard deviation of the β°-LBLUE

For $\mathbf{f} \in \mu(\mathbf{X}')$ the β°-LBLUE of $\mathbf{f}'\beta$ we have

$$\widehat{\mathbf{f}'\beta}(\mathbf{Y}) = \mathbf{f}'(\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta^{\circ})\mathbf{X})^{-}\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta^{\circ})\mathbf{Y},$$

and its dispersion at β° is

$$\mathscr{D}_{\beta^{\circ}}(\widehat{\mathbf{f}'\beta}(\mathbf{Y})) = \mathbf{f}'(\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta^{\circ})\mathbf{X})^{-}\mathbf{f}.$$

So the standard deviation of the β°-LBLUE of $\mathbf{f}'\beta$ at β° is

$$\sigma_{\beta^{\circ}} = \sqrt{\mathbf{f}'(\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta^{\circ})\mathbf{X})^{-}\mathbf{f}}.$$

Since $\mathbf{\Sigma}^{-1}(\min) - \mathbf{\Sigma}^{-1}(\beta^{\circ})$ and $\mathbf{\Sigma}^{-1}(\beta^{\circ}) - \mathbf{\Sigma}^{-1}(\max)$ are p.s.d. matrices, $\sigma_{\beta^{\circ}}$ is in the interval

$$\sqrt{\mathbf{f}'(\mathbf{X}'\mathbf{\Sigma}^{-1}(\min)\mathbf{X})^{-}\mathbf{f}} \leqslant \sigma_{\beta^{\circ}} \leqslant \sqrt{\mathbf{f}'(\mathbf{X}'\mathbf{\Sigma}^{-1}(\max)\mathbf{X})^{-}\mathbf{f}}. \tag{16}$$

Example 5.1. Using the results of Example 4.2 we obtain from (16) intervals for the standard deviation σ_{β° of the β°-LBLUE of β (for various b) in the model given in Example 2.6:

b	σ_{β°
10^{-6}	$\langle 0.0254827; 0.0254828\rangle$
10^{-3}	$\langle 0.02586; 0.02594\rangle$
10^{-2}	$\langle 0.0292; 0.0300\rangle$
10^{-1}	$\langle 0.0605; 0.0676\rangle$
1	$\langle 0.340; 0.404\rangle$
10	$\langle 3.04; 3.68\rangle$

6. Properly recorded standard deviation and properly recorded β°-LBLUe

Now we write a sufficient condition for obtaining the properly recorded standard deviation of the β°-LBLUE of $\mathbf{f}'\beta$ (for $\mathbf{f} \in \mu(\mathbf{X}')$, i.e. for a linearly estimable linear functional of β) and also for obtaining the properly recorded β°-LBLUe of such an $\mathbf{f}'\beta$.

According to (16) we have the properly recorded standard deviation $\underline{\sigma}_{\beta^\circ}$ of the β°-LBLUE of $\mathbf{f}'\beta$ for $\mathbf{f} \in \mu(\mathbf{X}')$ if the properly recorded numbers $\sqrt{\mathbf{f}'(\mathbf{X}'\mathbf{\Sigma}^{-1}(\max)\mathbf{X})^{-}\mathbf{f}}$ and $\sqrt{\mathbf{f}'(\mathbf{X}'\mathbf{\Sigma}^{-1}(\min)\mathbf{X})^{-}\mathbf{f}}$ (as standard deviations) are the same. This number is also $\underline{\sigma}_{\beta^\circ}$ we are looking for.

From (15) and (16) we obtain also the properly recorded β°-LBLUe of $\mathbf{f}'\beta$ (for $\mathbf{f} \in \mu(\mathbf{X}')$) if

$$\sqrt{\mathbf{f}'(\mathbf{X}'\mathbf{D}\mathbf{X})^{-}\mathbf{f}} \leqslant 0.005\sqrt{\mathbf{f}'(\mathbf{X}'\mathbf{\Sigma}^{-1}(\min)\mathbf{X})^{-}\mathbf{f}}. \tag{17}$$

It is easy to see that

$$\begin{aligned}\mathbf{D} = [\gamma(\mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta)\mathbf{X})^{-}\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta)\mathbf{y})'\mathbf{\Sigma}^{-1}(\min)\\ \times (\mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta)\mathbf{X})^{-}\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta)\mathbf{y})]^{-1}\mathbf{\Sigma}^{-1}(\max),\end{aligned}$$

and so (17) is satisfied for $\beta \in \mathscr{B}_\circ$ if

$$\begin{aligned}&(\gamma(\mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta)\mathbf{X})^{-}\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta)\mathbf{y})'\mathbf{\Sigma}^{-1}(\min)\\ &\qquad \times (\mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta)\mathbf{X})^{-}\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta)\mathbf{y}))^{1/2}\\ &\qquad \leqslant 0.005\sqrt{\frac{\mathbf{f}'(\mathbf{X}'\mathbf{\Sigma}^{-1}(\min)\mathbf{X})^{-}\mathbf{f}}{\mathbf{f}'(\mathbf{X}'\mathbf{\Sigma}^{-1}(\max)\mathbf{X})^{-}\mathbf{f}}}.\end{aligned} \tag{18}$$

The wanted sufficient condition for the properly recorded $\widetilde{\mathbf{f}'\boldsymbol\beta}(\mathbf{y})$ to be the properly recorded $\boldsymbol\beta^\circ$-LBLUe of $\mathbf{f}'\boldsymbol\beta$ is (18). Of course the constant 0.005 could be in many cases much greater.

Remark 6.1. Of course the left hand sides of (17) and (18) depend on $\boldsymbol\beta$. But if for a $\boldsymbol\beta \in \mathscr{B}_o$ (17) or (18) is satisfied, we can obtain the properly recorded $\boldsymbol\beta^\circ$-LBLUe of $\mathbf{f}'\boldsymbol\beta$ (even if we do not know the true value $\boldsymbol\beta^\circ$) from (15), i.e. using $\widetilde{\mathbf{f}'\boldsymbol\beta}(\mathbf{y})$ and rounding.

7. Bounds for the $\boldsymbol\beta^\circ$-LBLUE

If we consider $\widehat{\mathbf{X}\boldsymbol\beta}(\mathbf{Y})$ (the $\boldsymbol\beta^\circ$-LBLUE of $\mathbf{X}\boldsymbol\beta$) as a random variable and $\widetilde{\mathbf{X}\boldsymbol\beta}(\mathbf{Y}) \in \mathscr{T}$, another unbiased estimator of $\mathbf{X}\boldsymbol\beta$, then (12) yields

$$(19)\qquad P_{\boldsymbol\beta^\circ}\{\mathbf{y} \in \mathbb{R}^n : \|\widehat{\mathbf{X}\boldsymbol\beta}(\mathbf{y}) - \widetilde{\mathbf{X}\boldsymbol\beta}(\mathbf{y})\|^2_{\boldsymbol\Sigma^{-1}(\boldsymbol\beta^\circ)} \leqslant \gamma\|\mathbf{y} - \widehat{\mathbf{X}\boldsymbol\beta}(\mathbf{y})\|^2_{\boldsymbol\Sigma^{-1}(\boldsymbol\beta^\circ)}\} = 1.$$

As $\mathbf{Y} \sim N_n(\mathbf{X}\boldsymbol\beta^\circ, \boldsymbol\Sigma(\boldsymbol\beta^\circ))$, $\|\mathbf{Y} - \widehat{\mathbf{X}\boldsymbol\beta}(\mathbf{Y})\|^2_{\boldsymbol\Sigma^{-1}(\boldsymbol\beta^\circ)}$ has $\chi^2_{n-R(\mathbf{X})}$ distribution. Let $\chi^2_{n-R(\mathbf{X})}(\alpha)$ be the $(1-\alpha)$ quantile of χ^2 distribution with $n - R(\mathbf{X})$ degrees of freedom ($R(\mathbf{X})$ is the rank of the matrix $\mathbf{X}$). We can write

$$P_{\boldsymbol\beta^\circ}\left\{\mathbf{y} \in \mathbb{R}^n : \frac{1}{\gamma}\|\widehat{\mathbf{X}\boldsymbol\beta}(\mathbf{y}) - \widetilde{\mathbf{X}\boldsymbol\beta}(\mathbf{y})\|^2_{\boldsymbol\Sigma^{-1}(\boldsymbol\beta^\circ)} \leqslant \chi^2_{n-R(\mathbf{X})}(\alpha)\right\} \geqslant 1-\alpha$$

and also

$$\begin{aligned}P_{\boldsymbol\beta^\circ}\{\mathbf{y} \in \mathbb{R}^n : (\widehat{\mathbf{X}\boldsymbol\beta}(\mathbf{y}) - \widetilde{\mathbf{X}\boldsymbol\beta}(\mathbf{y}))'[\gamma\chi^2_{n-R(\mathbf{X})}(\alpha)\boldsymbol\Sigma(\max)]^{-1}\\ \times (\widehat{\mathbf{X}\boldsymbol\beta}(\mathbf{y}) - \widetilde{\mathbf{X}\boldsymbol\beta}(\mathbf{y})) \leqslant 1\} \geqslant 1-\alpha.\end{aligned}$$

Denoting

$$\mathbf{C} = [\gamma\chi^2_{n-R(\mathbf{X})}(\alpha)\boldsymbol\Sigma(\max)]^{-1}$$

we obtain in the same way as in Section 4 that for every linear functional $\mathbf{f}'\boldsymbol\beta$ with $\mathbf{f} \in \mu(\mathbf{X}')$ the following relation holds:

$$P_{\boldsymbol\beta^\circ}\{\mathbf{y} \in \mathbb{R}^n : (\widehat{\mathbf{f}'\boldsymbol\beta}(\mathbf{y}) - \widetilde{\mathbf{f}'\boldsymbol\beta}(\mathbf{y}))^2 \leqslant \mathbf{f}'(\mathbf{X}'\mathbf{C}\mathbf{X})^-\mathbf{f}\} \geqslant 1-\alpha,$$

i.e.

$$\begin{aligned}(20)\qquad P_{\boldsymbol\beta^\circ}\Big\{\mathbf{y} \in \mathbb{R}^n : \widetilde{\mathbf{f}'\boldsymbol\beta}(\mathbf{y}) - \sqrt{\mathbf{f}'(\mathbf{X}'\mathbf{C}\mathbf{X})^-\mathbf{f}} \leqslant \widehat{\mathbf{f}'\boldsymbol\beta}(\mathbf{y})\\ \leqslant \widetilde{\mathbf{f}'\boldsymbol\beta}(\mathbf{y}) + \sqrt{\mathbf{f}'(\mathbf{X}'\mathbf{C}\mathbf{X})^-\mathbf{f}}\Big\} \geqslant 1-\alpha.\end{aligned}$$

The last relation gives us bounds for the $\boldsymbol\beta^\circ$-LBLUE of $\mathbf{f}'\boldsymbol\beta$ using another (available) estimator $\widetilde{\mathbf{f}'\boldsymbol\beta}(\mathbf{Y}) \in \mathscr{T}_\mathbf{f}$.

Example 7.1. According to (20) we obtain in model (1) given in Example 2.6 for various b the following bounds for the β°-LBLUE of β:

b	β° – LBLUE
10^{-6}	$\langle 0.9675416; 0.9675433\rangle$
10^{-3}	$\langle 1.00984; 1.01152\rangle$
10^{-2}	$\langle 0.99440; 1.01074\rangle$
10^{-1}	$\langle 1.03868; 1.18401\rangle$
1	$\langle 0.31998; 1.56168\rangle$
10	$\langle -12.791; -0.991\rangle$

8. Confidence region for $\mathbf{f}'\beta$

The random variable $(\mathbf{Y}-\mathbf{X}\beta^\circ)'\mathbf{\Sigma}^{-1}(\beta^\circ)(\mathbf{Y}-\mathbf{X}\beta^\circ)$ has χ^2_n distribution. It is easy to see that

$$P_{\beta^\circ}\{\mathbf{y}\in\mathbb{R}^n : (\mathbf{y}-\mathbf{X}\beta^\circ)'\mathbf{\Sigma}^{-1}(\max)(\mathbf{y}-\mathbf{X}\beta^\circ)\leqslant \chi^2_n(\alpha)\}\geqslant 1-\alpha$$

or, equivalently,

$$P_{\beta^\circ}\{\mathbf{y}\in\mathbb{R}^n : (\mathbf{y}-\mathbf{X}\beta^\circ)'[\chi^2_n(\alpha)\mathbf{\Sigma}(\max)]^{-1}(\mathbf{y}-\mathbf{X}\beta^\circ)\leqslant 1\}\geqslant 1-\alpha.$$

Let $\mathbf{f}\in\mu(\mathbf{X}')$ (i.e. $\mathbf{f}=\mathbf{X}'\mathbf{u_f}$). We have

$$\begin{aligned}(21)\qquad P_{\beta^\circ}\{\mathbf{y}\in\mathbb{R}^n : &\left(\mathbf{u}_\mathbf{f}'\mathbf{X}(\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta)\mathbf{X})^{-}\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta)(\mathbf{y}-\mathbf{X}\beta^\circ)\right)^2\\ &\leqslant \mathbf{u}_\mathbf{f}'\mathbf{X}(\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta)\mathbf{X})^{-}\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta)\chi^2_n(\alpha)\\ &\quad\times\mathbf{\Sigma}(\max)\mathbf{\Sigma}^{-1}(\beta)\mathbf{X}(\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta)\mathbf{X})^{-}\mathbf{X}'\mathbf{u_f}\}\geqslant 1-\alpha.\end{aligned}$$

We denote the matrix

$$(\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta)\mathbf{X})^{-}\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta)\chi^2_n(\alpha)\mathbf{\Sigma}(\max)\mathbf{\Sigma}^{-1}(\beta)\mathbf{X}(\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta)\mathbf{X})^{-}$$

as $\mathbf{\Gamma}_{\max}$. So we rewrite (21) as

$$(22)\qquad P_{\beta^\circ}\left\{\mathbf{y}\in\mathbb{R}^n : \widetilde{\mathbf{f}'\beta}(\mathbf{y})-\sqrt{\mathbf{f}'\mathbf{\Gamma}_{\max}\mathbf{f}}\leqslant \mathbf{f}'\beta^\circ\leqslant\widetilde{\mathbf{f}'\beta}(\mathbf{y})+\sqrt{\mathbf{f}'\mathbf{\Gamma}_{\max}\mathbf{f}}\right\}\geqslant 1-\alpha,$$

which is the $(1-\alpha)$ confidence region we have been looking for.

Remark 8.1. As in the previous sections, using an arbitrary $\beta \in \mathscr{B}_o$ we obtain the estimator $\widetilde{\mathbf{f}'\beta} \in \mathscr{T}_\mathbf{f}$ and $\mathbf{\Gamma}_{\max}$ (depending on β chosen) and from (22) the $(1-\alpha)$ confidence region for $\mathbf{f}'\beta^\circ$.

Example 8.2. According to (22) we obtain in model (1) given in Example 2.6 for various b the following $(1-\alpha)$ confidence regions for β:

b	β
10^{-6}	$\langle 0.85850; 1.07658\rangle$
10^{-3}	$\langle 0.89967; 1.12169\rangle$
10^{-2}	$\langle 0.87414; 1.13101\rangle$
10^{-1}	$\langle 0.82201; 1.40069\rangle$
1	$\langle -0.791; 2.673\rangle$
10	$\langle -22.64; 8.86\rangle$

9. Confidence region for $\mathbf{f}'\beta$ using Cleveland's result

Let us obtain the confidence region using Cleveland's result (12). As

$$P_{\beta^\circ}\{\mathbf{y} \in \mathbb{R}^n : (\mathbf{y} - \mathbf{X}\beta^\circ)'\mathbf{\Sigma}^{-1}(\beta^\circ)(\mathbf{y} - \mathbf{X}\beta^\circ) \leqslant \chi_n^2(\alpha)\} = 1 - \alpha,$$

we have

$$P_{\beta^\circ}\{\mathbf{y} \in \mathbb{R}^n : (\mathbf{y} - \mathbf{X}\beta^\circ)'[\chi_n^2(\alpha)\mathbf{\Sigma}(\beta^\circ)]^{-1}(\mathbf{y} - \mathbf{X}\beta^\circ) \leqslant 1\} = 1 - \alpha.$$

Again as in the preceding section we obtain

$$P_{\beta^\circ}\Big\{\mathbf{y} \in \mathbb{R}^n : \widehat{\mathbf{f}'\beta}(\mathbf{y}) - \sqrt{\chi_n^2(\alpha)\mathbf{f}'(\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta^\circ)\mathbf{X})^-\mathbf{f}} \leqslant \mathbf{f}'\beta^\circ \leqslant \widehat{\mathbf{f}'\beta}(\mathbf{y}) + \sqrt{\chi_n^2(\alpha)\mathbf{f}'(\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta^\circ)\mathbf{X})^-\mathbf{f}}\Big\} = 1 - \alpha$$

and as $\mathbf{f}' \in \mu(\mathbf{X}')$ satisfies

$$\mathbf{f}'(\mathbf{X}'\mathbf{\Sigma}^{-1}(\max)\mathbf{X})^-\mathbf{f} \geqslant \mathbf{f}'(\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta^\circ)\mathbf{X})^-\mathbf{f},$$

we see that

$$P_{\beta^\circ}\Big\{\mathbf{y} \in \mathbb{R}^n : \widehat{\mathbf{f}'\beta}(\mathbf{y}) - \sqrt{\chi_n^2(\alpha)\mathbf{f}'(\mathbf{X}'\mathbf{\Sigma}^{-1}(\max)\mathbf{X})^-\mathbf{f}} \leqslant \mathbf{f}'\beta^\circ \leqslant \widehat{\mathbf{f}'\beta}(\mathbf{y}) + \sqrt{\chi_n^2(\alpha)\mathbf{f}'(\mathbf{X}'\mathbf{\Sigma}^{-1}(\max)\mathbf{X})^-\mathbf{f}}\Big\} \geqslant 1 - \alpha.$$

According to (15), for every $\widetilde{\mathbf{f}'\boldsymbol{\beta}}(\mathbf{y}) \in \mathscr{T}_{\mathbf{f}}$ we have

$$P_{\boldsymbol{\beta}^\circ}\Big\{\mathbf{y} \in \mathbb{R}^n : \widetilde{\mathbf{f}'\boldsymbol{\beta}}(\mathbf{y}) - \sqrt{\mathbf{f}'(\mathbf{X}'\mathbf{D}\mathbf{X})^-\mathbf{f}} \leqslant \widehat{\mathbf{f}'\boldsymbol{\beta}}(\mathbf{y}) \leqslant \widetilde{\mathbf{f}'\boldsymbol{\beta}}(\mathbf{y}) + \sqrt{\mathbf{f}'(\mathbf{X}'\mathbf{D}\mathbf{X})^-\mathbf{f}}\Big\} = 1.$$

Let us denote

$$\mathscr{A} = \Big\{\mathbf{y} \in \mathbb{R}^n : \widehat{\mathbf{f}'\boldsymbol{\beta}}(\mathbf{y}) - \sqrt{\chi^2_n(\alpha)\mathbf{f}'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\max)\mathbf{X})^-\mathbf{f}} \leqslant \mathbf{f}'\boldsymbol{\beta}^\circ \leqslant \widehat{\mathbf{f}'\boldsymbol{\beta}}(\mathbf{y}) + \sqrt{\chi^2_n(\alpha)\mathbf{f}'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\max)\mathbf{X})^-\mathbf{f}}\Big\}$$

$$\mathscr{D} = \Big\{\mathbf{y} \in \mathbb{R}^n : \widetilde{\mathbf{f}'\boldsymbol{\beta}}(\mathbf{y}) - \sqrt{\mathbf{f}'(\mathbf{X}'\mathbf{D}\mathbf{X})^-\mathbf{f}} \leqslant \widehat{\mathbf{f}'\boldsymbol{\beta}}(\mathbf{y})\Big\}$$

and

$$\mathscr{F} = \Big\{\mathbf{y} \in \mathbb{R}^n : \widehat{\mathbf{f}'\boldsymbol{\beta}}(\mathbf{y}) \leqslant \widetilde{\mathbf{f}'\boldsymbol{\beta}}(\mathbf{y}) + \sqrt{\mathbf{f}'(\mathbf{X}'\mathbf{D}\mathbf{X})^-\mathbf{f}}\Big\}.$$

Using Bonferroni's inequality we obtain

$$\begin{aligned}(23)\quad & P_{\boldsymbol{\beta}^\circ}\Big\{\mathbf{y} \in \mathbb{R}^n : \widetilde{\mathbf{f}'\boldsymbol{\beta}}(\mathbf{y}) - \sqrt{\mathbf{f}'(\mathbf{X}'\mathbf{D}\mathbf{X})^-\mathbf{f}} - \sqrt{\chi^2_n(\alpha)\mathbf{f}'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\max)\mathbf{X})^-\mathbf{f}} \\ & \leqslant \mathbf{f}'\boldsymbol{\beta}^\circ \leqslant \widetilde{\mathbf{f}'\boldsymbol{\beta}}(\mathbf{y}) + \sqrt{\mathbf{f}'(\mathbf{X}'\mathbf{D}\mathbf{X})^-\mathbf{f}} + \sqrt{\chi^2_n(\alpha)\mathbf{f}'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\max)\mathbf{X})^-\mathbf{f}}\Big\} \\ & \geqslant P(\mathscr{A} \cap \mathscr{D} \cap \mathscr{F}) \geqslant 1 - \alpha.\end{aligned}$$

So we have obtained another $(1-\alpha)$ confidence region for $\mathbf{f}'\boldsymbol{\beta}$ (using Cleveland's result).

Remark 9.1. Of course, as well as in the previous sections, the $(1-\alpha)$ confidence region (23) depends on the $\boldsymbol{\beta} \in \mathscr{B}_\circ$ chosen.

Example 9.2. According to (23) we obtain in model (1) given in Example 2.6 for various b the following $(1-\alpha)$ confidence regions for β (using Cleveland's result):

b	β
10^{-6}	$\langle 0.85850; 1.07658\rangle$
10^{-3}	$\langle 0.89878; 1.12258\rangle$
10^{-2}	$\langle 0.86792; 1.13723\rangle$
10^{-1}	$\langle 0.74467; 1.47802\rangle$
1	$\langle -1.169; 3.050\rangle$
10	$\langle -26.32; 12.54\rangle$

10. Comparison of the two obtained $(1-\alpha)$confidence regions

Lemma 10.1. $\mathbf{X}\boldsymbol{\Gamma}_{\max}\mathbf{X}' - \mathbf{X}(\mathbf{X}'[\chi_n^2(\alpha)\boldsymbol{\Sigma}(\max)]^{-1}\mathbf{X})^{-}\mathbf{X}'$ *is a p.s.d. matrix.*

Proof. It is obvious that the matrix

$$\mathbf{Z} = \mathbf{X}[(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta})\mathbf{X})^{-}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}) - (\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\max)\mathbf{X})^{-}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\max)]\boldsymbol{\Sigma}(\max)$$
$$\times [(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta})\mathbf{X})^{-}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}) - (\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\max)\mathbf{X})^{-}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\max)]'\mathbf{X}'\chi_n^2(\alpha)$$

is a p.s.d. matrix. We have

$$\begin{aligned}\mathbf{Z} = {} & \mathbf{X}\boldsymbol{\Gamma}_{\max}\mathbf{X}' \\ & - \chi_n^2(\alpha)\mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta})\mathbf{X})^{-}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta})\boldsymbol{\Sigma}(\max)\boldsymbol{\Sigma}^{-1}(\max)\mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\max)\mathbf{X})^{-}\mathbf{X}' \\ & - \chi_n^2(\alpha)\mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\max)\mathbf{X})^{-}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\max)\boldsymbol{\Sigma}(\max)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta})\mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta})\mathbf{X})^{-}\mathbf{X}' \\ & + \chi_n^2(\alpha)\mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\max)\mathbf{X})^{-}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\max)\boldsymbol{\Sigma}(\max) \\ & \times \boldsymbol{\Sigma}^{-1}(\max)\mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\max)\mathbf{X})^{-}\mathbf{X}' \\ = {} & \mathbf{X}\boldsymbol{\Gamma}_{\max}\mathbf{X}' - \mathbf{X}(\mathbf{X}'[\chi_n^2(\alpha)\boldsymbol{\Sigma}(\max)]^{-1}\mathbf{X})^{-}\mathbf{X}'.\end{aligned}$$

The lemma is proved. □

For $\mathbf{f} \in \mu(\mathbf{X}')$ (i.e. $\mathbf{f} = \mathbf{X}'\mathbf{u_f}$) we have

$$\begin{aligned}\mathbf{f}'\boldsymbol{\Gamma}_{\max}\mathbf{f} &= \mathbf{u}_\mathbf{f}'\mathbf{X}\boldsymbol{\Gamma}_{\max}\mathbf{X}'\mathbf{u_f} \\ &\geqslant \mathbf{u}_\mathbf{f}'\mathbf{X}(\mathbf{X}'[\chi_n^2(\alpha)\boldsymbol{\Sigma}(\max)]^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{u_f} \\ &= \mathbf{f}'(\mathbf{X}'[\chi_n^2(\alpha)\boldsymbol{\Sigma}(\max)]^{-1}\mathbf{X})^{-}\mathbf{f} \geqslant 0.\end{aligned}$$

Let us denote

$$L_\mathbf{f} = \sqrt{\mathbf{f}'\boldsymbol{\Gamma}_{\max}\mathbf{f}} - \sqrt{\mathbf{f}'(\mathbf{X}'[\chi_n^2(\alpha)\boldsymbol{\Sigma}(\max)]^{-1}\mathbf{X})^{-}\mathbf{f}}.$$

Lemma 10.2. *Let* $\mathbf{f} \in \mu(\mathbf{X}')$ *and* $\widetilde{\mathbf{X}\boldsymbol{\beta}}(\mathbf{Y}) \in \mathcal{T}$. *Then*

$$(24)\quad P_{\boldsymbol{\beta}^\circ}\{\mathbf{y} \in \mathbb{R}^n : \mathbf{f}'(\mathbf{X}'[\gamma(\mathbf{y} - \widetilde{\mathbf{X}\boldsymbol{\beta}}(\mathbf{y}))'\boldsymbol{\Sigma}^{-1}(\min)(\mathbf{y} - \widetilde{\mathbf{X}\boldsymbol{\beta}}(\mathbf{y}))\boldsymbol{\Sigma}(\max)]^{-1}\mathbf{X})^{-}\mathbf{f} \geqslant L_\mathbf{f}^2\} \geqslant \varphi,$$

where

$$\varphi = P\Big\{\chi^2_{n-R(\mathbf{X})} \geqslant \frac{L_\mathbf{f}^2}{\gamma\mathbf{f}'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\max)\mathbf{X})^{-}\mathbf{f}}\Big\}.$$

P r o o f. From (12) we obtain for every $\widetilde{\mathbf{X}\beta}(\mathbf{Y}) \in \mathscr{T}$ that

$$P_{\beta^\circ}\Big\{\mathbf{y} \in \mathbb{R}^n : (\mathbf{y} - \widehat{\mathbf{X}\beta}(\mathbf{y}))'\boldsymbol{\Sigma}^{-1}(\beta^\circ)(\mathbf{y} - \widehat{\mathbf{X}\beta}(\mathbf{y})) \leqslant (\mathbf{y} - \widetilde{\mathbf{X}\beta}(\mathbf{y}))'\boldsymbol{\Sigma}^{-1}(\beta^\circ)(\mathbf{y} - \widetilde{\mathbf{X}\beta}(\mathbf{y})) \leqslant (\mathbf{y} - \widetilde{\mathbf{X}\beta}(\mathbf{y}))'\boldsymbol{\Sigma}^{-1}(\min)(\mathbf{y} - \widetilde{\mathbf{X}\beta}(\mathbf{y}))\Big\} = 1.$$

Because of the $\chi^2_{n-R(\mathbf{X})}$ distribution of

$$(\mathbf{Y} - \widehat{\mathbf{X}\beta}(\mathbf{Y}))'\boldsymbol{\Sigma}^{-1}(\beta^\circ)(\mathbf{Y} - \widehat{\mathbf{X}\beta}(\mathbf{Y})),$$

we obtain that

$$P_{\beta^\circ}\Big\{\mathbf{y} \in \mathbb{R}^n : (\mathbf{y} - \widetilde{\mathbf{X}\beta}(\mathbf{y}))'\boldsymbol{\Sigma}^{-1}(\min)(\mathbf{y} - \widetilde{\mathbf{X}\beta}(\mathbf{y})) \geqslant \frac{L^2_{\mathbf{f}}}{\gamma \mathbf{f}'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\max)\mathbf{X})^{-}\mathbf{f}}\Big\} \geqslant \varphi,$$

which is equivalent to (24). □

Relation (24) gives us the lower bound for the probability

$$P_{\beta^\circ}\Big\{\mathbf{y} \in \mathbb{R}^n : \sqrt{\mathbf{f}'\boldsymbol{\Gamma}_{\max}\mathbf{f}} - \sqrt{\mathbf{f}'(\mathbf{X}'[\chi^2_n(\alpha)\boldsymbol{\Sigma}(\max)]^{-1}\mathbf{X})^{-}\mathbf{f}} \leqslant \mathbf{f}'(\mathbf{X}'[\gamma(\mathbf{y} - \widetilde{\mathbf{X}\beta}(\mathbf{y}))'\boldsymbol{\Sigma}^{-1}(\min)(\mathbf{y} - \widetilde{\mathbf{X}\beta}(\mathbf{y}))\boldsymbol{\Sigma}(\max)]^{-1}\mathbf{X})^{-}\mathbf{f}\Big\},$$

i.e. for the probability that the $(1-\alpha)$ confidence region given in (22) is smaller than the $(1-\alpha)$ confidence region given in (23) (using Cleveland's result).

Example 10.3. As the value $\eta = \frac{L^2_{\mathbf{f}}}{\gamma \mathbf{f}'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\max)\mathbf{X})^{-}\mathbf{f}}$ in model (1) given in Example 2.6 for various b is

b	η
10^{-6}	1.201×10^{-13}
10^{-3}	2.863×10^{-8}
10^{-2}	1.673×10^{-6}
10^{-1}	9.000×10^{-6}
1	7.309×10^{-7}
10	5.124×10^{-10}

we see that for all investigated values b be $(1-\alpha)$ confidence region (22) is smaller than the $(1-\alpha)$ confidence region (23) (using Cleveland's result) with probability (at least) $\varphi = P\{\chi^2_9 \geqslant \eta\} \doteq 1$. This is in full agreement with the results of Examples 8.2 and 9.2.

Let $\mathbf{Y} \sim N_n(\mathbf{X}\beta^\circ, \mathbf{\Sigma}(\beta^\circ))$. It is obvious that

$$(\mathbf{Y} - \mathbf{X}\beta^\circ)'\mathbf{\Sigma}^{-1}(\beta^\circ)(\mathbf{Y} - \mathbf{X}\beta^\circ)$$

has χ_n^2 distribution. For $0 \leqslant \varepsilon \leqslant 1$ we have

$$P_{\beta^\circ}\{\mathbf{y} \in \mathbb{R}^n : (\mathbf{y} - \mathbf{X}\beta^\circ)'\mathbf{\Sigma}^{-1}(\beta^\circ)(\mathbf{y} - \mathbf{X}\beta^\circ) \leqslant \chi_n^2(\varepsilon)\} = 1 - \varepsilon.$$

We also have

$$P_{\beta^\circ}\{\mathbf{y} \in \mathbb{R}^n : |\mathbf{f}'(\mathbf{y} - \mathbf{X}\beta^\circ)| \leqslant \sqrt{\mathbf{f}'\chi_n^2(\varepsilon)\mathbf{\Sigma}(\beta^\circ)\mathbf{f}} \quad \text{for all } \mathbf{f} \in \mathbb{R}^n\} = 1 - \varepsilon$$

and therefore

$$P_{\beta^\circ}\{\mathbf{y} \in \mathbb{R}^n : (\mathbf{y} - \mathbf{e}_i'\mathbf{X}\beta^\circ)^2 \leqslant \chi_n^2(\varepsilon)\sigma^2(a + b|\mathbf{e}_i'\mathbf{X}\beta^\circ|)^2 \ , i = 1, 2, \dots, n\} \geqslant 1 - \varepsilon.$$

For $\mathbf{X}\beta \in \mathbb{R}^n$ let us denote

$$S_{\mathbf{X}\beta} = \{\mathbf{y} \in \mathbb{R}^n : (\mathbf{y} - \mathbf{e}_i'\mathbf{X}\beta)^2 \leqslant \chi_n^2(\varepsilon)\sigma^2(a + b|\mathbf{e}_i'\mathbf{X}\beta|)^2, \ i = 1, 2, \dots, n\}$$

and for $\mathbf{y} \in \mathbb{R}^n$ let

$$T_{\mathbf{y}} = \{\mathbf{X}\beta \in \mathbb{R}^n : (\mathbf{y} - \mathbf{e}_i'\mathbf{X}\beta)^2 \leqslant \chi_n^2(\varepsilon)\sigma^2(a + b|\mathbf{e}_i'\mathbf{X}\beta|)^2, \ i = 1, 2, \dots, n\}.$$

For every $\beta \in \mathbb{R}^k$ we have

$$P_\beta\{\mathbf{y} \in \mathbb{R}^n : \mathbf{y} \in S_{\mathbf{X}\beta}\} = P_\beta\{\mathbf{y} \in \mathbb{R}^n : \mathbf{X}\beta \in T_{\mathbf{y}}\}.$$

The probability that $T_{\mathbf{y}}$ covers $\mathbf{X}\beta^\circ$ is greater than equal to $1 - \varepsilon$.

We see that with such a probability

$$(\mathbf{y} - \mathbf{e}_i'\mathbf{X}\beta^\circ)^2 \leqslant \chi_n^2(\varepsilon)\sigma^2(a + b|\mathbf{e}_i'\mathbf{X}\beta^\circ|)^2$$

for $i = 1, 2, \dots, n$ i.e.

$$(\mathbf{e}_i'\mathbf{X}\beta^\circ)^2[1 - \chi_n^2(\varepsilon)\sigma^2 b^2] + \mathbf{e}_i'\mathbf{X}\beta^\circ(-2)[y_i \pm ab\sigma^2\chi_n^2(\varepsilon)] + y_i^2 - a^2\sigma^2\chi_n^2(\varepsilon) \leqslant 0$$

(the sign + or − corresponds to the sign of $\mathbf{e}_i'\mathbf{X}\beta^\circ$).

Let

$$1 - \chi_n^2(\varepsilon)\sigma^2 b^2 > 0. \tag{25}$$

It is easy to show that with probability at least $1 - \varepsilon$ for $i = 1, 2, \ldots, n$

$$\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}^\circ \in \mathscr{A}_i = \Big\langle \max\Big(0; \frac{y_i + ab\sigma^2\chi_n^2(\varepsilon) - \sqrt{\chi_n^2(\varepsilon)}\sigma|a + by_i|}{1 - \chi_n^2(\varepsilon)\sigma^2 b^2}\Big), \max\Big(0; \frac{y_i + ab\sigma^2\chi_n^2(\varepsilon) + \sqrt{\chi_n^2(\varepsilon)}\sigma|a + by_i|}{1 - \chi_n^2(\varepsilon)\sigma^2 b^2}\Big)\Big\rangle \cup \Big\langle \min\Big(\frac{y_i - ab\sigma^2\chi_n^2(\varepsilon) - \sqrt{\chi_n^2(\varepsilon)}\sigma|a - by_i|}{1 - \chi_n^2(\varepsilon)\sigma^2 b^2}; 0\Big), \min\Big(\frac{y_i - ab\sigma^2\chi_n^2(\varepsilon) + \sqrt{\chi_n^2(\varepsilon)}\sigma|a - by_i|}{1 - \chi_n^2(\varepsilon)\sigma^2 b^2}; 0\Big)\Big\rangle. \tag{26}$$

From (26) we obtain $\underline{\gamma_i}$ and $\overline{\gamma_i}$, $i = 1, 2, \ldots, n$, then $\underline{\gamma}$, $\overline{\gamma}$ and γ for evaluation of (11) and also for evaluation of $\boldsymbol{\Sigma}(\min)$ and $\boldsymbol{\Sigma}(\max)$. So, using in (6) any matrix

$$\boldsymbol{\Sigma}_\delta^\star = \sigma^2 \begin{pmatrix} (a + b|\delta_1|)^2 & 0 & \ldots & 0 \\ 0 & (a + b|\delta_2|)^2 & \ldots & 0 \\ \vdots & & \ddots & \\ 0 & & \ldots & (a + b|\delta_n|)^2 \end{pmatrix},$$

where $\delta_i \in \mathscr{A}_i$, $i = 1, 2, \ldots, n$, we obtain results (15), (16), (18) with probability greater than or equal to $1 - \varepsilon$. We only note that in the case when (25) is not satisfied, $\mathscr{A}_i$, $i = 1, 2, \ldots, n$ are infinitely large.

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