Gejza Wimmer

Linear-quadratic estimators in a special structure of the linear model

Applications of Mathematics, Vol. 40 (1995), No. 2, 81–105

Persistent URL: http://dml.cz/dmlcz/134282

40 (1995) APPLICATIONS OF MATHEMATICS No. 2, 81–105

LINEAR-QUADRATIC ESTIMATORS IN A SPECIAL STRUCTURE OF THE LINEAR MODEL

GEJZA WIMMER, Bratislava

(Received March 18, 1991)

Summary. The paper deals with the linear model with uncorrelated observations. The dispersions of the values observed are linear-quadratic functions of the unknown parameters of the mean (measurements by devices of a given class of precision). Investigated are the locally best linear-quadratic unbiased estimators as improvements of locally best linear unbiased estimators in the case that the design matrix has none, one or two linearly dependent rows.

Keywords: parametric estimation, linear model with variances depending on the mean value parameters, locally best linear-quadratic unbiased estimator (LBLQUE)

AMS classification: 62F10, 62F99

INTRODUCTION

Let us have the linear model $(\tilde{\mathbf{Y}}, \tilde{\mathbf{X}}\beta, \tilde{\Sigma})$, where the vector of observations $\tilde{\mathbf{Y}}_{n,1}$ has its mean value $\mathscr{E}(\tilde{\mathbf{Y}}) = \tilde{\mathbf{X}}\beta$ ($\tilde{\mathbf{X}}_{n,k}$ is a known design matrix and $\beta_{k,1} \in \mathbb{R}^k$ is the vector of unknown parameters). The covariance matrix of the vector $\tilde{\mathbf{Y}}$ is

$$\tilde{\Sigma} = \sigma^2 \tilde{\Sigma}(\beta) = \sigma^2 \begin{pmatrix} (a + b|\mathbf{e}_1'\tilde{\mathbf{X}}\beta|)^2 & 0 & \dots & 0 \\ 0 & (a + b|\mathbf{e}_2'\tilde{\mathbf{X}}\beta|)^2 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & \dots & & (a + b|\mathbf{e}_n'\tilde{\mathbf{X}}\beta|)^2 \end{pmatrix},$$

where a, b and σ^2 are known positive constants, $\mathbf{e}_i'$ is the transpose of the i-th unit vector. We meet this model in the case of the linear model with uncorrelated measurements which are performed by a measuring device whose dispersion characteristic is linear-quadratically dependent on the measured value (see [1], [5] etc.).

There exist some iterative algorithms for solving the problem of obtaining an estimate of a linear functional of the unknown parameter β in the above mentioned model (see e.g. [2], [3], [6] etc.) but the statistical properties of such estimators (except some asymptotical properties) are totally unknown.

In the paper [7] the author investigated the β_0-locally best linear unbiased estimator (β_0-LBLUE) and the uniformly best linear unbiased estimator (UBLUE) of a linear functional $\mathbf{f}'\beta$ of parameters β in model considered.

In the paper [8] the reader can find necessary and sufficient conditions for existence of the β_0-locally best linear-quadratic unbiased estimators (β_0-LBLQUE) of the functionals $\sigma^2(a + b|\mathbf{e}_i'\widetilde{\mathbf{X}}\beta|)^2$, $i = 1, 2, \ldots, n$, in the above mentioned model if $\widetilde{\mathbf{Y}}$ is normally distributed and $R(\widetilde{\mathbf{X}})$ (the rank of the matrix $\widetilde{\mathbf{X}}$) is n ($\leqslant k$) or $n-1$ ($\leqslant k$).

In the present paper the β_0-LBLQUE of the linear functional $\mathbf{f}'\beta$ of parameters β is investigated in the cases $R(\widetilde{\mathbf{X}}) = n \leqslant k$, $R(\widetilde{\mathbf{X}}) = n - 1 \leqslant k$ and $R(\widetilde{\mathbf{X}}) = n - 2 \leqslant k$ under the assumption that $\widetilde{\mathbf{Y}}$ is normally distributed.

1. Preliminaries

Let us rearrange the rows of the matrix $\widetilde{\mathbf{X}}$ to obtain the matrix

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{R(\mathbf{X}),R(\mathbf{X})} \\ \mathbf{E} \end{pmatrix} \mathbf{X}_1.$$

$\mathbf{X}_1$ is a matrix of order $R(\mathbf{X}) \times k$, $\mathbf{X}_2 = \mathbf{E}\mathbf{X}_1$, where $\mathbf{E} = \mathbf{X}_2\mathbf{X}_1'(\mathbf{X}_1, \mathbf{X}_1')^{-1}$ is of order $\big(n - R(\mathbf{X})\big) \times R(\mathbf{X})$.

In the same way we rearrange the coordinates of $\widetilde{\mathbf{Y}}$ and the rows of the matrix $\widetilde{\Sigma}(\beta)$. We obtain the vector $\mathbf{Y}$ and its covariance matrix

$$\Sigma = \sigma^2\Sigma(\beta) = \sigma^2 \begin{pmatrix} \Sigma_1(\beta) & \mathbf{O} \\ \mathbf{O} & \Sigma_2(\beta) \end{pmatrix},$$

where

$$\Sigma_1 = \begin{pmatrix} (a + b|\mathbf{e}_1'\mathbf{X}_1\beta|)^2 & 0 & \ldots & 0 \\ 0 & (a + b|\mathbf{e}_2'\mathbf{X}_1\beta|)^2 & \ldots & 0 \\ \vdots & & \ddots & \\ 0 & \ldots & & (a + b|\mathbf{e}_{R(\mathbf{X})}'\mathbf{X}_1\beta|)^2 \end{pmatrix}$$

and

$$\Sigma_2 = \begin{pmatrix} (a + b|\mathbf{e}_1'\mathbf{E}\mathbf{X}_1\beta|)^2 & 0 & \ldots & 0 \\ 0 & (a + b|\mathbf{e}_2'\mathbf{E}\mathbf{X}_1\beta|)^2 & \ldots & 0 \\ \vdots & & \ddots & \\ 0 & \ldots & & (a + b|\mathbf{e}_{n-R(\mathbf{X})}'\mathbf{E}\mathbf{X}_1\beta|)^2 \end{pmatrix}.$$

Further, we assume that $\mathbf{Y}$ is normally distributed. We have obtained the model

$$(\mathbf{Y}, \mathbf{X}\beta, \Sigma). \tag{1.1}$$

Let us denote by $\mathscr{D}$ the class of matrices $\mathbf{B}_{n,n}$ satisfying the following three conditions

$$\forall\{\beta \in \mathbb{R}^k\} \quad \operatorname{Tr} \mathbf{B} \begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\beta| & 0 & \dots & 0 \\ 0 & |\mathbf{e}_2'\mathbf{X}\beta| & \dots & 0 \\ \vdots & & \ddots & \\ 0 & \dots & & |\mathbf{e}_n'\mathbf{X}\beta| \end{pmatrix} = 0, \tag{1.2}$$

$$\operatorname{Tr} \mathbf{B} = 0, \tag{1.3}$$

$$\mathbf{X}'\Big(\mathbf{B} + \sigma^2 b^2 \sum_{i=1}^{n} \mathbf{e}_i\mathbf{e}_i'\mathbf{B}\mathbf{e}_i\mathbf{e}_i'\Big)\mathbf{X} = \mathbf{O} \tag{1.4}$$

($\operatorname{Tr} \mathbf{B}$ is the trace of $\mathbf{B}$ i.e. $\sum_{i=1}^{n} \mathbf{e}_i'\mathbf{B}\mathbf{e}_i$.)

Lemma 1.1. *The random variable* $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ *is in model* (1) *the* β_0-LBLQUE *of its mean value (in the class of linear-quadratic estimators) iff there exists a vector* $\mathbf{z} \in \mathbb{R}^n$ *such that*

$$\mathbf{a} = -(\mathbf{A} + \mathbf{A}')\mathbf{X}\beta_0 + (\mathbf{X}')^-_{m(\Sigma(\beta_0))}\mathbf{X}'\mathbf{z} \tag{1.5}$$

and

$$\forall\{\mathbf{D} \in \mathscr{D}\} \quad \operatorname{Tr}(\mathbf{D} + \mathbf{D}')\{\sigma^2\Sigma(\beta_0)(\mathbf{A} + \mathbf{A}')\Sigma(\beta_0) + 2\mathbf{X}\beta_0\mathbf{z}'\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\Sigma(\beta_0)\} = 0, \tag{1.6}$$

where $((\mathbf{X}')^-_{m(\Sigma(\beta_0))}$ *is an arbitrary but fixed minimum* $\Sigma(\beta_0)$*-norm g-inverse of the matrix* $\mathbf{X}'$*, i.e. a matrix satisfying the relations* $\mathbf{X}'(\mathbf{X}')^-_{m(\Sigma(\beta_0))}\mathbf{X}' = \mathbf{X}'$ *and* $((\mathbf{X}')^-_{m(\Sigma(\beta_0))}\mathbf{X}')'\Sigma(\beta_0) = \Sigma(\beta_0)(\mathbf{X}')^-_{m(\Sigma(\beta_0))}\mathbf{X}')$.

Proof. See in [8], Theorem 1.3. □

Lemma 1.2. *The random variable* $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ *ia an unbiased estimator of the linear functional* $\mathbf{f}'\beta$ *of a parameter* β *iff*

$$\mathbf{f} = \mathbf{X}'\mathbf{a} \tag{1.7}$$

and

$$\mathbf{A} \in \mathscr{D}. \tag{1.8}$$

P r o o f. The random variable $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ is an unbiased estimator of $\mathbf{f}'\beta$ iff

$$\forall\{\beta \in \mathbb{R}^k\} \quad \mathscr{E}_\beta(\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}) = \mathbf{f}'\beta,$$

i.e. iff

$$\begin{aligned}
\forall\{\beta \in \mathbb{R}^k\} \quad & \mathbf{a}'\mathbf{X}\beta + \beta'\mathbf{X}'\mathbf{A}\mathbf{X}\beta + \sigma^2 \operatorname{Tr} \mathbf{A}\Sigma(\beta) \\
&= \mathbf{a}'\mathbf{X}\beta + \beta'\mathbf{X}'\mathbf{A}\mathbf{X}\beta + \sigma^2 a^2 \operatorname{Tr} \mathbf{A} \\
&+ 2ab\sigma^2 \operatorname{Tr} \mathbf{A} \begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\beta| & 0 & \dots & 0 \\ 0 & |\mathbf{e}_2'\mathbf{X}\beta| & \dots & 0 \\ \vdots & & \ddots & \\ 0 & \dots & & |\mathbf{e}_n'\mathbf{X}\beta| \end{pmatrix} \\
&+ \sigma^2 b^2 \sum_{i=1}^n \mathbf{e}_i'\mathbf{A}\mathbf{e}_i(\mathbf{e}_i'\mathbf{X}\beta)^2 = \mathbf{f}'\beta.
\end{aligned}$$

This is equivalent to the following three conditions:

$$\sum_{i=1}^n \mathbf{e}_i'\mathbf{A}\mathbf{e}_i = 0, \tag{1.9}$$

$$\forall\{\beta \in \mathbb{R}^k\} \quad (\mathbf{X}'\mathbf{a} - \mathbf{f})' + 2ab\sigma^2 \sum_{i=1}^n \mathbf{e}_i'\mathbf{A}\mathbf{e}_i|\mathbf{e}_i'\mathbf{X}\beta| = 0 \tag{1.10}$$

and

$$\forall\{\beta \in \mathbb{R}^k\} \quad \beta'\mathbf{X}'\Big(\mathbf{A} + \sigma^2 b^2 \sum_{i=1}^n \mathbf{e}_i\mathbf{e}_i'\mathbf{A}\mathbf{e}_i\mathbf{e}_i'\Big)\mathbf{X}\beta = 0. \tag{1.11}$$

Let us analyze the condition (1.10). We see that

$$\forall\{\beta \in \mathbb{R}^k\} \quad (\mathbf{X}'\mathbf{a} - \mathbf{f})'\beta + 2ab\sigma^2 \sum_{i=1}^n \mathbf{e}_i'\mathbf{A}\mathbf{e}_i|\mathbf{e}_i'\mathbf{X}\beta| = 0,$$

but also

$$(\mathbf{X}'a - \mathbf{f})'(-\beta) + 2ab\sigma^2 \sum_{i=1}^n \mathbf{e}_i'\mathbf{A}\mathbf{e}_i|\mathbf{e}_i'\mathbf{X}\beta| = 0.$$

This implies

$$\mathbf{X}'\mathbf{a} - \mathbf{f} = 0 \tag{1.12}$$

and also

$$\text{(1.13)} \qquad \forall\{\beta \in \mathbb{R}^k\} \quad \sum_{i=1}^{n} \mathbf{e}_i'\mathbf{A}\mathbf{e}_i|\mathbf{e}_i'\mathbf{X}\beta| = 0.$$

Condition (1.11) is equivalent to

$$\text{(1.14)} \qquad \mathbf{X}'\Big(\mathbf{A} + \mathbf{A}' + 2\sigma^2 b^2 \sum_{i=1}^{n} \mathbf{e}_i\mathbf{e}_i'\mathbf{A}\mathbf{e}_i\mathbf{e}_i'\Big)\mathbf{X} = 0.$$

It is obvious that conditions (1.9), (1.12), (1.13) and (1.14) are necessary and sufficient for $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ to be an unbiased estimator of $\mathbf{f}'\beta$.

Because of the equality

$$\mathbf{Y}'\mathbf{B}\mathbf{Y} = \mathbf{Y}\frac{\mathbf{B}+\mathbf{B}'}{2}\mathbf{Y},$$

which is true for every $n \times n$ matrix $\mathbf{B}$, we obtain that

$$\begin{aligned}
&\Bigg\{\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y} \colon \operatorname{Tr}\mathbf{A}\begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\beta| & 0 & \dots & 0 \\ 0 & |\mathbf{e}_2'\mathbf{X}\beta| & \dots & 0 \\ \vdots & & \ddots & \\ 0 & \dots & & |\mathbf{e}_n'\mathbf{X}\beta| \end{pmatrix} = 0 \quad \forall\{\beta \in \mathbb{R}^k\}, \\
&\qquad \operatorname{Tr}\mathbf{A} = 0, \quad \mathbf{X}'\Big(\mathbf{A} + \mathbf{A}' + 2\sigma^2 b^2 \sum_{i=1}^{n} \mathbf{e}_i\mathbf{e}_i'\mathbf{A}\mathbf{e}_i\mathbf{e}_i'\Big)\mathbf{X} = \mathbf{O}, \ \mathbf{X}'\mathbf{a} = \mathbf{f}\Bigg\} \\
&= \Bigg\{\mathbf{b}'\mathbf{Y} + \mathbf{Y}'\mathbf{D}\mathbf{Y} \colon \operatorname{Tr}\mathbf{D}\begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\beta| & 0 & \dots & 0 \\ 0 & |\mathbf{e}_2'\mathbf{X}\beta| & \dots & 0 \\ \vdots & & \ddots & \\ 0 & \dots & & |\mathbf{e}_n'\mathbf{X}\beta| \end{pmatrix} = 0 \quad \forall\{\beta \in \mathbb{R}^k\}, \\
&\qquad \operatorname{Tr}\mathbf{D} = 0, \quad \mathbf{X}\Big(\mathbf{D} + \sigma^2 b^2 \sum_{i=1}^{n} \mathbf{e}_i\mathbf{e}_i'\mathbf{D}\mathbf{e}_i\mathbf{e}_i'\Big)\mathbf{X} = \mathbf{O}, \ \mathbf{X}'\mathbf{b} = \mathbf{f}\Bigg\} \\
&= \{\mathbf{b}'\mathbf{Y} + \mathbf{Y}'\mathbf{D}\mathbf{Y} \colon \mathbf{X}'\mathbf{b} = \mathbf{f}, \ \mathbf{D} \in \mathscr{D}\}.
\end{aligned}$$

(We note that here $\mathbf{D}$ need not be a symmetric matrix.) The lemma is proved. □

An easy consequence of Lemma 1.1 and Lemma 1.2 is the next theorem.

Theorem 1.3. *The random variable* $\mathbf{a}'\mathbf{Y}+\mathbf{Y}'\mathbf{A}\mathbf{Y}$ *is in model* (1) *the* β_0-LBLQUE *of the linear functional* $\mathbf{f}'\beta$ *of a parameter* β *if* (1.5), (1.6), (1.7) *and* (1.8) *are satisfied.*

P r o o f is easy and therefore omitted. □

For our further analysis we still need some other lemmas

Lemma 1.4. *If* $\mathbf{V}_{n,n}$ *is a positive definite matrix and* $\mathbf{Z}_{n,k}$ *an arbitrary one then*
(i) $\mu(\mathbf{VZ}) \cap \operatorname{Ker} \mathbf{Z}' = \{0\}$ *and*
(ii) $R(\mathbf{VZ}) + R(\operatorname{Ker} \mathbf{Z}') = n$, *where* $\mu(\mathbf{C}_{s,t}) = \{\mathbf{Cu} \colon \mathbf{u} \in \mathbb{R}^t\}$ *and* $\operatorname{Ker} \mathbf{C} = \{\mathbf{w} \colon \mathbf{w} \in \mathbb{R}^t$ *and* $\mathbf{Cw} = 0\}$.

Proof. (i) $\mathbf{a} \in \mu(\mathbf{VZ}) \cap \operatorname{Ker} \mathbf{Z}' \Rightarrow \exists\{\xi \in \mathbb{R}^k\}\ \alpha = \mathbf{VZ}\xi$ and $\mathbf{Z}'\alpha = 0 \Rightarrow \exists\{\xi \in \mathbb{R}^k\}\ \alpha = \mathbf{VZ}\xi$ and $\mathbf{Z}'\mathbf{VZ}\xi = 0 \Rightarrow \{\xi \in \mathbb{R}^k\}\ \alpha = \mathbf{VZ}\xi$ and $\mathbf{Z}'\xi = 0 \Rightarrow \alpha = 0$.
The reverse implication is trivial.
(ii) We have

$$R(\mathbf{Z}) \geqslant R(\mathbf{VZ}) \geqslant R(\mathbf{V}^{-1}\mathbf{VZ}) = R(\mathbf{Z})$$

and that is why $R(\mathbf{VZ}) = R(\mathbf{Z})$. Now we easily obtain (ii). □

Corollary 1.5. *Let* $\mathbf{V}_{n,n}$ *be a positive definite matrix and* $\mathbf{Z}_{n,k}$ *an arbitrary one. Then*

$$\forall\{\mathbf{z} \in \mathbb{R}^n\}\exists\{\mathbf{x} \in \mathbb{R}^k\} \quad \textit{and}\ \exists\{\mathbf{y} \in \operatorname{Ker} \mathbf{Z}'\},$$

so that

$$\mathbf{VZx} + \mathbf{y} = \mathbf{z}.$$

Proof. This is an easy consequence of Lemma 1.4. □

Lemma 1.6. *For the linear functional* $\mathbf{f}'\beta$ *of a parameter* β *there exists the* β_0*-locally best linear unbiased estimator* (β_0-LBLUE) *iff* $\mathbf{f} \in \mu(\mathbf{X}')$.

Proof. See Lemma 2.4 in [7]. □

Lemma 1.7. *The* β_0-LBLUE *of* $\mathbf{f}'\beta$ *(for* $\mathbf{f} \in \mu(\mathbf{X}')$*) is* $\mathbf{f}'[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\mathbf{Y}$, *where* $(\mathbf{X}')^-_{m(\Sigma(\beta_0))}$ *is an arbitrary but fixed minimum* $\Sigma(\beta_0)$*-norm g-inverse of the matrix* $\mathbf{X}'$. *The dispersion of the* β_0-LBLUE *of* $\mathbf{f}'\beta$ *for* $\beta = \beta_0$ *is* $\sigma^2\mathbf{f}'[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\Sigma(\beta_0) \times (\mathbf{X}')^-_{m(\Sigma(\beta_0))}\mathbf{f}$ *and is invariant of the choice of* $(\mathbf{X}')^-_{m(\Sigma(\beta_0))}$.

Proof. See [7], Lemma 2.4 and its proof. □

Lemma 1.8. *If* $\mathbf{X}'\mathbf{DX} = \mathbf{O}$ *for all* $\mathbf{D} \in \mathscr{D}$ *then*

$$\{\mathbf{f} \colon \exists\beta_0\text{-}\mathit{LBQLUE}\ \text{for}\ \mathbf{f}'\beta\} = \mu(\mathbf{X}'). \tag{1.15}$$

Further, the dispersion of the β_0-LBLQUE *of* $\mathbf{f}'\beta$ *is the same as the dispersion of the* β_0-LBLUE *of* $\mathbf{f}'\beta$ *at* $\beta = \beta_0$.

Proof. According to Theorem 1.3 $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ is the β_0-LBLQUE of $\mathbf{f}'\beta$ iff (1.5), (1.6), (1.7) and (1.8) hold. So we see that if $\mathbf{X}'\mathbf{D}\mathbf{X} = \mathbf{O}$ for all $\mathbf{D} \in \mathscr{D}$ then for every $\mathbf{z} \in \mathbb{R}^n$

$$\mathbf{z}'\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\mathbf{Y} + \mathbf{Y}'\mathbf{O}\mathbf{Y} \tag{1.16}$$

is the β_0-LBLQUE of $\mathbf{f}'\beta = \mathbf{z}'\mathbf{X}\beta$. That is why (1.15) is true. Due to Lemma 1.7 (1.16) is also the β_0-LBLQUE of $\mathbf{f}'\beta$ and the second assertion of the lemma is true. The lemma is proved. □

2. $R(\mathbf{X}) = n \leqslant k$

Theorem 2.1. *If in model* (1.1) $R(\mathbf{X}) = n \leqslant k$, *then* (1.15) *is true and the dispersion of the* β_0*-LBLQUE of* $\mathbf{f}'\beta$ *is the same as the dispersion of the* β_0*-LBLUE of* $\mathbf{f}'\beta$ *for* $\beta = \beta_0$.

Proof. According to Lemma 2.1 in [8] $\mathbf{D} \in \mathscr{D}$ in model (1.1) for $R(\mathbf{X}) = n \leqslant k$ iff $\mathbf{e}_i'\mathbf{D}\mathbf{e}_i = \mathbf{O}$, $i = 1, 2, \ldots, n$ and $\mathbf{X}'\mathscr{D}\mathbf{X} = \mathbf{O}$. Now the proof is an easy consequence of Lemma 1.8. □

3. $R(\mathbf{X}) = n - 1 \leqslant k$

3.1. Case $\mathbf{E} = \gamma\mathbf{e}_s'$.

Theorem 3.1.1. *If in model* (1.1) $\mathbf{E} = \gamma\mathbf{e}_s'$, $\gamma \neq 0$, $s \in \{1, 2, \ldots, n-1\}$ *then* (1.15) *is true and the dispersion of the* β_0*-LBLQUE of* $\mathbf{f}'\beta$ *is the same as the dispersion of the* β_0*-LBLUE of* $\mathbf{f}'\beta$ *at* $\beta = \beta_0$.

Proof. (i) If $|\gamma| \neq 1$ then, according to Lemma 3.1 in [8], $\mathbf{D} \in \mathscr{D}$ iff $\mathbf{e}_i'\mathbf{D}\mathbf{e}_i = \mathbf{O}$, $i = 1, 2, \ldots, n$ and $\mathbf{X}'\mathbf{D}\mathbf{X} = \mathbf{O}$.

(ii) If $|\gamma| = 1$ then, according to Lemma 4.1 in [8], $\mathbf{D} \in \mathscr{D}$ iff $\mathbf{e}_i'\mathbf{D}\mathbf{e}_i = \mathbf{O}$, $i \notin \{s, n\}$, $\mathbf{e}_s'\mathbf{D}\mathbf{e}_s + \mathbf{e}_n'\mathbf{D}\mathbf{e}_n = \mathbf{O}$ and $\mathbf{X}'\mathbf{D}\mathbf{X} = \mathbf{O}$.

In both cases the proof is an easy consequence of Lemma 1.8. □

3.2. Case $\mathbf{E} = \sum_{i=1}^{t} \gamma_i\mathbf{e}_{s_i}'$, $t \geqslant 2$.

Theorem 3.2.1. *If in model* (1.1) $\mathbf{E} = \sum_{i=1}^{t} \gamma_i\mathbf{e}_{s_i}'$, $\gamma_i \neq 0$, $s_i \in \{1, 2, \ldots, n-1\}$ *for* $i = 1, 2, \ldots, t$, $t \geqslant 2$ *then* (1.15) *is true and the dispersion of the* β_0*-LBLQUE of* $\mathbf{f}'\beta$ *is the same as the dispersion of the* β_0*-LBLUE of* $\mathbf{f}'\beta$ *at* $\beta = \beta_0$.

Proof. According to Lemma 5.1 in [8] in this case $\mathbf{D} \in \mathscr{D}$ iff $\mathbf{e}_i'\mathbf{D}\mathbf{e}_i = \mathbf{O}$, $i = 1$, 2, …, n and $\mathbf{X}'\mathbf{D}\mathbf{X} = \mathbf{O}$.

The proof is again an easy consequence of Lemma 1.8. □

4. $R(\mathbf{X}) = n - 2 \leqslant k$

4.1. Case $\mathbf{E} = \binom{\gamma \mathbf{e}_s'}{\delta \mathbf{e}_s'}$. Relying on the previous methods we can prove the following lemma:

Lemma 4.1.1. *If in model* (1.1) $\mathbf{E} = \binom{\gamma \mathbf{e}_s'}{\delta \mathbf{e}_s'}$, $s \in \{1, 2, \ldots, n-2\}$ *where*
(i) $|\gamma| = 1$, $|\delta| \neq 1$, $\delta \neq 0$,
(ii) $|\gamma| \neq 1$, $\gamma \neq 0$, $|\delta| = 1$,
(iii) $|\gamma| = |\delta| \neq 1$, $\gamma \neq 0$, $\delta \neq 0$ *or*
(iv) $|\gamma| \neq 1$, $|\delta| \neq 1$, $\gamma \neq \delta$, $\gamma \neq 0$, $\delta \neq 0$ *then* $\mathbf{D} \in \mathscr{D}$ *iff*

$$\mathbf{e}_i'\mathbf{D}\mathbf{e}_i = \mathbf{O}, \quad i \notin \{s, n-1\},$$
$$\mathbf{e}_s'\mathbf{D}\mathbf{e}_s + \mathbf{e}_{n-1}'\mathbf{D}\mathbf{e}_{n-1} = \mathbf{O}$$

and

$$\mathbf{X}'\mathbf{D}\mathbf{X} = \mathbf{O}$$

in the case (i),

$$\mathbf{e}_i'\mathbf{D}\mathbf{e}_i = \mathbf{O}, \quad i \notin \{s, n\},$$
$$\mathbf{e}_s'\mathbf{D}\mathbf{e}_s + \mathbf{e}_n'\mathbf{D}\mathbf{e}_n = \mathbf{O}$$

and

$$\mathbf{X}'\mathbf{D}\mathbf{X} = \mathbf{O}$$

in the case (ii),

$$\mathbf{e}_i'\mathbf{D}\mathbf{e}_i = \mathbf{O}, \quad i \notin \{n-1, n\},$$
$$\mathbf{e}_{n-1}'\mathbf{D}\mathbf{e}_{n-1} + \mathbf{e}_n'\mathbf{D}\mathbf{e}_n = \mathbf{O}$$

and

$$\mathbf{X}'\mathbf{D}\mathbf{X} = \mathbf{O}$$

in the case (iii),

(4.1.1) $$\mathbf{e}_i'\mathbf{D}\mathbf{e}_i = \mathbf{O}, \quad i \notin \{s, n-1, n\},$$

(4.1.2) $$\mathbf{e}_s'\mathbf{D}\mathbf{e}_s + \frac{|\delta| - |\gamma|}{1 - |\gamma|}\mathbf{e}_n'\mathbf{D}\mathbf{e}_n = \mathbf{O},$$

(4.1.3) $$\mathbf{e}_{n-1}'\mathbf{D}\mathbf{e}_{n-1} + \frac{1 - |\delta|}{1 - |\gamma|}\mathbf{e}_n'\mathbf{D}\mathbf{e}_n = \mathbf{O}$$

and

$$\mathbf{X}'\mathbf{D}\mathbf{X} + \sigma^2 b^2(|\gamma| - |\delta|)(1 - |\delta|)\mathbf{X}'\mathbf{e}_s\mathbf{e}_s'\mathbf{X}\mathbf{e}_n'\mathbf{D}\mathbf{e}_n = \mathbf{O} \tag{4.1.4}$$

in the case (iv).

Thus, we have also obtained the proof of the next theorem:

Theorem 4.1.2. *If in model* (1.1) $\mathbf{E} = \binom{\gamma \mathbf{e}_s'}{\delta \mathbf{e}_s'}$, $s \in \{1, 2, \dots, n-1\}$ *where*
(i) $|\gamma| = 1$, $|\delta| \neq 1$, $\delta \neq 0$,
(ii) $|\gamma| \neq 1$, $\gamma \neq 1$, $|\delta| = 1$, *or*
(iii) $|\gamma| = |\delta| \neq 1$, $\gamma \neq 0$, $\delta \neq 0$ *then* (1.15) *is true and the dispersion of the* β_0-LBLQUE *of* $\mathbf{f}'\beta$ *is the same as the dispersion of the* β_0-LBLUE *of* $\mathbf{f}'\beta$ *for* $\beta = \beta_0$.

The case with $|\gamma| \neq 1$, $|\delta| \neq 1$, $\gamma \neq \delta$, $\gamma \neq 0$, $\delta \neq 0$ is rather different and needs a deeper analysis.

If we denote by $\otimes$ the symbol for the Kronecker product (see e.g. [4], p. 11), and if

$$\operatorname{vec} \mathbf{A}_{n,m} = (a_{11}, a_{21}, \dots, a_{n1}, a_{12}, a_{22}, \dots, a_{n2}, \dots, a_{1m}, a_{2m}, \dots, a_{nm})'$$

denotes the vector formed by columns of the matrix $\mathbf{A}$, then using the formulas

$$\operatorname{Tr} \mathbf{AB} = (\operatorname{vec} \mathbf{B}')' \operatorname{vec} \mathbf{A}$$

and

$$\operatorname{vec} \mathbf{ABC} = (\mathbf{C}' \otimes \mathbf{A}) \operatorname{vec} \mathbf{B}$$

we obtain that (4.1.1)–(4.1.4) are equivalent to

$$\mathbf{Y}_* \operatorname{vec} \mathbf{D} = \mathbf{O}, \tag{4.1.5}$$

where $\mathbf{Y}_*$ is a $(k^2 + n - 1) \times n^2$ matrix of the form

$$\mathbf{Y}_* = \begin{pmatrix} \mathbf{e}_1' \otimes \mathbf{e}_1' \\ \vdots \\ \mathbf{e}_{s-1}' \otimes \mathbf{e}_{s-1}' \\ \mathbf{e}_s' \otimes \mathbf{e}_s' + \frac{|\delta| - |\gamma|}{1 - |\gamma|}(\mathbf{e}_n' \otimes \mathbf{e}_n') \\ \mathbf{e}_{s+1}' \otimes \mathbf{e}_{s+1}' \\ \vdots \\ \mathbf{e}_{n-1}' \otimes \mathbf{e}_{n-2}' \\ (\mathbf{e}_{n-1}' \otimes \mathbf{e}_{n-1}') + \frac{1 - |\delta|}{1 - |\gamma|}(\mathbf{e}_n' \otimes \mathbf{e}_n') \\ (\mathbf{X}' \otimes \mathbf{X}')(\mathbf{I} + \sigma^2 b^2(|\gamma| - |\delta|)(1 - |\delta|)(\mathbf{e}_s\mathbf{e}_n' \otimes \mathbf{e}_s\mathbf{e}_n')) \end{pmatrix}.$$

A consequence of Lemma 4.1.1, case (iv) is

$$\mathbf{D} \in \mathscr{D} \Leftrightarrow \mathbf{Y}_* \operatorname{vec} \mathbf{D} = \mathbf{O} \Leftrightarrow \operatorname{vec} \mathbf{D} \in \operatorname{Ker} \mathbf{Y}_*. \tag{4.1.6}$$

Now let us analyze condition (1.6). We have

$$\begin{aligned} \forall\{\mathbf{D} \in \mathscr{D}\} \quad & \operatorname{Tr}(\mathbf{D} + \mathbf{D}')\{\sigma^2\Sigma(\beta_0)(\mathbf{A} + \mathbf{A}')\Sigma(\beta_0) \\ & + 2\mathbf{X}\beta_0\mathbf{z}'\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\Sigma(\beta_0)\} = 0 \\ & \Leftrightarrow \exists\{\delta \in \mathbb{R}^{k^2+n-1}\} \\ & \big(\Sigma(\beta_0) \otimes \Sigma(\beta_0)\big) \operatorname{vec}(\mathbf{A} + \mathbf{A}') - 2b^2(|\gamma| - |\delta|)(1 - |\delta|) \\ & \quad \times \mathbf{e}_s'\mathbf{X}\beta_0\mathbf{e}_s'\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\Sigma(\beta_0)\mathbf{z}(\mathbf{e}_n \otimes \mathbf{e}_n) = \mathbf{Y}_s'\delta. \end{aligned} \tag{4.1.7}$$

According to Corollary 1.5 and (4.1.6) for every $\mathbf{z} \in \mathbb{R}^n$ there exist $\psi \in \mathbb{R}^{k^2+n-1}$ and $\mathbf{C}_\mathbf{z} \in \mathscr{D}$ such that

$$\begin{aligned} & \big(\Sigma(\beta_0) \otimes \Sigma(\beta_0)\big) \operatorname{vec} \mathbf{C}_\mathbf{z} - 2b^2(|\gamma| - |\delta|)(1 - |\delta|) \\ & \quad \times \mathbf{e}_s'\mathbf{X}\beta_0\mathbf{e}_s'\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\Sigma(\beta_0)\mathbf{z}(\mathbf{e}_n \otimes \mathbf{e}_n) = \mathbf{Y}_*'\psi. \end{aligned} \tag{4.1.8}$$

If we denote by $\mathbf{I}^*$ the $n^2 \times n^2$ matrix for which the assertion

$$\forall\{\mathbf{A}_{n,n}\} \quad \operatorname{vec} \mathbf{A}' = \mathbf{I}^* \operatorname{vec} \mathbf{A}$$

is valid then (4.1.1)–(4.1.4) are equivalent to

$$\mathbf{Y}_*\mathbf{I}^* \operatorname{vec} \mathbf{D} = \mathbf{O}. \tag{4.1.9}$$

We have

$$\mathbf{D} \in \mathscr{D} \Leftrightarrow \mathbf{Y}_*\mathbf{I}^* \operatorname{vec} \mathbf{D} = \mathbf{O} \Leftrightarrow \operatorname{vec} \mathscr{D} \in \operatorname{Ker} \mathbf{Y}_*\mathbf{I}^*. \tag{4.1.10}$$

According to (4.1.6) and (4.1.10)

$$\operatorname{Ker} \mathbf{Y}_* = \operatorname{Ker} \mathbf{Y}_*\mathbf{I}^*$$

and that is why

$$\mu(\mathbf{Y}_*') = \mu\big((\mathbf{I}^*)'\mathbf{Y}_*'\big) = \mu(\mathbf{I}^*\mathbf{Y}_*'). \tag{4.1.11}$$

From (4.1.8) we have that for every $\mathbf{z} \in \mathbb{R}^n$ there exist $\psi \in \mathbb{R}^{k^2+n-1}$, $\mathbf{C_z} \in \mathscr{D}$ and $\varphi \in \mathbb{R}^{k^2+n-1}$ such that

$$\text{(4.1.12)} \quad \begin{aligned} &\mathbf{I}^*(\Sigma(\beta_0) \otimes \Sigma(\beta_0)) \operatorname{vec} \mathbf{C_z} - \mathbf{I}^* 2b^2(|\gamma| - |\delta|)(1 - |\delta|) \\ &\times \mathbf{e}_s'\mathbf{X}\beta_0\mathbf{e}_s'\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\Sigma(\beta_0)\mathbf{z}(\mathbf{e}_n \otimes \mathbf{e}_n) = \mathbf{I}^*\mathbf{Y}_*'\psi = \mathbf{Y}_*'\varphi. \end{aligned}$$

That is why (by virtue of (4.1.8) and (4.1.12)) for every $\mathbf{z} \in \mathbb{R}^n$ there exist $\mathbf{A_z} = \frac{\mathbf{C_z}}{2} \in \mathscr{D}$, $\frac{1}{2}\psi \in \mathbb{R}^{k^2+n-1}$ and $\frac{1}{2}\varphi \in \mathbb{R}^{k^2+n-1}$ (i.e. $\delta = \frac{1}{2}(\psi + \varphi)$) such that

$$\text{(4.1.13)} \quad \begin{aligned} &(\Sigma(\beta_0) \otimes \Sigma(\beta_0)) \operatorname{vec}(\mathbf{A_z} + \mathbf{A_z}') - 2b^2(|\gamma| - |\delta|)(1 - |\delta|) \\ &\times \mathbf{e}_s'\mathbf{X}\beta_0\mathbf{e}_s'\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\Sigma(\beta_0)\mathbf{z}(\mathbf{e}_n \otimes \mathbf{e}_n) \\ &= \mathbf{Y}_*'\frac{1}{2}(\psi + \varphi) = \mathbf{Y}_*'\delta. \end{aligned}$$

We have obtained that $\forall\{\mathbf{z} \in \mathbb{R}^n\}$ $\exists\{\mathbf{A_z} \in \mathscr{D}\}$ such that

$$\text{(4.1.14)} \quad \begin{aligned} \forall\{\mathbf{D} \in \mathscr{D}\} \quad &\operatorname{Tr}(\mathbf{D} + \mathbf{D}')\{\sigma^2\Sigma(\beta_0)(\mathbf{A}_z + \mathbf{A}_z')\Sigma(\beta_0) \\ &+ 2\mathbf{X}\beta_0\mathbf{z}'\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\Sigma(\beta_0)\} = 0. \end{aligned}$$

Let us return to find the β_0-LBLQUE of $\mathbf{f}'\beta$ in the investigated case $|\gamma| \neq 1$, $|\delta| \neq 1$, $\gamma \neq \delta$, $\gamma \neq 0$ and $\delta \neq 0$. According to Theorem 1.3 $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ is the β_0-LBLQUE of $\mathbf{f}'\beta$ iff (1.5), (1.6), (1.7) and (1.8) hold. We have proved that (1.6) is equivalent to (4.1.7) and that (4.1.14) is true. So

$$\text{(4.1.15)} \quad \begin{aligned} \mathscr{F} &= \{\mathbf{f}\colon \exists\beta_0\text{-LBLQUE for } \mathbf{f}'\beta\} \\ &= \{\mathbf{X}'\mathbf{a}\colon \mathbf{a} = -(\mathbf{A} + \mathbf{A}')\mathbf{X}\beta_0 + (\mathbf{X}')^-_{m(\Sigma(\beta_0))}\mathbf{X}'\mathbf{z}, \\ &\quad \mathbf{A} \in \mathscr{D} \text{ satisfies (4.1.14)}, \ \mathbf{z} \in \mathbb{R}^n\}. \end{aligned}$$

Theorem 4.1.3. *If in model* (1.1) $\mathbf{E} = \binom{\gamma\mathbf{e}_s'}{\delta\mathbf{e}_s'}$ *where* $s \in \{1, 2, \dots, n-2\}$, $|\gamma| \neq 1$, $|\delta| \neq 1$, $\gamma \neq 0$, $\gamma \neq \delta$ *and* $\delta \neq \gamma$ *then* $\mathscr{F} = \mu(\mathbf{X}')$, *i.e.* (1.15) *is true. Further, the dispersion of the* β_0-LBLQUE *of* $\mathbf{f}'\beta$ *is not greater than the dispersion of the* β_0-LBLUE *of* $\mathbf{f}'\beta$ *at* $\beta = \beta_0$.

Proof. As (4.1.13) is true, from (4.1.4) we obtain

$$\text{(4.1.16)} \quad \begin{aligned} \mathbf{X}'\mathbf{a} = &-\mathbf{X}'(\mathbf{A} + \mathbf{A}')\mathbf{X}\beta_0 + \mathbf{X}'\mathbf{z} = \sum_{\substack{i=1 \\ i\neq s}}^{n-2} z_i\mathbf{X}'\mathbf{e}_i \\ &+ [\sigma^2 b^2(|\gamma| - |\delta|)(1 - |\delta|)\mathbf{e}_s'\mathbf{X}\beta_0\mathbf{e}_n'(\mathbf{A} + \mathbf{A}')\mathbf{e}_n \\ &+ z_s + \gamma z_{n-1} + \delta z_n]\mathbf{X}\mathbf{e}_s. \end{aligned}$$

If we denote by $\mathbf{A}_{\mathbf{e}_i}$ the matrix corresponding to $\mathbf{z} = \mathbf{e}_i$, $i = 1, 2, \ldots, n$, in (4.1.14) then

$$A_{\mathbf{e}_{n-1}} + \mathbf{A}'_{\mathbf{e}_{n-1}} = \gamma(\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}'_{\mathbf{e}_s}), \quad \mathbf{A}_{\mathbf{e}_n} + \mathbf{A}'_{\mathbf{e}_n} = \delta(\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}'_{\mathbf{e}_s})$$

and

$$\mathbf{A} + \mathbf{A}' = \sum_{i=1}^{n} z_i(\mathbf{A}_{\mathbf{e}_i} + \mathbf{A}'_{\mathbf{e}_i}).$$

From (4.1.16) we have

$$\begin{aligned}(4.1.17)\quad \mathbf{X}'\mathbf{a} = &\sum_{\substack{i=1\\ i\neq s}}^{n-2} z_i \mathbf{X}'\mathbf{e}_i \\ &+ \sigma^2 b^2(|\gamma| - |\delta|)(1 - |\delta|)\mathbf{e}'_s\mathbf{X}\beta_0 \sum_{\substack{i=1\\ i\neq s}}^{n-2} z_i \mathbf{e}'_n(\mathbf{A}_{\mathbf{e}_i} + \mathbf{A}'_{\mathbf{e}_i})\mathbf{e}_n \mathbf{X}'\mathbf{e}_s \\ &+ \left(1 + \sigma^2 b^2(|\gamma| - |\delta|)(1 - |\delta|)\mathbf{e}'_s\mathbf{X}\beta_0\mathbf{e}'_n(\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}'_{\mathbf{e}_s})\mathbf{e}_n\right) \\ &\times (z_s + \gamma z_{n-1} + \delta z_n)\mathbf{X}\mathbf{e}_s.\end{aligned}$$

As

$$\mu(\mathbf{X}') = \left\{\sum_{i=1}^{n-2} \xi_i \mathbf{X}'\mathbf{e}_i \colon \xi_i \in \mathbb{R},\ i = 1, 2, \ldots, n-2\right\},$$

$\mathscr{F} = \mu(\mathbf{X}')$ iff

$$(4.1.18)\qquad 1 + \sigma^2 b^2(|\gamma| - |\delta|)(1 - |\delta|)\mathbf{e}'_s\mathbf{X}\beta_0\mathbf{e}'_n(\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}'_{\mathbf{e}_s})\mathbf{e}_n \neq 0.$$

If

$$(4.1.19)\qquad 1 + \sigma^2 b^2(|\gamma| - |\delta|)(1 - |\delta|)\mathbf{e}'_s\mathbf{X}\beta_0\mathbf{e}'_n(\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}'_{\mathbf{e}_s})\mathbf{e}_n = 0$$

then

$$\begin{aligned}&2b^2(|\gamma| - |\delta|)(1 - |\delta|)\mathbf{e}'_s\mathbf{X}\beta_0\mathbf{e}'_s\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]' \\ &\quad \times \Sigma(\beta_0)\mathbf{e}_s\mathbf{e}'_n(\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}'_{\mathbf{e}_s})\mathbf{e}_n = -\frac{2}{\sigma^2}\mathbf{e}'_s\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\Sigma(\beta_0)\mathbf{e}_s\end{aligned}$$

and

$$\begin{aligned}0 &\leqslant [\mathrm{vec}(\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}'_{\mathbf{e}_s})]'\left(\Sigma(\beta_0) \otimes \Sigma(\beta_0)\right)\mathrm{vec}(\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}'_{\mathbf{e}_s}) \\ &= -\frac{2}{\sigma^2}\mathbf{e}'_s\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\Sigma(\beta_0)\mathbf{e}_s \leqslant 0\end{aligned}$$

(because of the semidefiniteness of the matrix $\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\Sigma(\beta_0))$. So $\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}'_{\mathbf{e}_s} = \mathbf{O}$ and this contradicts (4.1.19). That is why (4.1.18) is true and $\mathscr{F} = \mu(\mathbf{X}')$.

Let us now calculate the dispersion of the β_0-LBLQUE of the functional $\mathbf{f}'\beta$ (for $\mathbf{f} \in \mu(\mathbf{X}')$) at $\beta = \beta_0$. After a short computation we obtain

$$D_{\beta_0}(\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}) = \sigma^2\big((\mathbf{A} + \mathbf{A}')\mathbf{X}\beta_0 + \mathbf{a}'\big)'\Sigma(\beta_0)\mathbf{a}. \tag{4.1.20}$$

On the other hand, the dispersion of the β_0-LBLUE of this functional (according to Lemma 1.6 and Lemma 1.7) at $\beta = \beta_0$ is

$$\begin{aligned} &D_{\beta_0}\big(\mathbf{f}'[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\mathbf{Y}\big) \\ &\quad = \sigma^2\beta_0'\mathbf{X}'(\mathbf{A} + \mathbf{A}')\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\Sigma(\beta_0)(\mathbf{X}')^-_{m(\Sigma(\beta_0))}\mathbf{X}'(\mathbf{A} + \mathbf{A}')\mathbf{X}\beta_0 \\ &\qquad - \sigma^2\beta_0'\mathbf{X}'(\mathbf{A} + \mathbf{A}')\Sigma(\beta_0)(\mathbf{X}')^-_{m(\Sigma(\beta_0))}\mathbf{X}'\mathbf{z} + \sigma^2\big((\mathbf{A} + \mathbf{A}')\mathbf{X}\beta_0 + \mathbf{a}\big)'\Sigma(\beta_0)\mathbf{a}. \end{aligned}$$

So we have after a straightforward simplification

$$\begin{aligned} &D_{\beta_0}\big(\mathbf{f}'[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\mathbf{Y}\big) - D_{\beta_0}(\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}) \\ &\quad = \sigma^2\beta'\mathbf{X}'(\mathbf{A} + \mathbf{A}')\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\Sigma(\beta_0)(\mathbf{X}')^-_{m(\Sigma(\beta_0))}\mathbf{X}'(\mathbf{A} + \mathbf{A}')\mathbf{X}\beta_0 \\ &\qquad + \frac{1}{2}\sigma^4 \operatorname{Tr}(\mathbf{A} + \mathbf{A}')\Sigma(\beta_0)(\mathbf{A} + \mathbf{A}')\Sigma(\beta_0) \geqslant 0. \end{aligned}$$

The theorem is proved. □

How to obtain the β_0-LBLQUE of $\mathbf{f}'\beta$ for $\mathbf{f} \in \mu(\mathbf{X}')$? If $\gamma \in \mu(\mathbf{X}')$ then

$$\mathbf{f} = \sum_{i=1}^{n} \alpha_i\mathbf{X}'\mathbf{e}_i = \sum_{\substack{i=1\\ i\neq s}}^{n-2} \alpha_i\mathbf{X}'\mathbf{e}_i + (\alpha_s + \gamma\alpha_{n-1} + \delta\alpha_n)\mathbf{X}'\mathbf{e}_s.$$

Let $\mathbf{A}_{\mathbf{e}_1}$, $\mathbf{A}_{\mathbf{e}_2}$, ..., $\mathbf{A}_{\mathbf{e}_{n-2}}$ be matrices satisfying (4.1.14) for $\mathbf{z} = \mathbf{e}_i$, $i = 1, 2, \ldots, n-2$. If

$$z_i = \alpha_i \quad i = 1, 2, \ldots, s-1, s+1, \ldots, n-2,$$

$$z_s = \frac{(\alpha_s + \gamma\alpha_{n-1} + \delta\alpha_n) - \sigma^2 b^2(|\gamma| - |\delta|)(1 - |\delta|)\mathbf{e}_s'\mathbf{X}\beta_0 \sum_{\substack{i=1\\ i\neq s}}^{n-2} \alpha_i\mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_i} + \mathbf{A}'_{\mathbf{e}_i})\mathbf{e}_n}{1 + \sigma^2 b^2(|\gamma| - |\delta|)(1 - |\delta|)\mathbf{e}_s'\mathbf{X}\beta_0\mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}'_{\mathbf{e}_s})\mathbf{e}_n},$$

then for the vector $\mathbf{z} = (z_1, z_2, \ldots, z_{n-2}, 0, 0)'$, $\mathbf{A} = \sum_{i=1}^{n-2} z_i\mathbf{A}_{\mathbf{e}_i} \in \mathscr{D}$ is a matrix satisfying (4.1.14). Further,

$$\begin{aligned} &\mathbf{X}'[-(\mathbf{A} + \mathbf{A}')\mathbf{X}\beta_0 + (\mathbf{X}')^-_{m(\Sigma(\beta_0))}\mathbf{X}'\mathbf{z}] \\ &\quad = \sum_{\substack{i=1\\ i\neq s}}^{n-2} \alpha_i\mathbf{X}'\mathbf{e}_i + (\alpha_s + \gamma\alpha_{n-1}\delta\alpha_n)\mathbf{X}'\mathbf{e}_s = \mathbf{f}. \end{aligned}$$

That is why

$$\big(-(\mathbf{A}+\mathbf{A}')\mathbf{X}\beta_0+(\mathbf{X}')^-_{m(\Sigma(\beta_0))}\mathbf{X}'\mathbf{z}\big)'\mathbf{Y}+\mathbf{YAY}$$

is the β_0-LBLQUE of $\mathbf{f}'\beta$.

The last question is how to obtain $\mathbf{A}_{\mathbf{e}_i}$, $i = 1, 2, \ldots, n-2$ in (4.1.14). The answer is given in the next lemma and remark.

Lemma 4.1.4. *If we denote*

$$\big(\Sigma(\beta_0)\otimes\Sigma(\beta_0)\big)(\mathbf{I}-\mathbf{Y}_*^-\mathbf{Y}_*)=\mathbb{A}$$

and

$$\mathbb{A}\mathbb{A}'+\mathbf{Y}_*'\mathbf{Y}_*=\mathbb{B}$$

then $\mathbb{B}$ *is a positive definite matrix and*

$$\begin{aligned}(4.1.21)\qquad \operatorname{vec}\mathbf{C}_{\mathbf{z}} &= 2b^2(|\gamma|-|\delta|)(1-|\delta|)\mathbf{e}_s'\mathbf{X}\beta_0\mathbf{e}_s'\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\Sigma(\beta_0)\mathbf{z}\\ &\quad\times\big(\Sigma^{-1}(\beta_0)\otimes\Sigma^{-1}(\beta_0)\big)\mathbb{A}(\mathbb{A}'\mathbb{B}^{-1}\mathbb{A})^-\mathbb{A}'\mathbb{B}^{-1}(\mathbf{e}_n\otimes\mathbf{e}_n)\end{aligned}$$

is a solution to (4.1.8) *for* $\mathbf{z}\in\mathbb{R}^n$.

Proof is an easy consequence of Lemma 5.3.3 in [4] and Complement 1, p. 118 in [4].

Remark 4.1.5. If $\operatorname{vec}\mathbf{C}_{\mathbf{z}}$ is given in (4.1.21) then $\operatorname{vec}A_{\mathbf{z}}=\frac{1}{2}\operatorname{vec}(\mathbf{C}_{\mathbf{z}}+\mathbf{C}_{\mathbf{z}}')$ is a solution to (4.1.14) for $\mathbf{z}\in\mathbb{R}^n$. (See (4.1.8) and considerations below.)

Example 4.1.6. Let in model (1.1) the design matrix $\mathbf{X}$ be

$$\begin{pmatrix}1&2\\2&4\\3&6\end{pmatrix},$$

the coefficients being $a = 1$, $b = 1$ and $\sigma^2 = 1$. We want to estimate the linear functional $\beta_1+2\beta_2$ locally at $\beta_0=\binom{\beta_{01}}{\beta_{02}}=\binom{1}{1}$ by the locally best linear and linear-quadratic estimators and compare their dispersions.

According to Remark 2.5 in [7] the β_0-LBLUE is

$$\begin{aligned}&(1\ 2)\begin{pmatrix}0.053389&0.034866&0.025626\\0.106777&0.069732&0.051253\end{pmatrix}\mathbf{Y}\\&=0.266943Y_1+0.174330Y_2+0.128132Y_3.\end{aligned}$$

Its dispersion is 4.271083.

The β_0-LBLQUE is

$$0.264335Y_1 + 0.206709Y_2 + 0.107409Y_3$$
$$+\mathbf{Y}'\begin{pmatrix} 0.002487 & -0.000702 & -0.000516 \\ -0.000702 & -0.004974 & 0.000337 \\ -0.000516 & -0.000337 & 0.002487 \end{pmatrix}\mathbf{Y}$$

and its dispersion is 4.207355.

We see that the dispersion of the β_0-LBLQUE in the investigated case can be less than the dispersion of the β_0-LBLUE (at $\beta = \beta_0$).

4.2. Case $\mathbf{E} = \binom{\gamma\mathbf{e}'_s}{\delta\mathbf{e}'_l}$. As in Section 4.1 we give the basic results. Some of them are without proofs which are again based on the previous methods and considerations.

Lemma 4.2.1. *If in model* (1.1) $\mathbf{E} = \binom{\gamma\mathbf{e}'_s}{\delta\mathbf{e}'_l}$, $s \in \{1, 2, \dots, n-2\}$, $l \in \{1, 2, \dots, n-2\}$, $s \neq l$, *where*

(i) $|\gamma| = 1$, $|\delta| \neq 1$, $\delta \neq 0$,
(ii) $|\gamma| \neq 1$, $\gamma \neq 0$, $|\delta| = 1$,
(iii) $|\gamma| = |\delta| = 1$ *or*
(iv) $|\gamma| \neq 1$, $\gamma \neq 0$, $|\delta| \neq 1$, $\delta \neq 0$
then $\mathbf{D} \in \mathscr{D}$ *iff*

$$\mathbf{e}'_i\mathbf{D}\mathbf{e}_i = \mathbf{O}, \quad i \notin \{s, n-1\},$$
$$\mathbf{e}'_s\mathbf{D}\mathbf{e}_s + \mathbf{e}'_{n-1}\mathbf{D}\mathbf{e}_{n-1} = \mathbf{O}$$

and

$$\mathbf{X}'\mathbf{D}\mathbf{X} = \mathbf{O}$$

in the case (i),

$$\mathbf{e}'_i\mathbf{D}\mathbf{e}_i = \mathbf{O}, \quad i \notin \{l, n\},$$
$$\mathbf{e}'_l\mathbf{D}\mathbf{e}_l + \mathbf{e}'_n\mathbf{D}\mathbf{e}_n = \mathbf{O}$$

and

$$\mathbf{X}'\mathbf{D}\mathbf{X} = \mathbf{O}$$

in the case (ii),

$$\mathbf{e}'_i\mathbf{D}\mathbf{e}_i = \mathbf{O}, \quad i \notin \{s, l, n-1, n\},$$
$$\mathbf{e}'_s\mathbf{D}\mathbf{e}_s + \mathbf{e}'_{n-1}\mathbf{D}\mathbf{e}_{n-1} = \mathbf{O}$$
$$\mathbf{e}'_l\mathbf{D}\mathbf{e}_l + \mathbf{e}'_n\mathbf{D}\mathbf{e}_n = \mathbf{O}$$

and

$$\mathbf{X}'\mathbf{D}\mathbf{X} = \mathbf{O}$$

in the case (iii),

(4.2.1) $$\mathbf{e}_i'\mathbf{D}\mathbf{e}_i = \mathbf{O}, \quad i \notin \{s, l, n-1, n\},$$

(4.2.2) $$\mathbf{e}_s'\mathbf{D}\mathbf{e}_s = |\gamma|\frac{1-|\delta|}{1-|\gamma|}\mathbf{e}_n'\mathbf{D}\mathbf{e}_n,$$

(4.2.3) $$\mathbf{e}_l'\mathbf{D}\mathbf{e}_l = -|\delta|\mathbf{e}_n'\mathbf{D}\mathbf{e}_n,$$

(4.2.4) $$\mathbf{e}_{n-1}'\mathbf{D}\mathbf{e}_{n-1} = -\frac{1-|\delta|}{1-|\gamma|}\mathbf{e}_n'\mathbf{D}\mathbf{e}_n$$

and

(4.2.5) $$\mathbf{X}'\mathbf{D}\mathbf{X} + \sigma^2 b^2(1-|\delta|)\{|\gamma|\mathbf{X}'\mathbf{e}_s\mathbf{e}_s'\mathbf{X} - |\delta|\mathbf{X}'\mathbf{e}_l\mathbf{e}_l'\mathbf{X}\}\mathbf{e}_n'\mathbf{D}\mathbf{e}_n = \mathbf{O}$$

in the case (iv).

Theorem 4.2.2. *If in model* (1.1) $\mathscr{E} = \binom{\gamma\mathbf{e}_s'}{\delta\mathbf{e}_l'}$, $s \in \{1, 2, \ldots, n-2\}$, $i \in \{1, 2, \ldots, n-2\}$, $s \neq l$ *where*
(i) $|\gamma| = 1$, $|\delta| \neq 1$, $\delta \neq 0$,
(ii) $|\gamma| \neq 1$, $\gamma \neq 0$, $|\delta| = 1$, *or*
(iii) $|\gamma| = |\delta| = 1$,
then (1.15) *is true and the dispersion of the* β_0-LBLQUE *of* $\mathbf{f}'\beta$ *is the same as the dispersion of the*β_0-LBLUE *of* $\mathbf{f}'\beta$ *at* $\beta = \beta_0$.

The case with $|\gamma| \neq 1$, $|\delta| \neq 1$, $\gamma \neq 0$, $\delta \neq 0$ is again different and needs a deeper analysis.

As in Section 4.1 we obtain that conditions (4.2.1)–(4.2.5) are equivalent to

(4.2.6) $$\mathbf{Y}_{**} \operatorname{vec} \mathbf{D} = \mathbf{O},$$

where $\mathbf{Y}_{**}$ is $(k^2+n-1) \times n^2$ matrix of the form

$$\mathbf{Y}_{**} = \begin{pmatrix} \mathbf{e}_1' \otimes \mathbf{e}_1' \\ \vdots \\ \mathbf{e}_s' \otimes \mathbf{e}_s' - |\gamma|\frac{1-|\delta|}{1-|\gamma|}(\mathbf{e}_n' \otimes \mathbf{e}_n') \\ \vdots \\ \mathbf{e}_l' \otimes \mathbf{e}_l' + |\delta|(\mathbf{e}_n' \otimes \mathbf{e}_n') \\ \vdots \\ \mathbf{e}_{n-2}' \otimes \mathbf{e}_{n-2}' \\ (\mathbf{e}_{n-1}' \otimes \mathbf{e}_{n-1}' + \frac{1-|\delta|}{1-|\gamma|}(\mathbf{e}_n' \otimes \mathbf{e}_n') \\ (\mathbf{X}' \otimes \mathbf{X}')\{\mathbf{I} + \sigma^2 b^2(1-|\delta|)[|\gamma|(\mathbf{e}_s\mathbf{e}_n' \otimes \mathbf{e}_s\mathbf{e}_n') - |\delta|(\mathbf{e}_l\mathbf{e}_n' \otimes \mathbf{e}_l\mathbf{e}_n')]\} \end{pmatrix}.$$

A consequence of Lemma 4.2.1, case (iv) is that

$$\mathbf{D} \in \mathscr{D} \Leftrightarrow \mathbf{Y}_{**} \operatorname{vec} \mathbf{D} = \mathbf{O} \Leftrightarrow \operatorname{vec} \mathbf{D} \in \operatorname{Ker} \mathbf{Y}_{**}.$$

Following the considerations in Section 4.1 below the relation (4.1.6) we obtain that

$$\begin{aligned}
\forall\{\mathbf{D} \in \mathscr{D}\} \quad & \operatorname{Tr}(\mathbf{D}+\mathbf{D}')\{\sigma^2\Sigma(\beta_0)(\mathbf{A}+\mathbf{A}')\Sigma(\beta_0) \\
& + 2\mathbf{X}\beta_0\mathbf{z}'\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\Sigma(\beta_0)\} = 0 \\
\Leftrightarrow \exists\{\delta \in \mathbb{R}^{k^2+n-1}\} \quad & \big(\Sigma(\beta_0)\otimes\Sigma(\beta_0)\big)\operatorname{vec}(\mathbf{A}+\mathbf{A}') \\
& - 2b^2(1-|\delta|)\{|\gamma|\mathbf{e}_s'\mathbf{X}\beta_0\mathbf{z}'\Sigma(\beta_0)(\mathbf{X}')^-_{m(\Sigma(\beta_0))}\mathbf{X}'\mathbf{e}_s \\
& - |\delta|\mathbf{e}_l'\mathbf{X}\beta_0\mathbf{z}'\Sigma(\beta_0)(\mathbf{X}')^-_{m(\Sigma(\beta_0))}\mathbf{X}'\mathbf{e}_l\}(\mathbf{e}_n\otimes\mathbf{e}_n) = \mathbf{Y}'_{**}\delta.
\end{aligned}$$

We also obtain that $\forall\{\mathbf{z} \in \mathbb{R}^n\}$ $\exists\{\mathbf{A_z} \in \mathscr{D}\}$ such that

$$\begin{aligned}
\text{(4.2.7)} \qquad \forall\{\mathbf{D} \in \mathscr{D}\} \quad & \operatorname{Tr}(\mathbf{D}+\mathbf{D}')\{\sigma^2\Sigma(\beta_0)(\mathbf{A_z}+\mathbf{A}'_\mathbf{z})\Sigma(\beta_0) \\
& + 2\mathbf{X}\beta_0\mathbf{z}'\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma(\beta_0))}]'\Sigma(\beta_0)\} = 0.
\end{aligned}$$

Now let us find the β_0-LBLQUE of $\mathbf{f}'\beta$ in the investigated case $|\gamma| \neq 1$, $|\delta| \neq 1$, $\gamma \neq 0$, $\delta \neq 0$. According to Theorem 1.3 $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ is the β_0-LBLQUE of $\mathbf{f}'\beta$ iff (1.5), (1.6), (1.7) and (1.8) hold. We know that (4.2.7) is true. That is why

$$\begin{aligned}
\text{(4.2.8)} \qquad \mathscr{G} &= \{\mathbf{f}\colon \exists\beta_0\text{-LBLQUE for } \mathbf{f}'\beta\} \\
&= \{\mathbf{X}'\mathbf{a}\colon \mathbf{a} = -(\mathbf{A}+\mathbf{A}')\mathbf{X}\beta_0 + (\mathbf{X}')^-_{m(\Sigma(\beta_0))}\mathbf{X}'\mathbf{z}, \\
&\qquad \mathbf{A} \in \mathscr{D} \text{ satisfies (4.2.7)}, \mathbf{z} \in \mathbb{R}^n\}.
\end{aligned}$$

Theorem 4.2.3. *If in model* (1.1) $\mathbf{E} = \binom{\gamma\mathbf{e}_s'}{\delta\mathbf{e}_l'}$, $s \in \{1,2,\dots,n-2\}$, $l \in \{1,2,\dots,n-2\}$, $s \neq l$, $|\gamma| \neq 1$, $\gamma \neq 0$, $\delta \neq 1$, $\delta \neq 0$ *then* $\mathscr{G} = \mu(\mathbf{X}')$, *i.e.* (1.15) *is true. Further, the dispersion of the* β_0-LBLQUE *of* $\mathbf{f}'\beta$ *is not greater than the dispersion of the* β_0-LBLUE *of* $\mathbf{f}'\beta$ *at* $\beta = \beta_0$.

Proof. From (4.2.5) we obtain

$$\begin{aligned}
\text{(4.2.9)} \quad \mathbf{X}'\mathbf{a} = & -\mathbf{X}'(\mathbf{A}+\mathbf{A}')\mathbf{X}\beta_0 + \mathbf{X}'\mathbf{z} = \sum_{i\notin\{s,l,n-1,n\}} z_i\mathbf{X}'\mathbf{e}_i \\
& + \big(z_s + \gamma z_{n-1} + \sigma^2 b^2(1-|\delta|)|\gamma|\mathbf{e}_s'\mathbf{X}\beta_0\mathbf{e}_n'(\mathbf{A}+\mathbf{A}')\mathbf{e}_n\big)\mathbf{X}'\mathbf{e}_s \\
& + \big(z_l + \delta z_n - \sigma^2 b^2(1-|\delta|)|\delta|\mathbf{e}_l'\mathbf{X}\beta_0\mathbf{e}_n'(\mathbf{A}+\mathbf{A}')\mathbf{e}_n\big)\mathbf{X}'\mathbf{e}_l.
\end{aligned}$$

If we denote by $\mathbf{A}_{\mathbf{e}_i}$ the matrix in (4.2.7) corresponding to $\mathbf{z} = \mathbf{e}_i$, $i = 1, 2, \dots, n$ then

$$\mathbf{A}_{\mathbf{e}_{n-1}} + \mathbf{A}'_{\mathbf{e}_{n-1}} = \gamma(\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}'_{\mathbf{e}_s}), \quad \mathbf{A}_{\mathbf{e}_n} + \mathbf{A}'_{\mathbf{e}_n} = \delta(\mathbf{A}_{\mathbf{e}_l} + \mathbf{A}'_{\mathbf{e}_l})$$

and

$$\mathbf{A} + \mathbf{A}' = \sum_{i=1}^{n} z_i(\mathbf{A}_{\mathbf{e}_i} + \mathbf{A}'_{\mathbf{e}_i}).$$

From (4.9.2) we conclude

$$\begin{aligned}
(4.2.10)\quad \mathbf{X}'\mathbf{a} = & \sum_{i \notin \{s,l,n-1,n\}} z_i \mathbf{X}'\mathbf{e}_i \\
& + \Big\{\sigma^2 b^2 (1-|\delta|)|\gamma| \mathbf{e}_s'\mathbf{X}\beta_0 \sum_{i \notin \{s,l,n-1,n\}} z_i \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_i} + \mathbf{A}'_{\mathbf{e}_i})\mathbf{e}_n \\
& + \big(1 + \sigma^2 b^2 (1-|\delta|)|\gamma| \mathbf{e}_s'\mathbf{X}\beta_0 \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}'_{\mathbf{e}_s})\mathbf{e}_n\big)(z_s + \gamma z_{n-1}) \\
& + \sigma^2 b^2 (1-|\delta|)|\gamma| \mathbf{e}_s'\mathbf{X}\beta_0 \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_l} + \mathbf{A}'_{\mathbf{e}_l})\mathbf{e}_n (z_l + \delta z_n)\Big\}\mathbf{X}'\mathbf{e}_s \\
& + \Big\{ - \sigma^2 b^2 (1-|\delta|)|\delta| \mathbf{e}_l'\mathbf{X}\beta_0 \sum_{i \notin \{s,l,n-1,n\}} z_i \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_i} + \mathbf{A}'_{\mathbf{e}_i})\mathbf{e}_n \\
& + \big(- \sigma^2 b^2 (1-|\delta|)|\delta| \mathbf{e}_l'\mathbf{X}\beta_0 \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}'_{\mathbf{e}_s})\mathbf{e}_n\big)(z_s + \gamma z_{n-1}) \\
& + \big(1 - \sigma^2 b^2 (1-|\delta|)|\delta| \mathbf{e}_l'\mathbf{X}\beta_0 \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_l} + \mathbf{A}'_{\mathbf{e}_l})\mathbf{e}_n\big)(z_l + \delta z_n)\Big\}\mathbf{X}'\mathbf{e}_l.
\end{aligned}$$

In the investigated case

$$\mu(\mathbf{X}') = \Big\{ \sum_{i=1}^{n-2} \xi_i \mathbf{X}'\mathbf{e}_i \colon \xi_i \in \mathbb{R},\ i = 1, 2, \dots, n-2 \Big\}.$$

By virtue of (4.2.10) it means that $\mathscr{G} = \mu(\mathbf{X}')$ iff

$$\begin{aligned}
& [1 + \sigma^2 b^2 (1-|\delta|)|\gamma| \mathbf{e}_s'\mathbf{X}\beta_0 \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}'_{\mathbf{e}_s})\mathbf{e}_n] \\
& \qquad \times [1 - \sigma^2 b^2 (1-|\delta|)|\delta| \mathbf{e}_l'\mathbf{X}\beta_0 \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_l} + \mathbf{A}'_{\mathbf{e}_l})\mathbf{e}_n] \\
& \quad + \sigma^2 b^2 (1-|\delta|)|\gamma| \mathbf{e}_s'\mathbf{X}\beta_0 \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_l} + \mathbf{A}'_{\mathbf{e}_l}) \\
& \qquad \times \mathbf{e}_n \sigma^2 b^2 (1-|\delta|)|\delta| \mathbf{e}_l'\mathbf{X}\beta_0 \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}'_{\mathbf{e}_s})\mathbf{e}_n \neq 0,
\end{aligned}$$

or, equivalently, iff

$$\begin{aligned}
(4.2.11)\qquad & 1 + \sigma^2 b^2 (1-|\delta|)|\gamma| \mathbf{e}_s'\mathbf{X}\beta_0 \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}'_{\mathbf{e}_s})\mathbf{e}_n \\
& \quad - \sigma^2 b^2 (1-|\delta|)|\delta| \mathbf{e}_l'\mathbf{X}\beta_0 \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_l} + \mathbf{A}'_{\mathbf{e}_l})\mathbf{e}_n \neq 0.
\end{aligned}$$

If we suppose that (4.2.11) is not true, in the analyzed cases

(i) $\mathbf{e}_s'\mathbf{X}\beta_0 = \mathbf{e}_l'\mathbf{X}\beta_0 = 0$

(ii) $\mathbf{e}_s'\mathbf{X}\beta_0 = 0$, $\mathbf{e}_l'\mathbf{X}\beta_0 \neq 0$,

(iii) $\mathbf{e}_s'\mathbf{X}\beta_0 \neq 0$, $\mathbf{e}_l'\mathbf{X}\beta_0 = 0$, and

(iv) $\mathbf{e}_s'\mathbf{X}\beta_0 \neq 0$, $\mathbf{e}_l'\mathbf{X}\beta_0 \neq 0$

we obtain a contradiction and so we prove the first part of the theorem. The second part of the theorem about dispersion of the β_0-LBLQUE can be proved again in the same way as in Theorem 4.1.3 and the proof is omitted. □

How to obtain the β_0-LBLQUE of $\mathbf{f}'\beta$ for $\mathbf{f} \in \mu(\mathbf{X}')$ if $|\gamma| \neq 1$, $\gamma \neq 0$, $|\delta| \neq 1$, $\delta \neq 0$?

If $\mathbf{f} \in \mu(\mathbf{X}')$ then

$$\mathbf{f} = \sum_{i=1}^{n} \alpha_i \mathbf{X}'\mathbf{e}_i = \sum_{\substack{i=1 \\ i \notin \{s,l\}}}^{n-2} \alpha_i \mathbf{X}'\mathbf{e}_i + (\alpha_s + \gamma\alpha_{n-1})\mathbf{X}'\mathbf{e}_s + (\alpha_l + \delta\alpha_n)\mathbf{X}'\mathbf{e}_l.$$

Let $\mathbf{A}_{\mathbf{e}_1}, \mathbf{A}_{\mathbf{e}_2}, \ldots, \mathbf{A}_{\mathbf{e}_{n-2}}$ be the matrices satisfying (4.2.7) for $\mathbf{z} = \mathbf{e}_i$, $i = 1, 2, \ldots, n-2$. If

$$z_i = \alpha_i, \quad i = 1, 2, \ldots, s-1, s+1, \ldots, l-1, l+1, \ldots, n-2,$$

$$\begin{aligned} z_s = \{&1 + \sigma^2 b^2 (1 - |\delta|)[|\gamma| \mathbf{e}_s'\mathbf{X}\beta_0 \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}_{\mathbf{e}_s}')\mathbf{e}_n \\ &- |\delta| \mathbf{e}_l'\mathbf{X}\beta_0 \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_l} + \mathbf{A}_{\mathbf{e}_l}')\mathbf{e}_n]\}^{-1} \Big\{ [1 - \sigma^2 b^2 (1 - |\delta|)|\delta| \mathbf{e}_l'\mathbf{X}\beta_0 \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_l} + \mathbf{A}_{\mathbf{e}_l}')\mathbf{e}_n] \\ &\times \Big[\alpha_s + \gamma\alpha_{n-1} - \sigma^2 b^2 (1 - |\delta|)|\gamma| \mathbf{e}_s'\mathbf{X}\beta_0 \sum_{i \notin \{s,l,n-1,n\}} \alpha_i \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_i} + \mathbf{A}_{\mathbf{e}_i}')\mathbf{e}_n\Big] \\ &- [\sigma^2 b^2 (1 - |\delta|)[|\gamma| \mathbf{e}_s'\mathbf{X}\beta_0 \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_l} + \mathbf{A}_{\mathbf{e}_l}')\mathbf{e}_n] \\ &\times \Big[\alpha_l + \delta\alpha_n + \sigma^2 b^2 (1 - |\delta|)|\delta| \mathbf{e}_l'\mathbf{X}\beta_0 \sum_{i \notin \{s,l,n-1,n\}} \alpha_i \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_i} + \mathbf{A}_{\mathbf{e}_i}')\mathbf{e}_n\Big]\Big\} \end{aligned}$$

and

$$\begin{aligned} z_l = \{&1 + \sigma^2 b^2 (1 - |\delta|)[|\gamma| \mathbf{e}_s'\mathbf{X}\beta_0 \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}_{\mathbf{e}_s}')\mathbf{e}_n \\ &- |\delta| \mathbf{e}_l'\mathbf{X}\beta_0 \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_l} + \mathbf{A}_{\mathbf{e}_l}')\mathbf{e}_n]\}^{-1} \\ &\times \Big\{ [1 + \sigma^2 b^2 (1 - |\delta|)|\gamma| \mathbf{e}_s'\mathbf{X}\beta_0 \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}_{\mathbf{e}_s}')\mathbf{e}_n] \\ &\times \Big[\alpha_l + \delta\alpha_n + \sigma^2 b^2 (1 - |\delta|)|\delta| \mathbf{e}_l'\mathbf{X}\beta_0 \sum_{i \notin \{s,l,n-1,n\}} \alpha_i \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_i} + \mathbf{A}_{\mathbf{e}_i}')\mathbf{e}_n\Big] \\ &+ [\sigma^2 b^2 (1 - |\delta|)|\delta| \mathbf{e}_l'\mathbf{X}\beta_0 \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_s} + \mathbf{A}_{\mathbf{e}_s}')\mathbf{e}_n] \\ &\times \Big[\alpha_s + \gamma\alpha_{n-1} - \sigma^2 b^2 (1 - |\delta|)|\gamma| \mathbf{e}_s'\mathbf{X}\beta_0 \sum_{i \notin \{s,l,n-1,n\}} \alpha_i \mathbf{e}_n'(\mathbf{A}_{\mathbf{e}_i} + \mathbf{A}_{\mathbf{e}_i}')\mathbf{e}_n\Big]\Big\} \end{aligned}$$

then we can show that for the vector $\mathbf{z} = (z_1, z_2, \ldots, z_{n-2}, 0, 0)'$, $\mathbf{A} = \sum_{i=1}^{n-2} z_i \mathbf{A}_{\mathbf{e}_i} \in \mathscr{D}$ is a matrix satisfying (4.2.7) and $\mathbf{X}'[-(\mathbf{A}+\mathbf{A}')\mathbf{X}\beta_0 + (\mathbf{X}')^-_{m(\Sigma(\beta_0))}\mathbf{X}'\mathbf{z}] = \mathbf{f}$. That is why

$$\big(-(\mathbf{A}+\mathbf{A}')\mathbf{X}\beta_0 + (\mathbf{X}')^-_{m(\Sigma(\beta_0))}\mathbf{X}'\mathbf{z}\big)'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$$

is the β_0-LBLQUE of $\mathbf{f}'\beta$.

We only remark that one can solve (4.2.7) according to Lemma 4.1.4 and Remark 4.1.5 with $\mathbf{Y}_{**}$ instead of $\mathbf{Y}_*$ and

$$\begin{aligned}\operatorname{vec}\mathbf{C}_{\mathbf{z}} = 2b^2(1-|\delta|)\{&|\gamma|\mathbf{e}_s'\mathbf{X}\beta_0\mathbf{z}'\Sigma(\beta_0)(\mathbf{X}')^-_{m(\Sigma(\beta_0))}\mathbf{X}'\mathbf{e}_s\\ &-|\delta|\mathbf{e}_l'\mathbf{X}\beta_0\mathbf{z}'\Sigma(\beta_0)(\mathbf{X}')^-_{m(\Sigma(\beta_0))}\mathbf{X}'\mathbf{e}_l\}\\ &\times\big(\Sigma^{-1}(\beta_0)\otimes\Sigma^{-1}(\beta_0)\big)\mathbb{A}(\mathbb{A}'\mathbb{B}^{-1}\mathbb{A})^-\mathbb{A}'\mathbb{B}^{-1}(\mathbf{e}_n\otimes\mathbf{e}_n)\end{aligned}$$

instead of (4.1.21).

Example 4.2.4. Let in model (1.1) the design matrix $\mathbf{X}$ be

$$\begin{pmatrix}1 & 2\\ 1 & 3\\ 2 & 4\\ 3 & 9\end{pmatrix},$$

the coefficients being $a = 1$, $b = 1$ and $\sigma^2 = 1$. We want to estimate the linear functional $\beta_1 + 2.5\beta_2$ locally at $\beta_0 = \binom{\beta_{01}}{\beta_{02}} = \binom{1}{1}$ by the locally best linear and linear-quadratic estimators and compare their dispersions.

According to Remark 2.5 in [7] the β_0-LBLUE is

$$\begin{aligned}&(1\ \ 2.5)\begin{pmatrix}1.300855 & -0.857868 & 0.849558 & -0.380711\\ -0.433628 & 0.428934 & -0.283186 & 0.190355\end{pmatrix}\mathbf{Y}\\ &= 0.216785Y_1 + 0.214467Y_2 + 0.141593Y_3 + 0.0951765Y_4.\end{aligned}$$

Its dispersion is 4.415351.

The β_0-LBLQUE is

$$\begin{aligned}&0.215472Y_1 + 0.270347Y_2 + 0.165375Y_3 + 0.092210Y_4\\ &+\mathbf{Y}'\begin{pmatrix}0.005214 & 0.000000 & -0.001303 & 0.000000\\ 0.000000 & -0.003910 & 0.000000 & -0.003910\\ -0.001303 & 0.000000 & -0.002607 & 0.000000\\ 0.000000 & -0.003910 & 0.000000 & 0.001303\end{pmatrix}\mathbf{Y}\end{aligned}$$

and its dispersion is 3.025641.

We see that the dispersion of the β_0-LBLQUE in the investigated case is essentially lower than the dispersion of the β_0-LBLUE (at $\beta = \beta_0$).

4.3. Case $\mathbf{E} = \begin{pmatrix} \gamma \mathbf{e}'_s \\ \sum_{j=1}^t \delta_{l_j} \mathbf{e}'_{\delta_{l_j}} \end{pmatrix}$. As in the previous sections we give the main results without proofs.

Lemma 4.3.1. *If in model* (1.1) $\mathbf{E} = \begin{pmatrix} \gamma \mathbf{e}'_s \\ \sum_{j=1}^t \delta_{l_j} \mathbf{e}'_{\delta_{l_j}} \end{pmatrix}$, $s \in \{1, 2, \ldots, n-2\}$, $l_j \in \{1, 2, \ldots, n-2\}$, $d_j \neq 0$, $j = 1, 2, \ldots, t$, $t \geqslant 2$, *where*

(i) $|\gamma| = 1$ *or*

(ii) $|\gamma| \neq 1$, $\gamma \neq 0$ *then* $\mathbf{D} \in \mathscr{D}$ *iff*

$$\mathbf{e}'_s \mathbf{D} \mathbf{e}_i = \mathbf{O}, \quad i \notin \{s, n-1\},$$
$$\mathbf{e}'_s \mathbf{D} \mathbf{e}_s + \mathbf{e}'_{n-1} \mathbf{D} \mathbf{e}_{n-1} = \mathbf{O}$$

and

$$\mathbf{X}' \mathbf{D} \mathbf{X} = \mathbf{O}$$

in the case (i),

$$\mathbf{e}'_s \mathbf{D} \mathbf{e}_i = \mathbf{O}, \quad i = 1, 2, \ldots, n$$

and

$$\mathbf{X}' \mathbf{D} \mathbf{X} = \mathbf{O}$$

in the case (ii).

4.4. Case $\mathbf{E} = \begin{pmatrix} \sum_{i=1}^t \gamma_{s_i} \mathbf{e}'_{s_i} \\ k \sum_{i=1}^t \gamma_{s_i} \mathbf{e}'_{s_i} \end{pmatrix}$.

Lemma 4.4.1. *If in model* (1.1) $\mathbf{E} = \begin{pmatrix} \sum_{i=1}^t \gamma_{s_i} \mathbf{e}'_{s_i} \\ k \sum_{i=1}^t \gamma_{s_i} \mathbf{e}'_{s_i} \end{pmatrix}$, $s \in \{1, 2, \ldots, n-2\}$, $\gamma_{s_i} \neq 0$, $i = 1, 2, \ldots, t$, $t \geqslant 2$, *where*

(i) $k \neq 0$, $|k| \neq 1$ *or*

(ii) $|k| = 1$ *then* $\mathbf{D} \in \mathscr{D}$ *iff*

$$\mathbf{e}'_i \mathbf{D} \mathbf{e}_i = \mathbf{O}, \quad i = 1, 2, \ldots, n$$

and

$$\mathbf{X}' \mathbf{D} \mathbf{X} = \mathbf{O}$$

in the case (i) *and*

$$\mathbf{e}_i'\mathbf{D}\mathbf{e}_i = \mathbf{O}, \quad i \notin \{s, n-1\},$$
$$\mathbf{e}_{n-1}'\mathbf{D}\mathbf{e}_{n-1} + \mathbf{e}_n'\mathbf{D}\mathbf{e}_n = \mathbf{O}$$

and

$$\mathbf{X}'\mathbf{D}\mathbf{X} = \mathbf{O}$$

in the case (ii).

4.5. Case $\mathbf{E} = \begin{pmatrix} \sum_{i=1}^{t} \gamma_{s_i}\mathbf{e}_{s_i}' \\ \sum_{j=1}^{u} \delta_{l_j}\mathbf{e}_{l_j}' \end{pmatrix}$.

Lemma 4.5.1. *If in model* (1.1) $\mathbf{E} = \begin{pmatrix} \sum_{i=1}^{t} \gamma_{s_i}\mathbf{e}_{s_i}' \\ \sum_{j=1}^{u} \delta_{l_j}\mathbf{e}_{l_j}' \end{pmatrix}$, $s_i \in \{1, 2, \dots, n-2\}$, $\gamma_{s_i} \neq 0$, $i = 1, 2, \dots, t$, $t \geqslant 2$, $l_j \in \{1, 2, \dots, n-2\}$, $\delta_{l_j} \neq 0$, $j = 1, 2, \dots, u$, $u \geqslant 2$, $k \sum_{i=1}^{t} \gamma_{s_i}\mathbf{e}_{s_i}' \neq \sum_{j=1}^{u} \delta_{l_j}\mathbf{e}_{l_j}'$ *for all* $k \in \mathbb{R}$ *then*

$$\sum_{i=1}^{n} \mathbf{e}_i'\mathbf{B}\mathbf{e}_i |\mathbf{e}_i'\mathbf{X}\beta| = 0 \quad \forall\{\beta \in \mathbb{R}^k\}$$

iff

$$\mathbf{e}_i'\mathbf{B}\mathbf{e}_i = 0 \quad i = 1, 2, \dots, n.$$

Proof. Let us denote

$$\mathscr{A} = \{s_1, s_2, \dots, s_t\} \subset \{1, 2, \dots, n-2\},$$
$$\mathscr{B} = \{l_1, l_2, \dots, l_u\} \subset \{1, 2, \dots, n-2\}.$$

We have

$$(4.5.1) \sum_{i=1}^{n} \mathbf{e}_i'\mathbf{B}\mathbf{e}_i |\mathbf{e}_i'\mathbf{X}\beta| = 0 \quad \forall\{\beta \in \mathbb{R}^k\}$$
$$\Leftrightarrow \sum_{i \in \{\{1,2,\dots,n-2\} = \{\mathscr{A} \cup \mathscr{B}\}\}} \mathbf{e}_i'\mathbf{B}\mathbf{e}_i|u_i| + \sum_{i \in \{\mathscr{A} - \mathscr{B}\}} \mathbf{e}_i'\mathbf{B}\mathbf{e}_i|u_i|$$
$$+ \sum_{i \in \{\mathscr{A} - \mathscr{B}\}} \mathbf{e}_i'\mathbf{B}\mathbf{e}_i|u_i| + \sum_{i \in \{\mathscr{A} \cap \mathscr{B}\}} \mathbf{e}_i'\mathbf{B}\mathbf{e}_i|u_i| + \mathbf{e}_{n-1}'\mathbf{B}\mathbf{e}_{n-1}\Big|\sum_{i=1}^{t} \gamma_{s_i}u_{s_i}\Big|$$
$$+ \mathbf{e}_n'\mathbf{B}\mathbf{e}_n\Big|\sum_{j=1}^{u} \delta_{l_j}u_{l_j}\Big| = 0 \quad \forall\{\mathbf{u} = (u_1, u_2, \dots, u_{n-2})' \in \mathbb{R}^{n-2}\}.$$

(i) Let there be two (say s_1 and s_2) or more indices in $\mathscr{A} - \mathscr{B}$.

Because of $u \geqslant 2$, there are also two (say l_1 and l_2) or more other indices belonging to $\mathscr{B}$. For $\{\mathbf{u} \in \mathsf{R}^{n-2}: u_i = 0 \text{ for } i \neq s_1\}$ we have from (4.5.1) that

$$\mathbf{e}'_{s_1}\mathbf{B}\mathbf{e}_{s_1} + \mathbf{e}'_{n-1}\mathbf{B}\mathbf{e}_{n-1}|\gamma_{s_1}| = 0, \tag{4.5.2}$$

and for $\{\mathbf{u} \in \mathsf{R}^{n-2}: u_i = 0 \text{ for } i \neq s_2\}$ again

$$\mathbf{e}_{s_2}\mathbf{B}\mathbf{e}_{s_2} + \mathbf{e}'_{n-1}\mathbf{B}\mathbf{e}_{n-1}|\gamma_{s_2}| = 0. \tag{4.5.3}$$

So for $\left\{\mathbf{u} \in \mathsf{R}^{n-2}: u_i = 0 \text{ for } i \notin \{s_1, s_2\}\right\}$ we have the relation

$$\mathbf{e}'_{s_1}\mathbf{B}\mathbf{e}_{s_1}|u_{s_1}| + \mathbf{e}'_{s_2}\mathbf{B}\mathbf{e}_{s_2}|u_{s_2}| + \mathbf{e}'_{n-1}\mathbf{B}\mathbf{e}_{n-1}|\gamma_{s_1}u_{s_1} + \gamma_{s_2}u_{s_2}| = 0 \quad \forall\{u_{s_i} \in \mathsf{R}\ i = 1, 2\}. \tag{4.5.4}$$

Substituting (4.5.2) and (4.5.3) into (4.5.4) we have

$$e'_{n-1}\mathbf{B}\mathbf{e}_{n-1}[-|\gamma_{s_1}, u_{s_1}| - |\gamma_{s_2}u_{s_2}| + |\gamma_{s_1}u_{s_1} + \gamma_{s_2}u_{s_2}|] = 0 \quad \forall\{u_{s_i} \in \mathsf{R}\ i = 1, 2\},$$

which is satisfied iff

$$\mathbf{e}'_{n-1}\mathbf{B}\mathbf{e}_{n-1} = 0. \tag{4.5.5}$$

Now taking into account (4.5.5), for $\{\mathbf{u} \in \mathsf{R}^{n-2}: u_i = 0 \text{ for } i \neq l_1\}$ we obtain from (4.5.1) that

$$\mathbf{e}'_{l_1}\mathbf{B}\mathbf{e}_{l_1} + \mathbf{e}'_n\mathbf{B}\mathbf{e}_n|\delta_{l_1}| = 0 \tag{4.5.6}$$

and similarly

$$\mathbf{e}'_{l_2}\mathbf{B}\mathbf{e}_{l_2} + \mathbf{e}'_n\mathbf{B}\mathbf{e}_n|\delta_{l_2}| = 0. \tag{4.5.7}$$

For $\left\{\mathbf{u} \in \mathsf{R}^{n-2}: u_i = 0 \text{ for } i \neq \{l_1, l_2\}\right\}$ we obtain from (4.5.1)

$$\mathbf{e}'_{l_1}\mathbf{B}\mathbf{e}_{l_1}|u_{l_1}| + \mathbf{e}'_{l_2}\mathbf{B}\mathbf{e}_{l_2}|u_{l_2}| + \mathbf{e}'_n\mathbf{B}\mathbf{e}_n|\delta_{l_1}u_{l_1} + \delta_{l_2}u_{l_2}| = 0 \quad \forall\{u_{l_i} \in \mathsf{R}\ i = 1, 2\}. \tag{4.5.8}$$

Substituting (4.5.6) and (4.5.7) into (4.5.8) we have the condition

$$\mathbf{e}'_n\mathbf{B}\mathbf{e}_n = 0. \tag{4.5.9}$$

Considering (4.5.5) and (4.5.9) we easily obtain from (4.5.1) that also $\mathbf{e}'_i\mathbf{B}\mathbf{e}_i = 0$ for $i = 1, 2, \ldots, n-2$.

In the cases

(ii) there is only one index (say s_1) in $\mathscr{A} - \mathscr{B}$ and

(iii) $\mathscr{A} - \mathscr{B} = \emptyset$ we continue similarly and also obtain $\mathbf{e}'_i\mathbf{B}\mathbf{e}_i = 0$ for $i = 1, 2, \ldots, n$. The lemma is proved. □

Lemma 4.5.2. *If in model* (1.1) $\mathbf{E} = \begin{pmatrix} \sum_{i=1}^{t} \gamma_{s_i} \mathbf{e}'_{s_i} \\ \sum_{j=1}^{u} \delta_{l_j} \mathbf{e}'_{l_j} \end{pmatrix}$, $s_i \in \{1,2,\dots,n-2\}$, $\gamma_{s_i} \neq 0$, $i = 1, 2, \dots, t$, $t \geqslant 2$, $l_j \in \{1,2,\dots,n-2\}$, $\delta_{l_j} \neq 0$, $j = 1, 2, \dots, u$, $u \geqslant 2$, $k \sum_{i=1}^{t} \gamma_{s_i} \mathbf{e}'_{s_i} \neq \sum_{j=1}^{u} \delta_{l_j} \mathbf{e}'_{l_j}$ *for all* $k \in \mathbb{R}$ *then* $\mathbf{D} \in \mathscr{D}$ *iff*

$$\mathbf{e}'_i \mathbf{D} \mathbf{e}_i = 0, \quad i = 1,2,\dots,n$$

and

$$\mathbf{X}'\mathbf{D}\mathbf{X} = \mathbf{O}.$$

Proof follows from (1.2)–(1.4) and Lemma 4.5.1 is omitted.

A consequence of the considerations in Sections 4.3, 4.4 and 4.5 is the next theorem:

Theorem 4.5.3. *If in model* (1.1)

(i) $\mathbf{E} = \begin{pmatrix} \gamma \mathbf{e}'_s \\ \sum_{j=1}^{t} \delta_{l_j} \mathbf{e}'_{l_j} \end{pmatrix}$,

$s \in \{1,2,\dots,n-2\}$, $l_j \in \{1,2,\dots,n-2\}$, $\delta_j \neq 0$, $j = 1,2,\dots,t$, $t \geqslant 2$, $\gamma \neq 0$ *or*

(ii) $\mathbf{E} = \begin{pmatrix} \sum_{i=1}^{t} \gamma_{s_i} \mathbf{e}'_{s_i} \\ k \sum_{i=1}^{t} \gamma_{s_i} \mathbf{e}'_{s_i} \end{pmatrix}$,

$s \in \{1,2,\dots,n-2\}$, $\gamma_{s_i} \neq 0$, $i = 1,2,\dots,t$, $t \geqslant 2$, $k \neq 0$ *or*

(iii) $\mathbf{E} = \begin{pmatrix} \sum_{i=1}^{t} \gamma_{s_i} \mathbf{e}'_{s_i} \\ \sum_{j=1}^{u} \delta_{l_j} \mathbf{e}'_{l_j} \end{pmatrix}$,

$s_i \in \{1,2,\dots,n-2\}$, $\gamma_{s_i} \neq 0$, $i = 1, 2, \dots, t$, $t \geqslant 2$, $l_j \in \{1,2,\dots,n-2\}$, $\delta_{l_j} \neq 0$, $j = 1, 2, \dots, u$, $u \geqslant 2$, $k \sum_{i=1}^{t} \gamma_{s_i} \mathbf{e}'_{s_i} \neq \sum_{j=1}^{u} \delta_{l_j} \mathbf{e}'_{l_j}$ *for all* $k \in \mathbb{R}$ *then* (1.15) *is true and the dispersion of the* β_0-LBLQUE *of* $\mathbf{f}\beta$ *is the same as the dispersion of the* β_0-LBLUE *of* $\mathbf{f}'\beta$ at $\beta = \beta_0$.

Concluding remarks

We have investigated the β_0-LBLQUE of a linear functional of a parameter β in model (1.1) in all possible situations with none, one or two additional linear dependent measurements, i.e. if

(i) $R(\mathbf{X}) = n \leqslant k$,

(ii) $R(\mathbf{X}) = n - 1 \leqslant k$ and $\mathbf{E} = \gamma \mathbf{e}'_s$, $\gamma \neq 0$, $s \in \{1,2,\dots,n-1\}$ or $\mathbf{E} = \sum_{i=1}^{t} \gamma_i \mathbf{e}'_{s_i}$, $\gamma_i \neq 0$, $s_i \in \{1,2,\dots,n-1\}$ for $i = 1, 2, \dots, t$, $t \geqslant 2$ or

(iii) $R(\mathbf{X}) = n-2 \leqslant k$ and $\mathbf{E} = \binom{\gamma \mathbf{e}'_s}{\delta \mathbf{e}_s}$, $s \in \{1,2,\dots,n-2\}$, $|\gamma| \neq 0$, $\delta \neq 0$, or $\mathbf{E} = \binom{\gamma \mathbf{e}'_s}{\delta \mathbf{e}_l}$, $s \in \{1,2,\dots,n-2\}$, $l \in \{1,2,\dots,n-2\}$, $s \neq l$, $|\gamma| \neq 0$, $\delta \neq 0$, or $\mathbf{E} = \begin{pmatrix} \gamma \mathbf{e}'_s \\ \sum_{j=1}^t \delta_{l_j} \mathbf{e}'_{l_j} \end{pmatrix}$, $s \in \{1,2,\dots,n-2\}$, $l \in \{1,2,\dots,n-2\}$, $\delta_j \neq 0$, $j = 1,2,\dots,t$, $t \geqslant 2$, $\gamma \neq 0$, or $\mathbf{E} = \begin{pmatrix} \sum_{i=1}^t \gamma_{s_i} \mathbf{e}'_{s_i} \\ k \sum_{i=1}^t \gamma_{s_i} \mathbf{e}'_{s_i} \end{pmatrix}$, $s \in \{1,2,\dots,n-2\}$, $\gamma_{s_i} \neq 0$, $i = 1,2,\dots,t$, $t \geqslant 2$, $k \neq 0$, or $\mathbf{E} = \begin{pmatrix} \sum_{i=1}^t \gamma_{s_i} \mathbf{e}'_{s_i} \\ \sum_{j=1}^u \delta_{l_j} \mathbf{e}'_{l_j} \end{pmatrix}$, $s \in \{1,2,\dots,n-2\}$, $\gamma_{s_i} \neq 0$, $j = 1,2,\dots,t$, $t \geqslant 2$, $l_j \in \{1,2,\dots,n-2\}$, $\delta_{l_j} \neq 0$, $j = 1,2,\dots,u$, $u \geqslant 2$, $k \sum_{i=1}^t \gamma_{s_i} \mathbf{e}'_{s_i} \neq \sum_{j=1}^u \delta_{l_j} \mathbf{e}'_{l_j}$ for all $k \in \mathbb{R}$.

It was shown that only in two cases with special replicated observations: (a) $\mathbf{E} = \binom{\gamma \mathbf{e}'_s}{\delta \mathbf{e}'_s}$, $s \in \{1,2,\dots,n-2\}$, $|\gamma| \neq 1$, $|\delta| \neq 1$, $\gamma \neq \delta$, $\delta \neq \gamma$ and

(b) $\mathbf{E}\binom{\gamma \mathbf{e}'_s}{\delta \mathbf{e}'_s}$, $s \in \{1,2,\dots,n-2\}$, $l \in \{1,2,\dots,n-2\}$, $s \neq l$, $|\gamma| \neq 1$, $\gamma \neq 0$, $\delta \neq 1$, $\delta \neq 0$ the β_0-LBLQUE of $\mathbf{f}'\beta$ may have lower dispersion that the β_0-LBLUE of $\mathbf{f}'\beta$ at $\beta = \beta_0$. The class of linear functionals having β_0-LBLQUE is in all investigated cases the same as the class of linear functionals having β_0-LBLUE (Theorem 4.1.3 and Theorem 4.2.3). In the paper the problem of obtaining the β_0-LBLQUE of the linear functional of a parameter β is also solved.

References

[1] *V. Fajt*: Electrical measurements. SNTL/ALFA, Praha, 1978. (In Czech.)

[2] *Guido del Pino*: The unifying role of iterative generalized least squares in statistical algorithms. Statistical Science *4* (1980), 394–408.

[3] *J. Nelder and R. Wedderburn*: Generalized linear models. J. Roy. Statist. Soc. *Ser A* (1972), 370–384.

[4] *C.R. Rao and S.K. Mitra*: Generalized Inverse of Matrices and Its Applications. J. Wiley, New York, 1971.

[5] *K. Rinner and F. Benz*: Jordan/Eggert/Kneissl Handbuch der Vermessungskunde. Band VI, Stuttgart, 1966.

[6] *R. Wedderburn*: Quasi-likelihood functions, generalized linear models and the Gauss-Newton method. Biometrika *61* (1974), 439–447.

[7] *G. Wimmer*: Linear model with variance depending on the mean value. Mathematica Slovaca *42* (1992), 223–238.

[8] *G. Wimmer*: Estimation in a special structure of the linear model. Mathematica Slovaca *43* (1993), 221–264.

Author's address: *Gejza Wimmer*, Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia.