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Dag Lukkassen

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ON SOME SHARP BOUNDS FOR THE OFF-DIAGONAL ELEMENTS OF THE HOMOGENIZED TENSOR

DAG LUKKASSEN, Narvik

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Summary. In this paper we study bounds for the off-diagonal elements of the homogenized tensor for the stationary heat conduction problem. We also state that these bounds are sharp by proving a formula for the homogenized tensor in the case of laminate structures.

Keywords: stationary heat conduction problem, Y-periodicity, homogenized coefficients, bounds, laminate structures.

AMS classification: 35B27, 35Q99, 73B27, 73K20

1. INTRODUCTION

We consider the stationary heat conduction problem

$$\operatorname{div}(a \operatorname{grad} V) = f \quad \text{on } \Omega \subset \mathbb{R}^N,$$

where the heat conductivity tensor a is of the form λI and also periodic relative to a cell Y which is very small compared with Ω.. A direct numerical treatment of this problem leads to difficulties due to the rapid variations on micro scale level. However, if $\alpha \leqslant \lambda \leqslant \beta$ a.e. for some positive constants α and β, then there exists a homogeneous material with a constant heat conductivity tensor q whose overall response is close to that of the heterogeneous material under consideration. We can compute the *homogenized tensor* q by identifying and solving a certain variational problem called *the cell problem*. This is the key step in the homogenization procedure. Concerning this and other basic information in the homogenization theory we refer to the literature, e.g. the books [1] and [8].

Many different types of bounds for q have been found by several authors (see e.g. [2], [3], [5], [6] and [7] and the references therein). In this work we focus on the cell problem and prove upper and lower bounds for the off-diagonal elements of the homogenized tensor. In order to show that these bounds are sharp we also present an explicit formula for the homogenized tensor in the case of laminate structures.

2. Preliminaries

In the space $\mathbb{R}^N$, we consider a fixed parallelepiped $Y = \prod_{j=1}^{N}]0, x_j^0[$ (a Y-cell). Throughout this paper we assume that the thermal conductivity tensor a is Y-periodic and that $\forall x \in \mathbb{R}^N$, $a_{ij}(x) = \delta_{ij}\lambda(x)$, where $\alpha \leqslant \lambda \leqslant \beta$ a.e. for some positive constants α and β.

The weak formulation of the cell-problem takes the following form: Find χ^i in $H_{\text{per}}(Y)$ (= the space of all Y-periodic $\psi \in H^1(Y)$) such that

$$a_\lambda(\varphi, \chi^i) = f_\lambda(\varphi) \tag{1}$$

for all $\varphi \in H_{\text{per}}(Y)$. Here, $a_\lambda(\cdot,\cdot)$ and $f(\cdot)$ are defined by

$$a_\lambda(\varphi, \psi) = \int_Y \sum_{j=1}^{N} \lambda \frac{\partial \varphi}{\partial x_j} \frac{\partial \psi}{\partial x_j}\, \mathrm{d}x \text{ and } f_\lambda(\varphi) = \int_Y \lambda \frac{\partial \varphi}{\partial x_i}\, \mathrm{d}x.$$

It is also possible to prove (see e.g. [4], p. 50) that

$$L_\lambda(\varphi) \geqslant L_\lambda(\chi^i), L_\lambda(\varphi) = \frac{1}{2} a_\lambda(\varphi, \varphi) - f_\lambda(\varphi) \tag{2}$$

for all $\varphi \in H_{\text{per}}(Y)$ if and only if χ^i is a solution to (1). The solution χ^i is unique up to an additive constant.

The elements of the homogenized tensor q are given by

$$q_{ij} = \frac{1}{|Y|} \int_Y \left(\lambda \delta_{ij} - \lambda \frac{\partial \chi^i}{\partial x_j} \right) \mathrm{d}x. \tag{3}$$

(We do not use the summation convention in any part of this paper.)

3. The main results

In this section we present the main results of the paper. We let q_a and q_h denote the arithmetic and the harmonic mean of λ, respectively. First, we consider bounds for the off-diagonal elements of q.

Theorem 1. *If $r \neq s$, then*

$$|q_{rs}| \leqslant q_a - \frac{1}{2}(q_{rr} + q_{ss}).$$

The following result is a simple formula for the homogenized coefficients for the case of laminate structures.

Theorem 2. *Suppose that λ varies periodically only in one spatial direction, say ξ, where $|\xi|$ is the period. Then the homogenized coefficient q_{rs} is given by*

$$q_{rs} = \delta_{rs} q_a + \frac{\xi_r \xi_s}{\xi^2}(q_h - q_a). \tag{4}$$

Remark 1. If λ is only a function of a direction parallel to $e_r \pm e_s$, then Theorem 2 shows that q_{rs} actually equals $\pm(q_a - \frac{1}{2}(q_{rr} + q_{ss}))$. Thus, we can give non-trivial examples where equality occurs in the statement of Theorem 1.

In $\mathbb{R}^2$ the results of Theorem 2 can be translated into geometrical language. Let φ be the angle between e_s and ξ. In terms of φ, q_a and q_h we see that q_{rs} takes the following form:

$$q_{rs} = \frac{q_h - q_a}{2} \sin 2\varphi \quad \text{for } r \neq s,$$
$$q_{rs} = q_a \sin^2\varphi + q_h \cos^2\varphi \quad \text{for } r = s.$$

4. Proofs

Proof of Theorem 1. Consider the solutions of the cell-problem (1), χ^r and χ^s for $i = r$ and $i = s$, respectively. By putting $\varphi = \chi^k$ we get that

$$\int_Y \sum_{i=1}^{N} \lambda \Big(\frac{\partial \chi^k}{\partial x_i}\Big)^2 \,\mathrm{d}x = \int_Y \lambda \frac{\partial \chi^k}{\partial x_k} \,\mathrm{d}x \quad \text{for } k \in \{r, s\}. \tag{5}$$

Moreover, (2) gives that

$$\int_Y \Big(\frac{1}{2}\sum_{i=1}^N \lambda\Big(\frac{\partial \chi^r}{\partial x_i}\Big)^2 - \lambda\frac{\partial \chi^r}{\partial x_s}\Big)\,\mathrm{d}x \geqslant \int_Y \Big(\frac{1}{2}\sum_{i=1}^N \lambda\Big(\frac{\partial \chi^s}{\partial x_i}\Big)^2 - \lambda\frac{\partial \chi^s}{\partial x_s}\Big)\,\mathrm{d}x.$$

This also holds if we replace χ^r by $-\chi^r$. Using (5), we now get that

$$|q_{rs}| = \Big|\frac{1}{|Y|}\int_Y \lambda\frac{\partial \chi^r}{\partial x_s}\,\mathrm{d}x\Big| \leqslant \frac{1}{2|Y|}\Big(\int_Y \lambda\frac{\partial \chi^s}{\partial x_s}\,\mathrm{d}x + \int_Y \lambda\frac{\partial \chi^r}{\partial x_r}\,\mathrm{d}x\Big).$$

Hence,

$$|q_{rs}| \leqslant q_a - \frac{1}{2\,|Y|}\Big(\int_Y \Big(\lambda - \lambda\frac{\partial \chi^r}{\partial x_r}\Big)\,\mathrm{d}x + \int_Y \Big(\lambda - \lambda\frac{\partial \chi^s}{\partial x_s}\Big)\,\mathrm{d}x\Big) \leqslant q_a - \frac{1}{2}(q_{rr} + q_{ss}),$$

and the proof is complete. $\square$

P r o o f o f T h e o r e m 2. We first prove the following statement: *Suppose $f \in L^2(I)$, $I = [0,1]$, and that the following holds for every disjoint segments I_1 and I_2 of I: $|I_2|\int_{I_1} f\,\mathrm{d}x = |I_1|\int_{I_2} f\,\mathrm{d}x$. Then f is constant a.e.*

We state that

$$\int_V f\,\mathrm{d}x = \int_V k\,\mathrm{d}x \tag{6}$$

for every open or closed set $V \subseteq I$, where $k = |I|^{-1}\int_I f\,\mathrm{d}x$. This can be seen by observing that (6) holds for every segment, hence also for open sets since they are unions of countable collections of disjoint segments. Finally, (6) holds for closed sets since they are complements of open sets.

Fix $\varepsilon > 0$. Since I is bounded, $f \in L^1(I)$ and there is a continuous function g such that $\|f - g\|_1 < \varepsilon/2$. Let V be the open set $\{x \in I\colon g(x) > k\}$. By (6) it yields that

$$\int_I |g - k|\,\mathrm{d}x = \Big|\int_V g - f\,\mathrm{d}x\Big| + \Big|\int_{I\setminus V} g - f\,\mathrm{d}x\Big| \leqslant \int_I |g - f|\,\mathrm{d}x < \frac{\varepsilon}{2}.$$

The triangle inequality gives

$$\int_I |f - k|\,\mathrm{d}x \leqslant \int_I |f - g|\,\mathrm{d}x + \int_I |g - k|\,\mathrm{d}x < \varepsilon,$$

and since ε was arbitrarily chosen, $\|f - k\|_1 = 0$ which, in its turn, implies that $f = k$ a.e.

Let χ^s be the solution of (1) for $i = s$. Since λ varies only in the direction ξ any translation of χ^s orthogonal to ξ will be a solution to the cell problem. Hence, by uniqueness this implies that χ^s varies only in the direction ξ. Now we let ξ_k be a non-zero component of ξ and I_1 and I_2 two disjoint segments of $[0,1]$. In addition we choose a function $\varphi \in H_{\text{per}}(Y)$ defined within an arbitrary constant by

$$\nabla\varphi(\xi t) = \xi(|I_2|\,\chi_{I_1}(t) - |I_1|\,\chi_{I_2}(t)), \quad t \in [0,1]. \tag{7}$$

Here, χ_A denotes the usual characteristic function of the set A. Since χ^s, φ and λ vary only in the direction ξ, (1) may be written in the form

$$\int_{[0,1]} \Big(\sum_{n=1}^{N} \lambda \frac{\partial\varphi}{\partial x_n}\frac{\partial\chi^s}{\partial x_n}\Big((\xi t) - \lambda\frac{\partial\varphi}{\partial x_s}(\xi t)\Big)\Big)\,\mathrm{d}t = 0. \tag{8}$$

Since χ^s varies only in direction ξ, $\nabla\chi^s$ is parallel to ξ, i.e. there exists a real function η such that $\nabla\chi^s = \eta\xi$. This implies that $\partial\chi^s/\partial x_r = \eta\xi_r$ and $\partial\chi^s/\partial x_k = \eta\xi_k$. Hence,

$$\frac{\partial\chi^s}{\partial x_r} = \frac{\xi_r}{\xi_k}\frac{\partial\chi^s}{\partial x_k} \tag{9}$$

for all $r \in \{1,2,\ldots,N\}$. According to (7), (8) reduces to

$$|I_2|\int_{I_1}\Big(\frac{\lambda\xi^2}{\xi_k}\frac{\partial\chi^s}{\partial x_k} - \xi_s\lambda\Big)\,\mathrm{d}t = |I_1|\int_{I_2}\Big(\frac{\lambda\xi^2}{\xi_k}\frac{\partial\chi^s}{\partial x_k} - \xi_s\lambda\Big)\,\mathrm{d}t,$$

and the fact that this holds for every disjoint segments I_1 and I_2 of I, combined with (9) yields

$$\lambda\Big(\xi^2\frac{\partial\chi^s}{\partial x_r} - \xi_r\xi_s\Big) = \xi_r K_s \tag{10}$$

a.e. in Y, where K_s is a constant. Dividing by λ and recalling that χ^s takes the same values on opposite traces of Y, i.e. $\int_Y \partial\chi^s/\partial x_r = 0$, we get that $K_s = -q_h\xi_s$ and according to (3) it easily follows that

$$q_{rs} = \delta_{rs}q_a + \frac{\xi_r\xi_s}{\xi^2}(q_h - q_a).$$

The proof is complete. □

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References

[1] *A. Bensoussan, J. L. Lions, G. Papanicolaou*: Asymptotic analysis for periodic structures. North-Holland, Amsterdam, 1978.

[2] *L. V. Gibiansky*: Bounds on effective moduli of composite materials. Lecture notes, School on Homogenization, ICTP, Trieste, 1993.

[3] *Z. Hashin, S. Shtrikman*: A variational approach to the theory of effective magnetic permeability of multi phase materials.. J. Appl. Phys. *33* (1962), 3125–3131.

[4] *C. Johnson*: Numerical solution of partial differential equations by the finite element method. Studentlitteratur, Lund, 1987.

[5] *D. Lukkassen*: Upper and lower bounds for homogenized coefficients. Uspekhi Mat. Nauk *49* (1994), no. 4, 115; transl. as Russian Math. Surveys.

[6] *D. Lukkassen, L. E. Persson, P. Wall*: On some sharp bounds for the effective conductivity. Proceedings from the first International Conference on Composites Engineering (ICCE/1), New Orleans. 1994, pp. 855–856.

[7] *G. W. Milton*: On characterizing the set of possible effective tensors of composites: The Variational Method and the Translation Method. Comm. on Pure and Appl. Math. *XLIII* (1990), 63–125.

[8] *L. E. Persson, L. Persson, N. Svanstedt, J. Wyller*: The homogenization method: An introduction. Studentlitteratur, Lund, 1993.

Author's address: *Dag Lukkassen*, Department of Mathematics, Narvik Institute of Technology, 8501 Narvik, Norway, e-mail: `dl@tiur.sin.no`.