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A NOTE TO INDEPENDENT SETS IN SCHEDULING

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Summary. The paper studies the bus-journey graphs in the case when they are piecewise expanding and contracting (if described by fathers-sons relations starting with the greatest independent set of nodes). This approach can make it possible to solve the minimization problem of the total service time of crews.

Keywords: bus scheduling, crew scheduling, journey, graph, independent set, expanding graph, contracting graph

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INTRODUCTION

Let a bus time-table be given, that means, we have a set of bus journeys V with the elements of the form $v = (o_v, d_v, t_v, a_v)$, where o_v is the original (i.e. the initial station of v), d_v is the destination of v, t_v is the time of departure of v from o_v and a_v is the time of arrival of v into d_v. Moreover, let $m_{v,w}$ be the time which is necessary for a bus to get ready for to start the service on the journey w in o_w after having finished the service on the journey v in the point d_v. Let us define the graph $G = (V, H)$ where $(v, w) \in H$ if and only if $a_v + m_{vw} \leqslant t_w$, and let us call it the *bus-journey graph.* Hence any oriented path on G represents a schedule of a bus, i.e. a sequence of journeys realized by the same vehicle. This implies that the minimum number of paths covering the vertex set V represents the minimum necessary number of buses. Moreover, in the case of a strong assignment of drivers to buses (which is usual e.g. for any Czech bus company) a path represents a drivers' piece of work, which implies that the minimum sum of path lengths is equivalent to the minimum total service time of drivers.

It is not a new idea to utilize independent sets of journey graphs in bus scheduling. We can mention e.g. [1, 4, 6, 7]. In [7] maximum independent sets were used for a decomposition of the problem, in [4] cardinality of a maximum independent set is used as an objective function and in [1] it is used in order to accelerate the algorithm. The purpose of this paper is to discuss the role of independent sets in the case of piecewise expanding and contracting multipartite graphs. This approach may be used in minimization of the total length of covering paths and thus in minimization of the total service times of crews.

1. Preliminary. Throughout the whole paper we will suppose we are given a finite non-empty directed transitive graph $G = (V, H)$ without loops and cycles. V is the set of nodes, H is the set of arcs.

We note that the property of transitivity means that $h_1 = (w, u) \in H$ and $h_2 = (u, v) \in H$ implies $h_3 = (w, v) \in H$.

If M is an arbitrary set, then $|M|$ denotes the number of elements in M.

2. Definition. If $(w, v) \in H$ then w is called the *predecessor* of v and v is called the *successor* of w. If there exists no $u \in V$, $w \neq u \neq v$ $(w, u) \in H$, $(u, v) \in H$, then w is called the *father* (or the immediate predecessor) of v, and v is called the *son* (or the immediate successor) of w.

3. Definition. A subset W of V is called *independent*, if for arbitrary two nodes $w, v \in W$ we have $(w, v) \notin H$.

An independent set W is called *maximum* if it is not a proper subset of any independent set. W is called the *greatest* if there exists no independent set U with $|U| > |W|$.

4. Definition. Let $\mathbf{W} = \{W_k \colon k = 1, \dots, n\}$ be a partition of the node set V, let each W_k from $\mathbf{W}$ be an independent set. Then $\mathbf{W}$ is called an *N-partition* on G.

5. Definition. Let W be independent and for $v \in V$ let us have

(i) v is a son of some $w \in W$,

(ii) if v is a successor of some $y \in W$, then v is a son of y.

Then we say that v *immediately succeeds* W. We denote by $b(W)$ the set of all $v \in V$ immediately succeeding W.

If we put "precede" instead of "succeed" and "father" instead of "son", we get the definition of $v \in V$ *immediately preceding* W. The set of all $v \in V$ immediately succeeding W is denoted by $b(W)$, the set of all $v \in V$ immediately preceding W is denoted by $b_{-1}(W)$. Moreover, we denote

$$b^{(0)}(W) = W, \quad b^{(1)}(W) = b(W), \quad b^{(k)}(W) = b\big(b^{(k-1)}(W)\big),$$
$$b^{(-1)}(W) = b_{-1}(W), \quad b^{(-k)}(W) = b_{-1}\big(b^{(-k+1)}(W)\big), \quad k = 1, 2, \dots.$$

6. Lemma. *Let W be an independent node set in G. Then*
1. *$b(W)$ and $b_{-1}(W)$ are independent.*
2. *If W is maximum, then W is non-empty.*

Proof. 1. Indirectly, suppose there exist v, $x \in b(W)$, $(v, x) \in H$. Then there exists $w \in W$, $(w, v) \in H$ and hence x is a successor but not a son of w, which contradicts (ii) from Definition 5. The proof for $b_{-1}(W)$ is similar.

2. Since G is without loops, for any $v \in V$ the set $\{v\}$ is independent and thus any maximum independent set must be non-empty. □

7. Remark. If W is a maximum independent set, then $b(W)$ and $b_{-1}(W)$ need not be maximum as one can see from the graph

$$w \to v$$

$$y \to x.$$

For $W = \{w, x\}$ we have $b(W) = \{v\}$, $b_{-1}(W) = \{y\}$ but e.g. $b(W) \cup \{y\}$ and $b_{-1}(W) \cup \{w\}$ are independent.

8. Lemma. *Let W be a maximum independent node set in G and let m, n be non-negative integers, $b^{(n)}(W) \neq \emptyset$, $b^{(n+1)}(W) = \emptyset$, $b^{(-m)}(W) \neq \emptyset$, $b^{(-m-1)}(W) = \emptyset$. Then $\mathbf{B}(W) = \{b^{(k)}(w) : k = -m, \dots, n\}$ is an N-partition of V in G.*

Proof. It follows from Lemma 6 that all $b^{(k)}(W)$ are independent and the transitivity of G implies they are mutually disjoint. It remains to prove that their sum equals V. Indirectly: suppose

$$v \in V - \bigcup_{k=-m}^{n} b^{(k)}(W).$$

Since W is maximum there exists $w \in W$ such that $(w, v) \in H$ of $(v, w) \in H$ and because of the finiteness of G there must exist $k \in \{-m, \dots, n\}$ such that $v \in b^{(k)}(W)$, which gives a contradiction. □

9. Definition. Let $C = \{c_1, \dots, c_r\}$ be a set of chains in G such that any $v \in V$ belongs to at least one $c \in C$. Then we say that C is a *covering of the graph* G or that C *covers* G. Denote by $C(G)$ the class of all sets C covering G and by $\kappa(G)$ the minimum of cardinalities of the covering sets C.

If C covers G but no $C - \{c_k\}$ covers G than C is called a *minimum covering* *of* G. If C covers G and $|C| = \kappa(g)$ then C is called the smallest covering of G (it is obvious that the *smallest covering* is the minimum covering as well).

A covering C of G is called an exact covering if each $v \in V$ is incident to exactly one $c \in C$.

10. Remark. Since G is transitive, from every non-exact covering C of G it is possible to construct (at least 2) exact coverings of G in such a way that any $v \in V$ is left in one $c \in C$ and dropped from the others. Hence any consideration of coverings can be limited to exact coverings.

11. Lemma. *Let W be the greatest independent node set on G. Let $G' = (V', H')$ be a bipartite subgraph of G such that the first part of V' is W, the second is $b(W)$ and $(w, v) \in H$ iff w is the father of v. Then in G' there exists a matching M, $|M| = |b(W)|$.*

Proof. Obviously $|W| \geqslant |b(W)|$. Let M be the greatest matching on G'. Go ahead indirectly: let $|M| < |b(W)|$ which implies that there exist non-empty sets X in $b(W)$ and Y in W not incident to M. Let us define a new graph $G'' = (V', H'')$ where H'' contains the arcs from M and the reversed arcs (with the opposite directions) from $H' - M$. Because M is maximum, there exists no chain from X to Y on G''. Let $x \in X$, let us denote by $R(x) \subset W$ and $S(x) \subset b(W)$ the sets of vertices which are reachable from x in G''. Obviously

1. $R(x) \cap Y = 0$ (if not, M could not be maximum).
2. $S(x) \cap X = 0$ (the opposite contradicts the definition of X).
3. $|R(x)| = |S(x)|$.
4. $U = [W - R(x)] \cup S(x) \cup \{x\}$ is an independent set on G (if not, then there would exist an arc with the initial vertex in $W - R(x)$ and the end vertex in $S(x) \cup \{x\}$, which contradicts the definitions of x and $S(x)$).
5. $|U| = |W| + 1$, which contradicts the maximality of W. Then proofs is complete.

$\square$

12. Lemma. *Let W be an independent node set on G, let $G' = (V', H')$ be a bipartite graph defined similarly as in Lemma 11. Let there exist a matching M on G' with $|M| = |b(W)|$. Then W is the greatest independent node set on G'.*

Proof. Let U be an independent node set on G', let us denote $U_0 = U \cap W$, $U_1 = U \cap b(W)$. For each $u \in U_1$ there exists $w(u) \in W = U_0$ such that $\big(w(u), u\big) \in M$. Let us denote $U_2 = \{w(u) \colon u \in U_1\}$. Obviously $|U| = |U_0| + |U_1| = |U_0| + |U_2| \leqslant |W|$. Thus W is the greatest, q.e.d. $\square$

13. Theorem. *Let W be an independent node set on G, let $G' = (V', H')$ be a bipartite graph with the partition of V' into W and $b(W)$ and let $(w, v) \in H' \Leftrightarrow w \in W$, $v \in b(W)$, $(w, v) \in H$. Then W is the greatest independent node set in G' iff (i.e. if and only if) there exists matching M on G' with $|M| = |b(W)|$.*

Proof is an immediate consequence of Lemmas 11 and 12. □

Expansion and contradiction

14. Definition. Let $n > 1$ and let $\mathbf{W} = \{W_k : k = 1, \ldots, n\}$ be a sequence of mutually disjoint subsets of the node set V of G. For each $k = 1, \ldots, n$ let $G_k = (V_k, H_k)$ be a bipartite graphs for which $V_k = W_k \cup W_{k+1}$, $H_k = \{(w, v)H : w \in W_k, v \in W_{k+1}\}$ holds. If $\{W_k : k = 1, \ldots, n\}$ is not increasing and if for each $k = 1, \ldots, n$ there exists a matching M_k on G_k such that $|M_k| = |b(W_{k+1})|$ then we say that $\mathbf{W}$ is a *contracting sequence.*

If $\mathbf{W}' = \{W_{n-k+1} : k = 1, \ldots, n\}$ is a contracting sequence, then the sequence $\mathbf{W} = \{W_k : k = 1, \ldots, n\}$ is called an expanding sequence.

15. Lemma. *In the graph G let there exist an independent node set W and an integer $n > 0$ such that $V = \bigcup_{k=0}^{n} b^{(k)}(W)$ and the sequence $\mathbf{W} = \{b^{(k)}(W) : k = 0, \ldots, n\}$ is contracting. Then W is the greatest independent node set in G.*

Proof. Let U be an independent node set in G. Let us denote $U_k = U \cap b^{(k)}(W)$, $k = 0, \ldots, n$. Further, for each $k = 1, \ldots, n$ and each subset Y of $b^k(W)$ let us denote

$$a_k(Y) = \{x \in b^{(k-1)}(W) : \text{ ex. } y \in Y, (x, y) \in M_{k-1}\}$$

where M_{k-1} is a matching in the bipartite subgraph G_{k-1}, $|M_{k-1}| = |b^{(k)}(W)|$ (their existence follows from the properties of contradiction). Now, let us denote $Q_0 = U$, $Q_1 = a_1(U_1)$, $Q_2 = a_1(a_2(U_2))$, $\ldots$, $Q_n = a_1(a_2(\ldots a_n(U_n) \ldots))$. Obviously $Q_k \cap Q_j = \emptyset$ for $i + j$ (otherwise U would not be independent), $|Q_k| = |U_k|$, $k = 0, \ldots, n$ and thus

$$\sum_{k=0}^{n} |U_k| = \sum_{k=0}^{n} |Q_k| \leqslant |W|.$$

Since U was arbitrary, W is the greatest, q.e.d. □

16. Lemma. *In a graph G let there exist an independent node set W and an integer $n > 0$ fulfilling the conditions of Lemma 15. Let G_k be the corresponding bipartite graphs (Def. 14) and let M_k be the maximum matching on G_k, $k = 0, \ldots,$*

$n-1$. *Let* $k \in \{1,\dots,n\}$, $v \in b^{(k)}(W)$. *Then we denote by* $a_k(v)$ *that element of* $b^{(k-1)}(W)$ *for which* $\big(a_k(v),v\big) \in M_{k-1}$. *Let* C *be a set containing the following chains on* G*:*

$n)$ $\big(a_1\big(\dots a_n(v)\dots\big), a_2\big(\dots a_n(v)\dots\big),\dots,a_n(v),v\big)$ *for each* $v \in b^{(n)}(W)$,

$n-1)$ $\big(a_1\big(\dots a_{n-1}(v)\dots\big), a_2\big(\dots a_{n-1}(v)\dots\big),\dots,a_{n-1}(v),v\big)$ $\forall\ v \in b^{(n-1)}(w) - a_n\big(b^{(n)}(W)\big)$,

....

$1)$ $\big(a_1(v),v\big)$ *for each* $v \in b(W) - a_2\big(b^{(2)}(W) - a_2\big(a_3\big(b^{(3)}(W)\big)\big) - \dots - a_2\big(\dots a_n\big(b^{(n)}(W)\dots\big)\big)\big)$,

$0)$ $(v)\ \forall\, v \in W - a_1\big(b(W) - a_1\big(a_2\big(b^{(2)}(W)\big)\big) - \dots - a_1\big(a_2\big(\dots a_n\big(b^{(n)}(W)\dots\big)\big)\big)\big)$.

Then C *is the smallest covering of* G.

Proof. It follows from the construction that C is an exact covering of G, $|C| = |W|$. Since an independent set W cannot be covered by a smaller number of chains that $|W|$ the proof is complete. □

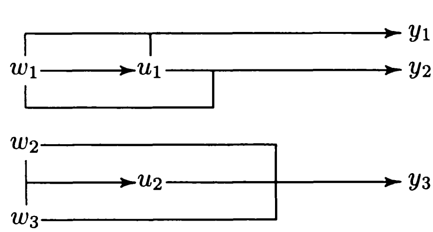
17. Remark. By virtue of the symmetric relation between contracting and expanding sequences (Def. 14), Lemma 15 and 16 remain valid if we replace "$b^{(k)}$" by "$b^{(-k)}$", "contracting" by "expanding" and the definition of C is properly adjusted.

This procedure of covering can be easily extended to graphs G with the node set

$$V = \bigcup_{k=-m}^{n} b^{(k)}(W)$$

where W is an independent set, $\{b^{(k)}(W)\colon k = 0,\dots,n\}$ is contracting and $\{b^{(k)}(W)\colon k = -m,\dots,0\}$ is expanding (the chains "to the left" are concatenated with those "to the right").

18. Remark. The case when $\{b^{(k)}\colon k = -m,\dots,0\}$ is contracting and $\{b^{(k)}(W)\colon k = 0,\dots,n\}$ is expanding is much more complicated. The main reason is the following: If we apply the construction of C (from Lemma 16) to $\{b^{(k)}(W)\colon k = -m,\dots,n\}$ which is contracting and afterwards to the expanding part $\{b^{(k)}(W)\colon k = 0,\dots,n\}$, the concatenation is easy only in the case of chains terminating and starting in $W = b^{(0)}(W)$. We cannot be sure we can connect chains, terminating in some $b^{(k)}(W)$, $k < 0$ with another one starting in $b^{(k)}(W)$, $k > 0$, e.g. in the graph



We can see that $W = \{u_1, u_2\}$, $b^{(-1)}(W) = \{w_1, w_2, w_3\}$, $b(W) = \{y_1, y_2, y_3\}$ $\{b^{(-1)}(W), W\}$ is contracting, $\{W, b(W)\}$ is expanding. Both the "left" and "right" parts can be covered by 3 chains—e.g.

"left": (w_1, u_1), (w_2), $(w_3,\ u_2)$,

"right": (u_1, y_1), (y_2), (u_2, y_3),

but for the covering of the whole graph G we need at least 4 chains. Naturally, the greatest independent node set is not $\{w_1, w_2, w_3\}$, but $\{w_2, w_3, y_1, y_2\}$.

If we add the arcs (w_2, u_1), (w_2, y_1) and (w_2, y_2) to the previous graph, we obtain another example: The "left" covering (w_1, u_1), (w_2, u_2), (w_3) can not be concatenated with the "right" one (u_1, y_2), (u_2, y_3), (y_1). On the another hand, "left" (w_1, u_1), (w_2), (w_3, u_2) with the "right" (u_1, y_1), (y_2), (u_2, y_3) can be concatenated to (w_1, u_1, y_1), (w_2, y_2), (w_3, u_2, y_3).

Hence the method of "contracting-expanding concatenation" is only heuristic (in contrast to the "expanding-contracting" case, when it is exact), which is, of course, much faster than the exact solution. Thus it is possible first to try the heuristic approach; if $|C| = \max\{|b(W)|\colon k = -m, \dots, n\}$, the result C is optimal.

If it is not true, one can use the general exact method. All these considerations lead to the case when V possesses a piecewise contracting and expanding decomposition. It is based on the following lemma.

19. Lemma. *Let W be a maximum independent node set in G. Then there exist integers $m \geqslant 0$, $n \geqslant 0$ such that*

$$V = \bigcup_{k=-m}^{n} b^{(k)}(W).$$

P r o o f. First, let us define

$$V_0 = \bigcup_{k=-\infty}^{\infty} b^{(k)}(W).$$

If $v \in V - V_0$, then $W \cup \{v\}$ is independent, which contradicts the maximality of W. Hence $V = V_0$. Since G is finite, m and n must exist, q.e.d. $\square$

20. Construction of a minimum exact covering of G.

1st phase: Find the greatest independent node set in G. (This is a well-known problem, described e.g. in [6].)

2nd phase: Find the numbers m, n from Lemma 19.

3rd phase: Divide the index set $\{-m, \dots, n\}$ into the parts

$$-m = j_0, \dots, j_1; j_1, \dots, j_2; \dots; j_{2r-1}, \dots, j_{2r} = n$$

such that for $i-1, \dots, r$, $W_{2i-1} = \{b^{(k)}(W)\colon k = j_{2(i-1)}, \dots, j_{2i-1}\}$ is expanding and $W_{2i} = \{b^k(W)\colon k = j_{2i-1}, \dots, j_{2i}\}$ is contracting. The existence of such numbers is a consequence of Lemma 11. (One can start with the greatest independent node set W and proceed to the left till it is expanding and to the right till it is contracting. Afterwards one can remove the just "elaborated" part of G and in the remainder find again the greatest independent node set, etc.)

4th phase: For the subgraphs defined by the node sets

$$V_i = \bigcup_{k=j_{2(i-1)}}^{j_{2i}} b^{(k)}(W), \quad i = 1, \dots, r$$

find exact coverings (following Lemma 16).

5th phase: For the subgraphs defined by the node sets

$$V_{i'} = \bigcup_{k=j_{2i-1}}^{j_{2i+1}} b^{(k)}(W), \quad i = 1, \dots, r-1$$

find a minimum covering by concatenating the parts of chains constructed in the 4th phase, incident with $V_{'i}$ (The solution is reached by the maximum matching of the bipartite graph constructed from the ends of chains from the left and the starts of chains to the right).

If all subgraphs possess a covering C_i with $|C_i| > |W|$, proceed to the 6th phase.

6th phase: Kill the parts of chains incident with $V_{'i}$ and solve the covering C_i of $V_{i'}$ exactly (as described e.g. in [6]). We certainly obtain $|C_i| \leqslant |W|$ (as proved e.g. in [6]).

7th phase: Write down the resulting chains for the whole G by concatenating the previous results.

21. Remark. One can say (right by) that the general exact method, used (if necessary) in the 6th phase could be used immediately at the beginning and thus the method 20 may seem inutile. It may seem so but it is not!

The first merit of the method is in the fact that it solves not only the "global" problem of the minimum covering (i.e. of the minimum number of buses), but also the problem of local minima (i.e. the minimum number of buses in some time period which makes it possible to minimize the service times for crews).

Moreover, the method prefers connecting fathers with sons in chains, which makes the structure of G more lucid (and in practice the pieces of work more effective).

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