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SHAPE OPTIMIZATION BY MEANS OF THE PENALTY METHOD WITH EXTRAPOLATION

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Summary. A model shape optimal design in R^2 is solved by means of the penalty method with extrapolation, which enables to obtain high order approximations of both the state function and the boundary flux, thus offering a reliable gradient for the sensitivity analysis. Convergence of the proposed method is proved for certain subsequences of approximate solutions.

Keywords: shape optimization, penalty method, extrapolation, finite elements

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INTRODUCTION

In optimal shape design, one usually requires the gradient of the cost functional with respect to design variables. The latter gradient can be frequently expressed by a boundary integral involving the boundary flux of the state function. Thus one needs an efficient and reliable method to compute both the solution of elliptic problems and its boundary flux. Such a method has been proposed for Dirichlet boundary value problems by J.T. King in 1974 [11] and developed by King and S.M. Serbin in [12], [13]. They called it penalty method with extrapolation. The author of the present paper extended the method to some axisymmetric 3-D problems in [8].

The aim of the present paper is to apply the penalty method with extrapolation to the sensitivity analysis for optimal shape design problems. We consider a simple model state problem, i.e., a Poisson equation in a bounded two-dimensional domain with a homogeneous Dirichlet boundary condition and two frequent cost functionals. In Section 1 the method of penalty and some error estimates given by Babuška in

[1] are extended to non-smooth domains with convex corners. We recall the main results of King and Serbin [11], [12], [13] on the penalty method with extrapolation in Section 2.

Basic definitions and relations of a model shape optimal design problem are displayed in Section 3 together with some sensitivity analysis. Then an application of the penalty method with extrapolation is proposed. In Section 4 we present some theoretical analysis of the method. We discuss the existence of a solution to the approximate optimization problems and prove the main convergence theorem. We show that having a sequence of solutions of the approximate optimization problem, then a subsequence exists, which converges to a solution of the original optimal shape problem.

1. The penalty method for plane domains with convex corners

The purpose of this section is to derive a priori error estimates for the penalty method applied to bounded two-dimensional domains with a piecewise smooth boundary. Thus we extend some results of Babuška [1], who assumed that the boundary is of class C^∞, to a class of domains with corners.

Henceforth we assume that

(A1) the domain $\Omega \subset \mathbb{R}^2$ is bounded, with Lipschitz-continuous boundary $\partial\Omega \equiv \Gamma$, which consists of a finite number of smooth arcs Γ_j, $j = 1, \dots, N$, of class $W^{2,\infty}$. In corner points, their tangents generate interior angles $\omega_j \in (0, \pi]$.

We employ the standard notation for Sobolev spaces $W^{k,2}(\Omega) \equiv H^k(\Omega)$, where $k \geqslant 0$ (k need not be integer), with the norms $\|\cdot\|_{k,\Omega}$; $H^0(\Omega) \equiv L^2(\Omega)$.

On the boundary Γ we define norms $\|\cdot\|_{s,\Gamma}$ in spaces $H^s(\Gamma)$, $s \geqslant 0$ (cf. [15], §5.2, p. 94 and (1.8) below). We denote

$$\int_\Omega fg\,\mathrm{d}x = (f, g), \qquad \int_{\partial\Omega} fg\,\mathrm{d}s = \langle f, g\rangle,$$

and the seminorm

$$|u|^2_{1,\Omega} = \sum_{i=1}^{2} \Big(\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_i}\Big).$$

Finally, we introduce the summation convention, i.e., any repeated index implies summation within the range 1, 2, unless another sum is indicated.

We shall need the following lemma which follows from a result of Kadlec [10].

Lemma 1.1. *Let the boundary $\partial\Omega$ fulfil the assumption* (A1), *let u be the weak solution of the problem*

$$(1.1)\qquad -\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $f \in L^2(\Omega)$, $a_{ij} \in C^{(0),1}(\bar{\Omega})$, $a_{12} = a_{21}$ and $a_{ij}t_it_j \geqslant c_0|t|^2$ $\forall t \in \mathbb{R}^2$, $\forall x \in \bar{\Omega}$.

Then $u \in H^2(\Omega)$ and

$$\|u\|_{2,\Omega} \leqslant C\|f\|_{0,\Omega}$$

with C independent of f.

Lemma 1.2. *Let Ω be a bounded domain with Lipschitz-continuous boundary Γ and let $\varepsilon \in (0, \frac{1}{2}]$. Then the trace mapping, which is defined for $u \in C(\bar{\Omega})$, has a unique continuous extension as an operator from $H^{1/2+\varepsilon}(\Omega)$ onto $H^{\varepsilon}(\Gamma)$.*

The *proof* follows immediately from Theorem 1.5.1.2 in the book [5], for $\varepsilon = s - 1/2$, $p = 2$, $k = 0$, $\ell = 0$.

In what follows, we denote

$$\frac{\partial u}{\partial \nu_A} = a_{ij}\nu_j\frac{\partial u}{\partial x_i},$$

the boundary flux, where ν_j are components of the unit outward normal to $\partial\Omega \equiv \Gamma$.

Lemma 1.3. *Let the boundary $\partial\Omega$ satisfy the assumption* (A1) *and let $w \in H^2(\Omega) \cap H^1_0(\Omega)$. Then*

$$\frac{\partial w}{\partial \nu_A} \in H^{1/2}(\Gamma)$$

and

$$(1.2)\qquad \Big\|\frac{\partial w}{\partial \nu_A}\Big\|_{1/2,\Gamma} \leqslant C\|w\|_{2,\Omega}.$$

Proof. First let us consider a small neighbourhood of a corner O (see Fig. 1). In general, we have

$$\frac{\partial w}{\partial \nu} = \frac{\partial w}{\partial \nu_A}\frac{1}{\nu_A\cdot\nu}, \qquad \nabla w = \frac{\partial w}{\partial \nu_A}\frac{1}{\nu_A\cdot\nu}\,\nu$$

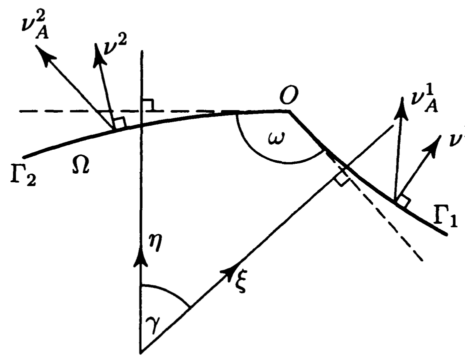

Fig. 1

where $\nu_A \cdot \nu = a_{ij}\nu_i\nu_j$.

Choosing a new coordinate system ξ, η in the directions of normals at the corner O, we easily obtain that

$$\frac{\partial w}{\partial \xi} = \frac{\partial w}{\partial \nu_A^1}\frac{\cos(\nu^1\xi)}{\nu_A^1 \cdot \nu^1}, \quad \frac{\partial w}{\partial \eta} = \frac{\partial w}{\partial \nu_A^1}\frac{\cos(\nu^1\eta)}{\nu_A^1 \cdot \nu^1} \quad \text{on } \Gamma_1,$$
$$\frac{\partial w}{\partial \xi} = \frac{\partial w}{\partial \nu_A^2}\frac{\cos(\nu^2\xi)}{\nu_A^2 \cdot \nu^2}, \quad \frac{\partial w}{\partial \eta} = \frac{\partial w}{\partial \nu_A^2}\frac{\cos(\nu^2\eta)}{\nu_A^2 \cdot \nu^2} \quad \text{on } \Gamma_2.$$

Let us define the two following constants

$$\alpha = \sin^{-2}\gamma[(\nu_A^1 \cdot \nu^1)|_0 - (\nu_A^2 \cdot \nu^2)|_0 \cos\gamma],$$
$$\beta = \sin^{-2}\gamma[(\nu_A^2 \cdot \nu^2)|_0 - (\nu_A^1 \cdot \nu^1)|_0 \cos\gamma],$$

Then

$$\alpha\frac{\partial w}{\partial \xi} + \beta\frac{\partial w}{\partial \eta} = \begin{cases} \frac{\partial w}{\partial \nu_A^1}\left(\frac{\alpha}{\nu_A^1 \cdot \nu^1}\cos(\nu^1\xi) + \frac{\beta}{\nu_A^1 \cdot \nu^1}\cos(\nu^1\eta)\right) & \text{on } \Gamma_1, \\ \frac{\partial w}{\partial \nu_A^2}\left(\frac{\alpha}{\nu_A^2 \cdot \nu^2}\cos(\nu^2\xi) + \frac{\beta}{\nu_A^2 \cdot \nu^2}\cos(\nu^2\eta)\right) & \text{on } \Gamma_2, \end{cases}$$

so that on $\Gamma_1 \cup \Gamma_2 \dot{-} O$ we may write

$$\frac{\partial w}{\partial \nu_A} = \left(\alpha\frac{\partial w}{\partial \xi} + \beta\frac{\partial w}{\partial \eta}\right)z \tag{1.3}$$

where the function z is defined as follows

$$z = \begin{cases} \left[\frac{\alpha}{\nu_A^1 \cdot \nu^1}\cos(\nu^1\xi) + \frac{\beta}{\nu_A^1 \cdot \nu^1}\cos(\nu^1\eta)\right]^{-1} & \text{on } \Gamma_1, \\ \left[\frac{\alpha}{\nu_A^2 \cdot \nu^2}\cos(\nu^2\xi) + \frac{\beta}{\nu_A^2 \cdot \nu^2}\cos(\nu^2\eta)\right]^{-1} & \text{on } \Gamma_2. \end{cases}$$

We can determine the value $z(O)$ in such a way, that the function z will be continuous at the point O and there exists a neighbourhood B_0 of O, such that $z \in C^{(0),1}(\Gamma_1 \cup \Gamma_2 \cap B_0)$.

Let us denote $y \equiv z^{-1}$, for the time being. If Γ_1 is the graph of a function $\xi(t)$, then $\xi \in W^{2,\infty}$ by assumption (A1) and $\cos(\nu^1\xi) = (1+\xi'^2)^{-1/2}$, $\cos(\nu^1\eta) = \cos(\gamma - (\nu^1\xi))$, $\nu_A^1 \cdot \nu^1 = \tilde{a}_{ij}\nu_i^1\nu_j^1 \in C^{(0),1}([0,b))$, for some $b > 0$. Consequently, $y \in C^{(0),1}(\Gamma_1 \cap B_{01})$, where B_{01} is a sufficiently small neighbourhood of the corner O. Moreover,

$$\lim_{t\to 0_+} y(t) = \frac{\alpha}{(\nu_A^1 \cdot \nu^1)|_0} + \frac{\beta}{(\nu_A^1 \cdot \nu^1)|_0}\cos\gamma = 1.$$

A similar analysis is true on $\Gamma_2 \cap B_{02}$ and we have again

$$\lim_{t\to 0_-} y(t) = \frac{\alpha}{(\nu_A^2 \cdot \nu^2)|_0}\cos\gamma + \frac{\beta}{(\nu_A^2 \cdot \nu^2)|_0} = 1.$$

Thus setting $z(O) = 1$, we obtain $z \in C^{(0),1}(\Gamma_1 \cup \Gamma_2 \cap B_0)$, where B_0 is a sufficiently small neighbourhood of the point O.

It is easy to verify that

$$u \in H^{1/2}(I), \quad \zeta \in C^{(0),1}(I) \implies u\zeta \in H^{1/2}(I) \tag{1.4}$$

and

$$\|u\zeta\|_{1/2,I} \leqslant C(\zeta)\|u\|_{1/2,I} \tag{1.5}$$

holds for any bounded interval $I = [a,b]$.

Since the traces of both $\partial w/\partial\xi$ and $\partial w/\partial\eta$ belong to $H^{1/2}(\Gamma)$, their linear combination $(\alpha\partial w/\partial\xi + \beta\partial w/\partial\eta)_\Gamma \in H^{1/2}(\Gamma)$. Applying (1.4), (1.5) to (1.3) and the Trace theorem, we arrive at

$$\frac{\partial w}{\partial\nu_A} \in H^{1/2}(\Gamma_1 \cup \Gamma_2 \cap B_0), \tag{1.6}$$

$$\left\|\frac{\partial w}{\partial\nu_A}\right\|_{\frac{1}{2},\Gamma_1\cup\Gamma_2\cap B_0} \leqslant \tilde{C}\left\|\alpha\frac{\partial w}{\partial\xi} + \beta\frac{\partial w}{\partial\eta}\right\|_{\frac{1}{2},\Gamma} \leqslant C\|w\|_{2,\Omega}. \tag{1.7}$$

Let us recall the definition of the norm in $H^{1/2}(\Gamma)$ (see [15—§5.2, p. 94]). There exists a finite covering of the boundary Γ by open sets B_i, $i = 1,\dots,m$ and

$$\left\|\frac{\partial w}{\partial\nu_A}\right\|_{1/2,\Gamma} = \left(\sum_{i=1}^{m}\left\|\frac{\partial w}{\partial\nu_A}\right\|_{1/2,\Delta_i}\right)^{1/2}, \tag{1.8}$$

where Δ_i are projections of $\Gamma \cap B_i$. Let the sets B_i be chosen for all corners in accordance with (1.6). For the remaining B_i, which cover the interiors of the arcs Γ_j, the assumptions $\Gamma_j \in W^{2,\infty}$ and $a_{ij} \in C^{(0),1}$ imply that the components of ν_A^j are Lipschitz continuous on $\Gamma_j \cap B_i$. Consequently,

$$\frac{\partial w}{\partial \nu_A}\Big|_{\Gamma_j \cap B_i} \in H^{1/2}(\Delta_i)$$

follows from (1.4).

Combining (1.8) with the estimates on Δ_i, we obtain (1.2). □

We introduce the second assumption

(A2) for any $h \in (0,1]$ there exists a finite dimensional subspace $V_h \subset H^1(\Omega)$ such that:

for any $u \in H^\ell(\Omega)$, $l = 1,2$, there exists $v_h \in V_h$ such that

$$\|u - v_h\|_{s,\Omega} \leqslant Ch^{\ell - s}\|u\|_{\ell,\Omega}$$

holds for all $s \in [0,1]$.

Lemma 1.4. *Suppose the assumptions* (A1), (A2) *are fulfilled and let* $w \in H^2(\Omega) \cap H_0^1(\Omega)$.

Then there exists $g_h \in V_h$ *such that*

$$\|w - g_h\|_{1,\Omega}^2 + \gamma h^{-1}\Big\|h\gamma^{-1}\frac{\partial w}{\partial \nu_A} + g_h\Big\|_{0,\Gamma}^2 \leqslant C(\gamma,\varepsilon)h^{2-\varepsilon}\|w\|_{2,\Omega}^2$$

holds for any positive γ, ε *with* $C(\gamma,\varepsilon)$ *independent of* h, w.

Proof. (see [1], where $\partial\Omega \in C^\infty$). Lemma 1.3 yields that $\frac{\partial w}{\partial \nu_A} \in H^{1/2}(\Gamma)$ and

$$\Big\|\frac{\partial w}{\partial \nu_A}\Big\|_{1/2,\Gamma} \leqslant C\|w\|_{2,\Omega}.$$

There exists a function $V \in H^1(\Omega)$ (see [15—Thm. 5.7, p. 103]) such that the trace $V|_\Gamma = \frac{\partial w}{\partial \nu_A}$ and

$$\|V\|_{1,\Omega} \leqslant \tilde{C}\Big\|\frac{\partial w}{\partial \nu_A}\Big\|_{1/2,\Gamma} \leqslant C\|w\|_{2,\Omega}. \tag{1.9}$$

By assumption (A2) and (1.9) there exists a function $\varphi_h \in V_h$ such that

$$\|V - \varphi_h\|_{s,\Omega} \leqslant \tilde{C}h^{1-s}\|V\|_{1,\Omega} \leqslant Ch^{1-s}\|w\|_{2,\Omega} \tag{1.10}$$

holds for all $s \in [0,1]$.

From Lemma 1.2 and (1.10) we obtain

$$\|V-\varphi_h\|_{0,\Gamma} \leqslant \|V-\varphi_h\|_{\varepsilon_0,\Gamma} \leqslant \tilde{C}(\varepsilon_0)\|V-\varphi_h\|_{1/2+\varepsilon_0,\Omega} \leqslant C(\varepsilon_0)h^{1/2-\varepsilon_0}\|w\|_{2,\Omega}. \tag{1.11}$$

By (A2) and Lemma 1.2 we deduce that there exists a $\xi_h \in V_h$ such that

$$\|w-\xi_h\|_{1,\Omega} \leqslant Ch\|w\|_{2,\Omega}, \tag{1.12}$$

$$\|\xi_h\|_{0,\Gamma} = \|\xi_h - w\|_{0,\Gamma} \leqslant \tilde{C}(\varepsilon_0)\|\xi_h - w\|_{1/2+\varepsilon_0,\Omega} \leqslant C(\varepsilon_0)h^{3/2-\varepsilon_0}\|w\|_{2,\Omega}. \tag{1.13}$$

If we set

$$g_h = \xi_h - h\gamma^{-1}\varphi_h,$$

then

$$\begin{aligned}
&\|w-g_h\|^2_{1,\Omega} + \gamma h^{-1}\Big\|h\gamma^{-1}\frac{\partial w}{\partial \nu_A} + g_h\Big\|^2_{0,\Gamma}\\
&= \|w-\xi_h + Vh\gamma^{-1} - h\gamma^{-1}(V-\varphi_h)\|^2_{1,\Omega}\\
&\quad + \gamma h^{-1}\Big\|h\gamma^{-1}\Big(\frac{\partial w}{\partial \nu_A} - V\Big) + h\gamma^{-1}(V-\varphi_h) + \xi_h\Big\|^2_{0,\Gamma}\\
&\leqslant C\big[\|w-\xi_h\|^2_{1,\Omega} + h^2\gamma^{-2}\|V-\varphi_h\|^2_{1,\Omega} + h^2\gamma^{-2}\|V\|^2_{1,\Omega} + \gamma h^{-1}\|\xi_h\|^2_{0,\Gamma}\\
&\quad + h\gamma^{-1}\|V-\varphi_h\|^2_{0,\Gamma}\big] \leqslant \tilde{C}(\gamma,\varepsilon_o)h^{2-2\varepsilon_0}\|w\|^2_{2,\Omega}
\end{aligned}$$

follows from the estimates (1.12), (1.10), (1.9), (1.13) and (1.11). □

Definition 1.1. Let us denote

$$a(u,v) = \Big(a_{ij}\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i}\Big).$$

We say that $u_h(\gamma) \in V_h$ is an *approximation of the solution by the penalty method*, if

$$a(u_h(\gamma),v) + \gamma h^{-1}\langle u_h(\gamma), v\rangle = (f,v) \quad \forall v \in V_h. \tag{1.14}$$

It is readily seen that for any $\gamma > 0$ there exists a unique solution of (1.14). In fact, the bilinear form

$$\mathcal{A}_\gamma(u,v) \equiv a(u,v) + \gamma h^{-1}\langle u,v\rangle$$

is symmetric, continuous and positive definite on $V_h \times V_h$, since

$$\mathcal{A}_\gamma(v,v) \geqslant c(\gamma)\|v\|^2_{1,\Omega} \tag{1.14'}$$

holds for all $v \in H^1(\Omega)$ due to the Friedrichs inequality.

Now everything is ready to prove the main

Theorem 1.1. *Suppose the assumptions* (A1), (A2), $f \in L^2(\Omega)$. *Let* u_0 *be the (weak) solution of the Dirichlet problem* (1.1) *and let* $u_h(\gamma)$ *be the approximation by the penalty method.*

Then $u_0 \in H^2(\Omega)$ *and the following estimates hold*

$$\|u_0 - u_h(\gamma)\|_{1,\Omega} \leqslant C(\gamma,\varepsilon)h^{1-\varepsilon}\|f\|_{0,\Omega}, \tag{1.15}$$

$$\|u_h(\gamma)\|_{0,\Gamma} \leqslant C(\gamma,\varepsilon)h\|f\|_{0,\Omega}, \tag{1.16}$$

$$\left\|\frac{\partial u_0}{\partial \nu_A} + \gamma h^{-1} u_h(\gamma)\right\|_{0,\Gamma} \leqslant C(\gamma,\varepsilon)h^{1/2-\varepsilon/2}\|f\|_{0,\Omega}, \tag{1.17}$$

where $\varepsilon > 0$ *is arbitrary.*

Proof. (Cf. [1] for the case $\partial\Omega \in C^\infty$). It is easy to verify that

$$F(v) = a(v,v) - 2(f,v) + \gamma h^{-1}\|v\|^2_{0,\Gamma}$$

is the potential associated with the penalty method (1.14). Let us introduce another functional for $v \in H^1(\Omega)$, namely

$$R(v) = a(u_0 - v, u_0 - v) + \gamma h^{-1}\left\|h\gamma^{-1}\frac{\partial u_0}{\partial \nu_A} + v\right\|^2_{0,\Gamma}.$$

By virtue of Lemma 1.1, u_0 belongs to $H^2(\Omega)$ and Lemma 1.3 implies that $\partial u_0/\partial \nu_A \in H^{1/2}(\Omega)$; thus the definition is senseful.

Since

$$a(u_0, v) = (f,v) + \left\langle \frac{\partial u_0}{\partial \nu_A}, v\right\rangle, \tag{1.18}$$

we may write

$$\begin{aligned} R(v) &= a(v,v) - 2(f,v) + \gamma h^{-1}\|v\|^2_{0,\Gamma} + h\gamma^{-1}\left\|\frac{\partial u_0}{\partial \nu_A}\right\|^2_{0,\Gamma} + a(u_0,u_0) \\ &= F(v) + K(u_0), \end{aligned}$$

where

$$K(u_0) = a(u_0,u_0) + h\gamma^{-1}\left\|\frac{\partial u_0}{\partial \nu_A}\right\|^2_{0,\Gamma}$$

is independent of v.

Consequently, we have

$$u_h(\gamma) = \operatorname*{argmin}_{V_h} F(v) = \operatorname*{argmin}_{V_h} R(v). \tag{1.19}$$

Lemma 1.4 yields the existence of a function $g_h \in V_h$ such that

$$R(g_h) \leqslant C\|u_0 - g_h\|_{1,\Omega}^2 + \gamma h^{-1}\left\|h\gamma^{-1}\frac{\partial u_0}{\partial \nu_A} + g_h\right\|_{0,\Gamma}^2$$
$$\leqslant C(\gamma,\varepsilon)h^{2-\varepsilon}\|u_0\|_{2,\Omega}^2$$

holds for any $\varepsilon > 0$.

From Lemma 1 it follows that

$$\|u_0\|_{2,\Omega} \leqslant C\|f\|_{0,\Omega}.$$

Making use of (1.19), we may write

$$R(u_h) \leqslant R(g_h) \leqslant C(\gamma,\varepsilon)h^{2-\varepsilon}\|f\|_{0,\Omega}^2. \tag{1.20}$$

Next we have

$$a(u_0 - u_h, u_0 - u_h) \leqslant R(u_h), \tag{1.21}$$

$$\left\|h\gamma^{-1}\frac{\partial u_0}{\partial \nu_A} + u_h\right\|_{0,\Gamma}^2 \leqslant \gamma^{-1}hR(u_h). \tag{1.22}$$

Using the triangle inequality, (1.22) and Lemma 1.3, we obtain

$$\begin{aligned}\|u_h(\gamma)\|_{0,\Gamma} &\leqslant \left\|u_h + h\gamma^{-1}\frac{\partial u_0}{\partial \nu_A}\right\|_{0,\Gamma} + h\gamma^{-1}\left\|\frac{\partial u_0}{\partial \nu_A}\right\|_{0,\Gamma}\\ &\leqslant [(\gamma^{-1}h)^{1/2}C^{1/2}(\gamma,\varepsilon)h^{1-\varepsilon/2} + Ch\gamma^{-1}]\|f\|_{0,\Omega}\\ &\leqslant C_1(\gamma,\varepsilon)h\|f\|_{0,\Omega}.\end{aligned} \tag{1.23}$$

Combining (1.20), (1.21), (1.23) and the Friedrichs inequality, we arrive at

$$\begin{aligned}C\|u_0 - u_h\|_{1,\Omega}^2 &\leqslant |u_0 - u_h|_{1,\Omega}^2 + \|u_0 - u_h\|_{0,\Gamma}^2\\ &\leqslant C(\gamma,\varepsilon)h^{2-\varepsilon}\|f\|_{0,\Omega}^2 + C_1^2(\gamma,\varepsilon)h^2\|f\|_{0,\Omega}^2 \leqslant C_2(\gamma,\varepsilon)h^{2-\varepsilon}\|f\|_{0,\Omega}^2.\end{aligned}$$

On the basis of (1.22) and (1.20) we deduce

$$\begin{aligned}\left\|\frac{\partial u_0}{\partial \nu_A} + \gamma h^{-1}u_h\right\|_{0,\Gamma} &= \gamma h^{-1}\left\|\gamma^{-1}h\frac{\partial u_0}{\partial \nu_A} + u_h\right\|_{0,\Gamma}\\ &\leqslant (\gamma h^{-1})^{1/2}C_3(\gamma,\varepsilon)h^{1-\varepsilon/2}\|f\|_{0,\Omega} = \tilde{C}(\gamma,\varepsilon)h^{1/2-\varepsilon/2}\|f\|_{0,\Omega}.\end{aligned}$$

□

2. Penalty method and extrapolation

The rate of convergence $O(h^{1-\varepsilon})$ of the penalty method can be increased by means of an extrapolation, i.e., by a suitable linear combination of several approximations $u_h(\gamma_i)$ with different γ_i, as was shown by J.T. King in [11]. To this end, however, a higher regularity of the solution u_0 is required together with C^∞-smoothness of the boundary $\partial\Omega$.

We define the k-th *extrapolate* as follows

$$u_h^{(k)} = \sum_{i=0}^{k} a_i u_h(\gamma_i), \quad k \geqslant 1, \tag{2.1}$$

where

$$0 < \gamma_0 < \ldots < \gamma_k,$$

and the coefficients a_i satisfy the linear system

$$\begin{aligned} &\sum_{i=0}^{k} a_i = 1, \\ &\sum_{i=0}^{k} a_i \gamma_i^{-j} = 0, \quad j = 1, \ldots, k. \end{aligned} \tag{2.2}$$

As the determinant of (2.2) is a Vandermonde, the system has a unique solution.

If the assumptions of Theorem 1.1 are fulfilled, then

$$\|u_h^{(k)} - u_0\|_{1,\Omega} \leqslant C(\gamma_0, \ldots, \gamma_k) h^{1-\varepsilon} \|f\|_{0,\Omega} \tag{2.3}$$

holds with arbitrary $\varepsilon > 0$.

To see this, we write

$$\begin{aligned} \|u_h^{(k)} - u_0\|_{1,\Omega} &= \Big\| \sum_{i=0}^{k} a_i \big(u_h(\gamma_i) - u_0\big) \Big\|_{1,\Omega} \\ &\leqslant \sum_{i=0}^{k} |a_i| \|u_h(\gamma_i) - u_0\|_{1,\Omega} \end{aligned}$$

end employ the estimates (1.15).

Consequently, the extrapolates converge at least as the approximations by penalty method.

In what follows, we assume that the boundary $\partial\Omega$ is of class C^∞ and the solution $u_0 \in H^s(\Omega)$, where $s \geqslant 3$. We present results of J.T. King and M. Serbin. For the proofs we refer to their papers [11], [12].

Consider the elliptic operator

$$Au \equiv -\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big), \tag{2.4}$$

where

$$a_{ij} = a_{ji} \in C^\infty(\bar{\Omega}). \tag{2.5}$$

Let k be an integer, $1 \leqslant k \leqslant s-2$. We define w_1 the solution of the following problem

$$Aw_1 = 0 \text{ in } \Omega, \quad w_1 = -\frac{\partial u_0}{\partial \nu_A} \text{ on } \Gamma, \tag{2.6}$$

for $2 \leqslant j \leqslant k$ let w_j be the solution of

$$Aw_j = 0 \text{ in } \Omega, \quad w_j = -\frac{\partial w_{j-1}}{\partial \nu_A} \text{ on } \Gamma. \tag{2.7}$$

We have (cf. [14])

$$\|w_j\|_{s-j,\Omega} \leqslant C\|u_0\|_{s,\Omega}. \tag{2.8}$$

In the paper [11] the following assumption is introduced.

(A3) Let $r \geqslant 2$ be an integer. For any $h \in (0,1)$ there exists a finite dimensional subspace $V_h^r \subset H^1(\Omega)$ such that for any $u \in H^s(\Omega)$, $2 \leqslant s \leqslant r$, there exists a constant C, independent of h and u, and $v_h \in V_h^r$ such that

$$\|u - v_h\|_{0,\Omega} + h\|u - v_h\|_{1,\Omega} \leqslant Ch^s\|u\|_{s,\Omega}. \tag{2.9}$$

For examples of such subspaces we refer the reader to the finite element method [3].

The following estimate can be established [11—Theorem 3.1].

Theorem 2.1. *Suppose $u_0 \in H^s(\Omega)$, $r \geqslant s \geqslant 3$, is the solution of the Dirichlet problem*

$$\begin{aligned} Au &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma; \end{aligned} \tag{2.10}$$

Let $u_h(\gamma)$ be the approximation of the solution by the penalty method (i.e., the solution of (1.14)*) and let us assume* (A3).

Then if k is any positive integer $k \leqslant s-2$, we have

$$\Big\| u_h(\gamma) - u_0 - \sum_{j=1}^{k} (\gamma^{-1}h)^j w_j \Big\|_{1,\Omega} \leqslant C(\gamma) h^{k+1} \|u_0\|_{s,\Omega}. \tag{2.11}$$

Corollary 2.1. *If the assumptions of Theorem 2.1 are fulfilled, then*

$$\|u_h^{(k)} - u_0\|_{1,\Omega} \leqslant C(\gamma_0, \ldots, \gamma_k) h^{k+1} \|u_0\|_{s,\Omega}. \tag{2.12}$$

Proof. On the basis of the formulas (2.1), (2.2) and (2.11) we obtain

$$\begin{aligned}\|u_h^{(k)} - u_0\|_{1,\Omega} &= \Big\| \sum_{i=0}^{k} a_i \Big(u_h(\gamma_i) - u_0 - \sum_{j=1}^{k} \gamma_i^{-j} h^j w_j \Big) \Big\|_{1,\Omega} \\ &\leqslant \sum_{i=0}^{k} |a_i| C(\gamma_i) h^{k+1} \|u_0\|_{s,\Omega},\end{aligned}$$

so that (2.12) follows. □

Remark 2.1. In [1] Babuška derived and King in [11] generalized some error estimates for the penalty method, where h^{-1} in (1.14) was replaced by a more general $h^{-\sigma}$, $\sigma \geqslant 1$. Not even for the best choice of the parameter σ, however, the error bound is quasioptimal [11].

The penalty method with extrapolation brings a remedy—it yields a *quasioptimal error estimate.* Indeed, if $s = r = k + 2$, (2.12) yields that

$$\|u_h^{(r-2)} - u_0\|_{1,\Omega} \leqslant C h^{r-1} \|u_0\|_{r,\Omega}. \tag{2.13}$$

Thus using finite elements with quadratic polynomials on the reference triangle according to Zlámal [16], for $k = 1$ and $r = s = 3$ we obtain

$$\|u_h^{(1)} - u_0\|_{1,\Omega} \leqslant C h^2 \|u_0\|_{2,\Omega}. \tag{2.14}$$

Remark 2.2. From Theorem 2.1 and (2.9) we can conclude that the dominant term in the error expansion of the penalty method is the term $\sum_{j=1}^{k} (\gamma^{-1}h)^j w_j$.

In the paper [12], King and Serbin proposed also an *approximation of the boundary flux* on the basis of the penalty method with extrapolation. To illustrate the motivation, let us recall that (1.18), i.e.

$$\left\langle \frac{\partial u_0}{\partial \nu_A}, v \right\rangle = a(u_0, v) - (f, v)$$

holds for all $v \in H^1(\Omega)$ and from the definition (1.14)

$$a(u_h(\gamma), v) - (f, v) = -\gamma h^{-1} \langle u_h(\gamma), v \rangle \quad \forall v \in V_h.$$

Comparing these relations, the authors of [12] define

$$e_h^{(0)} = -\gamma h^{-1} u_h(\gamma) \tag{2.15}$$

as the 0-th approximation of the boundary flux.

The k-th approximation of the boundary flux for $k \geqslant 1$ is defined by

$$e_h^{(k)} = -\sum_{i=0}^{k} a_i \gamma_i h^{-1} u_h(\gamma_i), \tag{2.16}$$

where the coefficients $a_0, \ldots, a_k$ are determined by the system (2.2) and $u_h(\gamma_i)$ is evaluated at the boundary.

If $\partial\Omega$ is of class C^∞, $u_0 \in H^s(\Omega)$, the coefficients of the operator A belong to $C^\infty(\bar{\Omega})$, the subspaces V_h^r satisfy (A3), and k equals at most $s-2$, then the following error estimate holds (see [12—(2.6)]):

$$\left\| \frac{\partial u_0}{\partial \nu_A} - e_h^{(k)} \right\|_{0,\Gamma} \leqslant C(\gamma_0, \ldots, \gamma_k) h^{k+1/2} \|u_0\|_{s,\Omega}. \tag{2.17}$$

If moreover $u_0 \in H^s(\Omega)$, where $s \geqslant k+3$, then even better estimate can be proven (see [12—(2.8)])

$$\left\| \frac{\partial u_0}{\partial \nu_A} - e_h^{(k)} \right\|_{0,\Gamma} \leqslant \tilde{C}(\gamma_0, \ldots, \gamma_k) h^{k+1} \|u_0\|_{s,\Omega}. \tag{2.18}$$

Remark 2.3. For the 0-th approximation (i.e., for penalty method without extrapolation) we easily deduce that

$$\left\| \frac{\partial u_0}{\partial \nu_A} - e_h^{(0)} \right\|_{0,\Gamma} \leqslant C(\gamma) h \|u_0\|_{3,\Omega},$$

provided $u_0 \in H^3(\Omega)$.

In fact, making use of the definition (2.15) and Theorem 2.1 for $k = 1$ we obtain

$$\left\| \frac{\partial u_0}{\partial \nu_A} - e_h^{(0)} \right\|_{0,\Gamma} = \| - w_1 + \gamma h^{-1} u_h(\gamma) \|_{0,\Gamma}$$
$$\leqslant C\gamma h^{-1} \| u_h(\gamma) - u_0 - \gamma^{-1} h w_1 \|_{1,\Omega} \leqslant C(\gamma) h \| u_0 \|_{3,\Omega}.$$

Supposing $u_0 \in H^2(\Omega)$ only, we have the error bound $O(h^{1/2-\varepsilon/2})$, as follows from (1.17).

Finally, let us present several practical features of the penalty method with extrapolation.

1. The approximants are not required to satisfy any boundary conditions.
2. Matrices of the linear system, which is equivalent to the penalty method (1.14), have the form $\mathcal{A} + \gamma B$, where $\mathcal{A}$ and B are symmetric, independent of the parameter γ and $\mathcal{A} + \gamma B$ is positive definite for any $\gamma > 0$. Consequently, the same basis for V_h^r is used in the determination of any extrapolate.
3. The linear system resulting from the penalty method (1.14) has condition number of order $O(h^{-2})$, i.e., the same as that for Galerkin method. The boundary weight γh^{-1} appears to be optimal in this sense, in comparison with $\gamma h^{-\sigma}$.

An important practical problem remains to be solved, namely, how to choose the parameters γ_i. In the paper [13], the authors present a series of computational experiments with a simple model problem on a square domain, with cubic splines and errors in $L^2(\Omega)$-norm. Their research resulted in the following conclusions.

In the penalty method the error is minimal for a sufficiently great $\gamma_{\text{opt}} = \gamma(h)$. Penalty method with extrapolation displays similar features. The error increases always when some optimal values of γ_i are exceeded.

The authors recommend therefore the penalty method with extrapolation for "moderately large" parameters (e.g., 10, 100, 1000) instead of the search for an optimal γ in the penalty method. The determination of suitable values of γ may easily be bested computationally by a standard procedure given in [4—p. 313].

3. Application of the penalty method with extrapolation to shape optimization problems

In shape optimal design, one of the most important question is to find the gradient of the cost functional with respect to design variables, i.e., the so called sensitivity analysis. We shall show that the penalty method with extrapolation can be useful in the computation of the above mentioned gradient.

First, let us recall the definition of a model shape optimal design problem. Let us consider a class of domains

$$\Omega(v) = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < v(y),\ 0 < y < 1\}, \tag{3.1}$$

where $v \in U_{ad}$ and

$$U_{ad} = \{v \in C^{(0),1}([0,1]),\ v_{\min} \leqslant v(y) \leqslant v_{\max},\ |\,\mathrm{d}v \,/\, \mathrm{d}y\,| \leqslant C_1 \text{ a.e.}\} \tag{3.2}$$

with given positive constants $v_{\min}$, $v_{\max}$, C_1. Assume that $v_{\min} > v_{\max}/2$.

Let us have the following state problem

$$\begin{aligned} Au &= f \quad \text{in } \Omega(v),\\ u &= 0 \quad \text{on } \partial\Omega(v), \end{aligned} \tag{3.3}$$

where A is an elliptic operator (2.4) with *constant* coefficients and $f \in L^2(\Omega_\delta)$, $\Omega_\delta = (0, \delta) \times (0, 1)$, $\delta \in (v_{\max}, 2v_{\min})$.

We introduce the two following cost functionals

$$J_1(v) = \int_{\Omega(v)} \big(u_0(v) - \bar{u}\big)^2 \,\mathrm{d}x\,, \tag{3.4}$$

$$J_2(v) = \int_{\Omega(v)} f u_0(v) \,\mathrm{d}x\,, \tag{3.5}$$

where $u_0(v)$ denotes the (unique) solution of the problem (3.3) and $\bar{u} \in C(\bar{\Omega}_\delta)$ is given.

It is well known from the sensitivity analysis [7], [9], that the Gâteaux differentials, i.e.

$$J_i'(v, \hat{v}) = \lim_{t \to 0} \big(J_i(v + t\hat{v}) - J_i(v)\big)/t$$

are

$$J_1'(v, \hat{v}) = \int_0^1 \Big(\frac{\partial u_0(v)}{\partial \nu}\frac{\partial z(v)}{\partial \nu_A} + \bar{u}^2\Big)\hat{v} \,\mathrm{d}y\,, \tag{3.6}$$

$$J_2'(v, \hat{v}) = \int_0^1 \frac{\partial u_0(v)}{\partial \nu}\frac{\partial u_0(v)}{\partial \nu_A}\hat{v} \,\mathrm{d}y\,. \tag{3.7}$$

Here $z(v)$ is the solution of an *adjoint problem*

$$\begin{aligned} Az &= 2\big(u_0(v) - \bar{u}\big) \quad \text{in } \Omega(v),\\ z &= 0 \quad \text{on } \partial\Omega(v) \end{aligned} \tag{3.8}$$

and all the functions in the integrals (3.6), (3.7) are evaluated on the variable part $\Gamma_0(v)$ of the boundary, i.e., on the graph of the function v.

Thus we are interested in an efficient method for computation of both the solution $u_0(v)$ in $\Omega(v)$ and of the derivatives $\partial u_0(v)/\partial \nu_A$, $\partial z(v)/\partial \nu_A$, $\partial u_0(v)/\partial \nu$ on $\Gamma_0(v)$. As we have seen in Section 2, the penalty method with extrapolation can be suitable, since it gives approximations of u_0 and $\partial u_0/\partial \nu_A$ (and $\partial z/\partial \nu_A$) of a higher order of accuracy.

Note that having $\partial u_0/\partial \nu_A$, we easily obtain $\partial u_0/\partial \nu$, since

$$\frac{\partial u_0}{\partial \nu} = (a_{ij}\nu_i\nu_j)^{-1}\frac{\partial u_0}{\partial \nu_A}. \tag{3.9}$$

In fact, due to the boundary condition $u = 0$ and introducing a new vector $b_i = a_{ij}\nu_j$, we may write at each regular point of $\Gamma_0(v)$

$$\frac{\partial u_0}{\partial \nu_A} = a_{ij}\nu_j\frac{\partial u_0}{\partial x_i} = b_i\frac{\partial u_0}{\partial x_i} = (b\cdot\nu)\frac{\partial u_0}{\partial \nu} + (b\cdot t)\frac{\partial u_0}{\partial s} = (b_i\nu_i)\frac{\partial u_0}{\partial \nu}.$$

Inserting

$$b_i\nu_i = a_{ij}\nu_i\nu_j,$$

we arrive at (3.9).

It remains to choose the subspaces $V_h^r(v)$ of $H^1(\Omega(v))$, satisfying the assumption (A3). To this aim we choose restrictions to $\Omega(v)$ of standard piecewise polynomial finite elements over a uniform partition $\mathcal{T}_h$ of the rectangular domain Ω_δ.

Let us introduce a finite-dimensional restriction of the set U_{ad} by means of the Bézier curves

$$F(\alpha)(y) = \sum_{i=0}^{n}\alpha_i\beta_i^{(n)}(y), \tag{3.12}$$

where

$$\beta_i^{(n)}(y) = \binom{n}{i}y^i(1-y)^{n-i};$$

F is the mapping $\mathbb{R}^{n+1} \to C([0,1])$.

Defining

$$\begin{aligned} U^{(n)} = \{\alpha \in \mathbb{R}^{n+1} \mid\ & v_{\min} \leqslant \alpha_i \leqslant v_{\max},\ i = 0,1,\dots,n, \\ & |\alpha_{i+1} - \alpha_i| \leqslant C_1/n,\ i = 0,1,\dots,n-1\}, \end{aligned} \tag{3.13}$$

we obtain that (cf. [9])

$$\alpha \in U^{(n)} \implies F(\alpha) \in U_{ad}.$$

Thus choosing a fixed n, instead of U_{ad} we shall deal with the set

$$U_{ad}^{(n)} \equiv F(U^{(n)}).$$

We shall solve the optimization problems

$$\alpha^{(i)} = \operatorname*{argmin}_{\alpha \in U^{(n)}} J_i\big(F(\alpha)\big), \quad i = 1, 2. \tag{3.14}$$

Let the family of partitions $\{\mathcal{T}_h\}$, $h \to 0$, be regular. Consider the subspaces $V_h^r\big(F(\alpha)\big)$, described above over the partitions $\mathcal{T}_h$ and denote

$$u_h(\alpha) = u_h^{(k)}(\alpha; \gamma_0, \dots, \gamma_k)$$

the solution by penalty method with extrapolation. Setting $r = p+1$ and $k = r-2 = p-1$, we may expect the best approximation, as follows from (2.13).

Let us define the functionals

$$\begin{aligned} J_{1h}(\alpha) &= j_1\big(\alpha, u_h(\alpha)\big) = \int_{\Omega_\alpha} \big(u_h(\alpha) - \bar{u}\big)^2 \,\mathrm{d}x\,, \\ J_{2h}(\alpha) &= j_2\big(\alpha, u_h(\alpha)\big) = \int_{\Omega_\alpha} f u_h(\alpha) \,\mathrm{d}x \end{aligned} \tag{3.15}$$

and the following *approximate optimization problems*

$$\alpha_h^{(i)} = \operatorname*{argmin}_{\alpha \in U^{(n)}} J_{ih}(\alpha), \quad (i = 1, 2). \tag{3.16}$$

Let us denote (cf. (2.16))

$$e_h(\alpha) = e_h^{(k)}(\alpha; \gamma_0, \dots, \gamma_k)$$

and let $e_h^z(\alpha)$ be an analogous approximation, where $u_h(\gamma_i)$ are replaced by $z_h(\gamma_i)$, corresponding to the right-hand side $f = 2u_h(\alpha) - 2\bar{u}$, cf. (3.8).

The formulas (3.6), (3.7) indicate that we can take the following integrals for the differentials of J_{ih}

$$\begin{aligned} \nabla J_{1h}(\alpha) \cdot \hat{\alpha} \equiv J'_{1h}(\alpha, \hat{\alpha}) &\doteq \int_0^1 [(a_{ij}\nu_i\nu_j)^{-1} e_h(\alpha) e_h^z(\alpha) + \bar{u}^2|_{x=F(\alpha)}] F(\hat{\alpha}) \,\mathrm{d}y\,, \\ \nabla J_{2h}(\alpha) \cdot \hat{\alpha} \equiv J'_{2h}(\alpha, \hat{\alpha}) &\doteq \int_0^1 \big[(a_{ij}\nu_i\nu_j)^{-1} \big(e_h(\alpha)\big)^2 F(\hat{\alpha})\big] \,\mathrm{d}y\,. \end{aligned}$$

We are going to show that the approximate optimization problems (3.16) have at least one solution for any fixed h.

Lemma 4.1. *Let $\alpha_m \to \alpha$ in $\mathbb{R}^{n+1}$, $\alpha_m \in U^{(n)}$, and let the uniform partition $\mathcal{T}_h$ of the rectangle Ω_δ be fixed, $h \leqslant \gamma/c_0$.*

If

$$G_p = \Omega\big(F(\alpha) - 1/p\big), \quad p = 2, 3, \dots.$$

and $u_h^0(\alpha_m)$ is the approximation by penalty method (i.e., solution of (1.14) with a fixed parameter γ), then

$$u_h^0(\alpha_m)\big|_{G_p} \to u_h^0(\alpha)\big|_{G_p} \quad \text{in } H^1(G_p), \text{ as } m \to \infty$$

holds for all $p > p_0(\alpha)$.

Sketch of the the Proof. Henceforth, we denote $\Omega = \Omega\big(F(\alpha)\big)$, $\Omega_m = \Omega\big((F(\alpha_m)\big)$, $u = u_h^0(\alpha)$, $u_m = u_h^0(\alpha_m)$. Using the extension

$$\tilde{u}(x, y) = u\big(2F(\alpha)(y) - x, y\big) \tag{4.1}$$

from $H^1(\Omega)$ onto $H^1(\Omega_\delta)$ and Lemma 4.2 below, we derive that

$$\|u_m\|_{1,\Omega_m} \leqslant C \quad \forall m. \tag{4.2}$$

For any fixed p we can choose a subsequence $\{u_{m_p}\}$, converging to a solution $\omega_p \in H^1(G_p)$ weakly in $H^1(G_p)$. We consider the sequence $\{u_{m_p}\}, \{u_{m_{p+1}}\}, \dots$ and choose the diagonal subsequence $\{u_{m_D}\} \equiv \{u_D\}$. We can define $\omega \in H^1(\Omega)$, by means of the restriction

$$\omega\big|_{G_p} = \omega_p.$$

Then

$$u_D\big|_{G_p} \to \omega\big|_{G_p} \quad \text{in } H^1(G_p), \text{ as } D \to \infty. \tag{4.3}$$

for all p great enough. Note that

$$\dim V_h^r(G_p) = \dim V_h^r(\Omega)$$

for all $p > p_o(\alpha)$, where

$$[p_0(\alpha)]^{-1} = \min\{\operatorname{dist}(\Gamma_\alpha, A), A \notin \Gamma_\alpha, A \in \text{ grid points of } \mathcal{T}_h\}. \tag{4.4}$$

Consequently, $\omega \in V_h^r(\Omega)$.

Next we can show that ω coincides with the solution $u_h^0(\alpha)$. To this end, we consider the definition of u_m for the diagonal subsequence u_D and pass to the limit with $D \to \infty$. After some tedious calculations, based on (4.3) and the convergence

$$F(\alpha_m) \to F(\alpha) \quad \text{in } C^1([0,1]),$$

we deduce that ω satisfies the equation (1.14). From the uniqueness of its solution we conclude that the whole sequence $\{u_m\}$ tends to ω in $H^1(G_p)$ for $p > p_0(\alpha)$. □

Theorem 4.1. *For any $h \leqslant c_0^{-1}\gamma_0$, there exists at least one solution of the approximate optimization problem* (3.16), *$i \in \{1, 2\}$.*

Proof. By definition (2.1) of $u_h(\alpha)$ and Lemma 4.1, we can see that the assertion of Lemma 4.1 holds also for the sequence $u_h(\alpha_m)$. Henceforth, let $\{\alpha_m\}$ be a minimizing sequence of $J_{ih}(\beta)$, i.e.,

$$\lim_{m\to\infty} J_{ih}(\alpha_m) = \inf_{\beta\in U^{(n)}} J_{ih}(\beta). \tag{4.5}$$

Since $U^{(n)}$ is compact, there exists a subsequence, denoted again by $\{\alpha_m\}$, such that

$$\alpha_m \to \alpha \quad \text{in } \mathbb{R}^{n+1},\ \alpha \in U^{(n)}. \tag{4.6}$$

For brevity, we denote $u := u_h(\alpha)$, $u_m = u_h(\alpha_m)$, $\Omega := \Omega(F(\alpha))$, $\Omega_m = \Omega(F(\alpha_m))$.

Let $i = 1$. For any $p > p_0(\alpha)$ and $m > m_0(p)$ we have

$$J_{1h}(\alpha_m) \geqslant \int_{G_p} (u_m - \bar{u})^2 \,\mathrm{d}x\,. \tag{4.7}$$

From Lemma 4.1 it follows that

$$u_m\big|_{G_p} \to u\big|_{G_p} \quad \text{in } H^1(G_p).$$

Passing to the limit with $m \to \infty$ and then with $p \to \infty$, we obtain

$$\lim_{m\to\infty} J_{1h}(\alpha_m) = \inf_{\beta\in U^{(n)}} J_{1h}(\beta) \geqslant \int_\Omega (u - \bar{u})^2 \,\mathrm{d}x \ = J_{1h}(\alpha). \tag{4.8}$$

Consequently, α is a solution of the problem (3.16).

Let $i = 2$. For $m > m_0(p)$ we may write

$$\begin{aligned} |J_{2h}(\alpha_m) - J_{2h}(\alpha)| &= |(f, u_m)_{\Omega_m} - (f, u_m)_{G_p}| \\ &\quad + |(f, u_m)_{G_p} - (f, u)_{G_p}| + |(f, u)_{G_p} - (f, u)_\Omega| \\ &= I_1 + I_2 + I_3, \end{aligned}$$

$I_1 \leqslant \|u_m\|_{0,\Omega_m} \|f\|_{0,\Omega_m - G_p} \to 0$, as $p \to \infty$, $m > m_0(p)$, $m \to \infty$, since the u_m are bounded by virtue of (4.1) and meas $(\Omega_m - G_p) \leqslant (1/p + \|F(\alpha_m) - F(\alpha)\|_\infty) \to 0$; $I_2 \to 0$ by virtue of Lemma 4.1; $I_3 \leqslant \|u\|_{0,\Omega} \|f\|_{0,\Omega - G_p} \to 0$, as $p \to \infty$.

Consequently, we have

$$\lim_{m\to\infty} J_{2h}(\alpha_m) = J_{2h}(\alpha) \tag{4.9}$$

so that α is a solution of the problem (3.16). □

Let us recall the extension $\tilde{u} \in H^1(\Omega_\delta)$ of any function $u \in H^1(\Omega_\alpha)$ by the relation

$$\tilde{u}(x,y) = u\big(2F(\alpha)(y) - x, y\big) \quad \text{on } \Omega_\delta - \Omega_\alpha.$$

Lemma 4.2. *There exists a constant $C_3 > 0$, independent of $\alpha \in U^{(n)}$ and h, such that*

$$a(\alpha; u, u) + \gamma h^{-1} \langle u, u\rangle_{\partial\Omega_\alpha} \geqslant C_3 \|\tilde{u}\|^2_{1,\Omega_\delta} \geqslant C_3 \|u\|^2_{1,\Omega_\alpha} \tag{4.10}$$

holds for all $u \in H^1(\Omega_\alpha)$ and $h \leqslant \gamma/c_0$.

Proof. We have

$$\begin{aligned} a(\alpha; u, u) + \gamma h^{-1} \|u\|^2_{0,\partial\Omega_\alpha} &\geqslant c_0 |u|^2_{1,\Omega_\alpha} + \gamma h^{-1} \|u\|^2_{0,\partial\Omega_\alpha} \\ &\geqslant \min(c_0, \gamma h^{-1})(|u|^2_{1,\Omega_\alpha} + \|u\|^2_{0,\partial\Omega_\alpha}). \end{aligned} \tag{4.11}$$

Since $|F'(\alpha)| \leqslant C_1$, we have

$$|\tilde{u}|^2_{1,\Omega_\delta} \leqslant (3 + 4C_1^2)|u|^2_{1,\Omega_\alpha} \tag{4.12}$$

so that

$$\|\tilde{u}\|^2_{1,\Omega_\delta} \leqslant C\|u\|^2_{1,\Omega_\alpha} \quad \forall u \in H^1(\Omega_\alpha), \tag{4.13}$$

holds with C independent of $\alpha \in U^{(n)}$.

Let $\Gamma_{\min} \subset \partial\Omega_\alpha$ be the straight line segment on the line $x = 0$. Then we have by (4.12)

$$\begin{aligned} C_0 \|\tilde{u}\|^2_{1,\Omega_\delta} &\leqslant |\tilde{u}|^2_{1,\Omega_\delta} + \|\tilde{u}\|^2_{0,\Gamma_{\min}} \\ &\leqslant \hat{C}|u|^2_{1,\Omega_\alpha} + \|\tilde{u}\|^2_{0,\partial\Omega_\alpha} \leqslant \hat{C}(|u|^2_{1,\Omega_\alpha} + \|u\|^2_{0,\partial\Omega_\alpha}). \end{aligned} \tag{4.14}$$

Combining (4.11), (4.14), we arrive at (4.10). □

Proposition 4.1. *Let $\alpha_h \to \alpha$ in $\mathbb{R}^{n+1}$, $\alpha_h \in U^{(n)}$, $h \to 0_+$. Let $\tilde{u}_h^0(\alpha_h)$ be the extension of the approximation $u_h^0(\alpha_h)$ by the penalty method on the domain $\Omega(F(\alpha_h))$, constructed according to* (4.1) *on Ω_δ. Then*

$$\tilde{u}_h^0(\alpha_h)\big|_{\Omega_\alpha} \rightharpoonup u_0(F(\alpha)) \quad \textit{(weakly) in } H^1(\Omega_\alpha), \text{ as } h \to 0_+,$$

where $\Omega_\alpha = \Omega(F(\alpha))$ and $u_0(F(\alpha))$ is the solution of the Dirichlet problem (3.3) *on the domain Ω_α.*

Proof. Lemma 4.2, (1.14) and (4.13) yield that

$$\|\tilde{u}_h^0(\alpha_h)\|_{1,\Omega_\delta} \leqslant CC_3^{-1}\|f\|_{0,\Omega_\delta}. \tag{4.15}$$

For brevity, let us denote $\Omega_h := \Omega(F(\alpha_h))$, $\Omega := \Omega(F(\alpha)) = \Omega_\alpha$, $u_h := u_h^0(\alpha_h)$, $u := u_0(F(\alpha))$.

Then a subsequence $\{\tilde{u}_{\hat{h}}\}$ and $u \in H^1(\Omega_\delta)$ exist such that

$$\tilde{u}_{\hat{h}} \rightharpoonup u \quad \text{in } H^1(\Omega_\delta) \text{ weakly, as } \hat{h} \to 0. \tag{4.16}$$

Let us show that $u\big|_\Omega$ coincides with the solution of the Dirichlet problem on Ω. Consider a $w \in H_0^1(\Omega)$. In what follows, we drop the hats over h. Let $Ew \in H_0^1(\Omega_\delta)$ be the extension of w by zero in $\Omega_\delta - \Omega$. There exists a sequence w_κ, $\kappa \to 0$, such that $w_\kappa \in C_0^\infty(\Omega)$, $\operatorname{supp} w_\kappa \subset \Omega$ and

$$\|w_\kappa - w\|_{1,\Omega} \to 0, \quad \text{as } \kappa \to 0. \tag{4.17}$$

Consider the Lagrange interpolate $\pi_h Ew_\kappa \in V_h^r(\Omega_\delta)$ of Ew_κ over the triangulation $\mathcal{T}_h(\alpha_h)$. Let κ be fixed, for a time being. We can insert $\pi_h Ew_\kappa$ into (1.14) to obtain

$$a(\alpha_h; u_h, \pi_h Ew_\kappa) = (f, \pi_h Ew_\kappa)_{\Omega_h}, \tag{4.18}$$

since $\pi_h Ew_\kappa = \pi_h w_\kappa = 0$ on $\Gamma(F(\alpha_h))$ holds for h small enough (i.e., if $h < \operatorname{dist}(\operatorname{supp} w_\kappa, \Gamma(F(\alpha_h)))$.

We shall pass to the limit with $h \to 0$. We introduce the functions $v_m = F(\alpha) - 1/m$, $m = 2, 3, \ldots$, and the domains

$$G_m = \{(x, y) \mid 0 < x < v_m,\ 0 < y < 1\}.$$

Then

$$G_m \subset \Omega_h \quad \text{for } h < h_1(m)$$

and we may write

$$(4.19)\qquad \begin{aligned}&|a(\alpha_h;u_h,\pi_h Ew_\kappa)-a(v_m;u,w_\kappa)|\\ &\quad=|a(v_m;u_h,w_\kappa)+a(v_m;u_h,\pi_h w_\kappa-w_\kappa)\\ &\qquad+\tilde{a}(\alpha_h-v_m;u_h,\pi_h Ew_\kappa)-a(v_m;u,w_\kappa)|\\ &\quad\leqslant|a(v_m;u_h-u,w_\kappa)|+|a(v_m;u_h,\pi_h w_\kappa-w_\kappa)|\\ &\qquad+|\tilde{a}(\alpha_h-v_m;u_h,\pi_h Ew_\kappa)|=I_1+I_2+I_3,\end{aligned}$$

where

$$\tilde{a}(\alpha_h-v_m;\cdot,\cdot)=a(\alpha_h,\cdot,\cdot)-a(v_m;\cdot,\cdot).$$

Consider a positive ε. From (4.16) we conclude that

$$(4.20)\qquad I_1<\varepsilon/6\quad\text{if } h<\bar{h}_1(\varepsilon,m).$$

To estimate I_2, we employ the assumption (A3) (2.9) on Ω_δ:

$$(4.21)\qquad \|\pi_h Ew_\kappa-Ew_\kappa\|_{1,\Omega_\delta}\leqslant Ch^{r-1}\|Ew_\kappa\|_{r,\Omega_\delta},\quad (r\geqslant 2).$$

Using (4.21) and (4.15), we obtain

$$(4.22)\qquad I_2\leqslant C\|u_h\|_{1,G_m}\|\pi_h w_\kappa-w_\kappa\|_{1,G_m}\leqslant\tilde{C}h^{r-1}\|Ew_\kappa\|_{r,\Omega_\delta}<\varepsilon/6$$

for $h<\bar{h}_2$. It remains to estimate I_3. First, we assume that

$$\|\pi_h Ew_\kappa\|_{1,T}\leqslant C\|Ew_\kappa\|_{r,T}$$

holds for all elements $T\in\mathcal{T}_h$. (This estimate is true for most finite element subspaces—see [3]).

Denote by G_m^h the smallest union of triangles $T\in\mathcal{T}_h(\alpha_h)$ such that $\Omega_h-G_m\subset G_m^h$. Obviously, we have

$$(4.23)\qquad \operatorname{meas}G_m^h\leqslant 1/m+2h+\|F(\alpha_h)-F(\alpha)\|_\infty$$

where $\|\cdot\|_\infty$ denotes the $C([0,1])$-norm. Consequently,

$$\|\pi_h Ew_\kappa\|^2_{1,\Omega_h-G_m}\leqslant\|\pi_h Ew_\kappa\|^2_{1,G_m^h}\leqslant C\|Ew_\kappa\|^2_{r,G_m^h}.$$

Using again (4.15), we may write

$$(4.24)\qquad I_3\leqslant C\|u_h\|_{1,\Omega_h}\|\pi_h Ew_\kappa\|_{1,\Omega_h-G_m}\leqslant C\|Ew_\kappa\|_{r,G_m^h}.$$

Combining (4.19), (4.20), (4.22) and (4.24), we arrive at the following estimate

$$|a(\alpha_h; u_h, \pi_h E w_\kappa) - a(v_m; u, w_\kappa)| \leqslant \varepsilon/3 + C\|Ew_\kappa\|_{r,G_m^h}$$

for $h < h_3(\varepsilon, m)$.

Then we obtain

$$\begin{aligned}&|a(\alpha_h; u_h, \pi_h E w_\kappa) - a(\alpha; u, w_\kappa)| \\ &\qquad \leqslant \varepsilon/3 + C\|Ew_\kappa\|_{r,G_m^h} + C\|u\|_{1,\Omega}\|w_\kappa\|_{1,\Omega - G_m}.\end{aligned}$$

Consequently, using also (4.23), we conclude that

$$\lim_{h\to 0} a(\alpha_h; u_h, \pi_h E w_\kappa) = a(\alpha; u, w_\kappa). \tag{4.25}$$

Next, we may write

$$\begin{aligned}&|(f, \pi_h E w_\kappa)_{\Omega_h} - (f, w_\kappa)_\Omega| \\ &\qquad \leqslant |(f, \pi_h E w_\kappa - E w_\kappa)_{\Omega_h}| + |(f, E w_\kappa)_{\Omega_h} - (f, w_\kappa)_\Omega| \\ &\qquad \leqslant C\|f\|_{0,\Omega_\delta} h^{r-1}\|Ew_\kappa\|_{r,\Omega_\delta} + \int_{\Delta(\Omega_h,\Omega)} |f||Ew_\kappa|\,\mathrm{d}x \to 0.\end{aligned}$$

Consequently,

$$\lim_{h\to 0} (f, \pi_h E w_\kappa)_{\Omega_h} = (f, w_\kappa)_\Omega. \tag{4.26}$$

Making use of (4.25) and (4.26) in (4.18), we are led to the relation

$$a(\alpha; u, w_\kappa) = (f, w_\kappa)_\Omega.$$

Passing to the limit with $\kappa \to 0$ and using (4.17), we obtain

$$a(\alpha; u, w) = (f, w)_\Omega. \tag{4.27}$$

It remains to verify that $u\big|_\Omega \in H_0^1(\Omega)$. We start with (1.14), which implies that

$$a(\alpha_h; u_h, u_h) + \gamma h^{-1}\|u_h\|^2_{0,\partial\Omega_h} = (f, u_h)_{\Omega_h}.$$

Using (4.15), we have

$$h^{-1}\gamma\|u_h\|^2_{0,\partial\Omega_h} \leqslant \|f\|_{0,\Omega_\delta}\cdot\|\tilde{u}_h\|_{0,\Omega_\delta} + C\|\tilde{u}_h\|^2_{1,\Omega_\delta} \leqslant \tilde{C},$$

so that

$$\|u_h\|^2_{0,\partial\Omega_h} \leqslant \tilde{C}\gamma^{-1}h \to 0, \quad \text{as } h \to 0. \tag{4.28}$$

We may write

$$\begin{aligned}\left|\|u_h\|^2_{0,\partial\Omega_h} - \|u\|^2_{0,\partial\Omega}\right| &\leqslant \left|\int_{\partial\Omega_h} u_h^2 \,\mathrm{d}s - \int_{\partial\Omega} \tilde{u}_h^2 \,\mathrm{d}s\right| + \left|\int_{\partial\Omega} (\tilde{u}_h^2 - u^2) \,\mathrm{d}s\right| \\ &= M_1 + M_2.\end{aligned} \tag{4.29}$$

Since the trace operator $H^1(\Omega) \to L^2(\partial\Omega)$ is compact, the convergence (4.16) implies that

$$\|\tilde{u}_h - u\|_{0,\partial\Omega} \to 0 \quad \text{as } h \to 0. \tag{4.30}$$

Consequently,

$$M_2 \leqslant \|\tilde{u}_h - u\|_{0,\partial\Omega}(\|\tilde{u}_h\|_{0,\partial\Omega} + \|u\|_{0,\partial\Omega}) \to 0. \tag{4.31}$$

Let us denote by Γ_h and Γ the graph of $F(\alpha_h)$ and $F(\alpha)$, respectively,

$$\tilde{u}_h\big|_{\Gamma_h} = \tilde{u}_h(\Gamma_h), \quad \tilde{u}_h\big|_{\Gamma} = \tilde{u}_h(\Gamma), \quad v_h = F(\alpha_h), \quad v = F(\alpha).$$

Then we may write

$$\begin{aligned} M_1 \leqslant &\left|\int_0^1 \tilde{u}_h^2(\Gamma_h)\left(1 + (v_h')^2\right)^{1/2} \mathrm{d}y - \int_0^1 \tilde{u}_h^2(\Gamma)\left(1 + (v')^2\right)^{1/2} \mathrm{d}y\right| \\ &+ \left|\int_{v(0)}^{v_h(0)} \tilde{u}_h^2 \,\mathrm{d}x\right| + \left|\int_{v(1)}^{v_h(1)} \tilde{u}_h^2 \,\mathrm{d}x\right| = M_{11} + M_{12} + M_{13};\end{aligned} \tag{4.32}$$

$$M_{11} \leqslant \int_0^1 |G| \,\mathrm{d}y\,, \tag{4.33}$$

where

$$\begin{aligned}|G| \leqslant &\ \tilde{u}_h^2(\Gamma_h)\left|\left(1 + (v_h')^2\right)^{1/2} - \left(1 + (v')^2\right)^{1/2}\right| \\ &+ |\tilde{u}_h^2(\Gamma_h) - \tilde{u}_h^2(\Gamma)|\left(1 + (v')^2\right)^{1/2} = G_1 + G_2.\end{aligned} \tag{4.34}$$

The estimate

$$\int_0^1 G_1 \,\mathrm{d}y \leqslant \|u_h\|^2_{0,\partial\Omega_h} \cdot \left\|\left(1 + (v_h')^2\right)^{1/2} - \left(1 + (v')^2\right)^{1/2}\right\|_\infty \to 0 \tag{4.35}$$

follows from (4.28) and the convergence $F(\alpha_h) \to F(\alpha)$ in C^1 (see Remark 4.1).

We have

$$(4.36) \qquad \int_0^1 G_2 \,\mathrm{d}y \leqslant (1 + C_1^2)^{1/2} \|\tilde{u}_h(\Gamma_h) - \tilde{u}_h(\Gamma)\|_0 \cdot \big(\|\tilde{u}_h(\Gamma_h)\|_0 + \|\tilde{u}_h(\Gamma)\|_0\big).$$

On the other hand

$$(4.37) \qquad \begin{aligned} \|\tilde{u}_h(\Gamma_h) - \tilde{u}_h(\Gamma)\|_0^2 &= \int_0^1 \mathrm{d}y \left(\int_{v(y)}^{v_h(y)} \partial \tilde{u}_h/\partial x \,\mathrm{d}x \right)^2 \\ &\leqslant \int_0^1 \mathrm{d}y \, \|v_h - v\|_\infty \left| \int_{v(y)}^{v_h(y)} (\partial \tilde{u}_h/\partial x)^2 \,\mathrm{d}x \right| \\ &\leqslant \|v_h - v\|_\infty \|\tilde{u}_h\|_{1,\Omega_\delta}^2 \to 0. \end{aligned}$$

follows from (4.15) and the uniform convergence $v_h \to v$.

By the trace theorem

$$(4.38) \quad \|\tilde{u}_h(\Gamma)\|_0^2 = \int_0^1 \big(\tilde{u}_h(\Gamma)\big)^2 \,\mathrm{d}y \leqslant \|\tilde{u}_h\|_{0,\partial\Omega}^2 \leqslant C \|\tilde{u}_h\|_{1,\Omega}^2 \leqslant C \|\tilde{u}_h\|_{1,\Omega_\delta}^2 \leqslant \tilde{C}^2.$$

From (4.37) and (4.38) we deduce

$$(4.39) \qquad \|u_h(\Gamma_h)\|_0 \leqslant \|\tilde{u}_h(\Gamma)\|_0 + \|\tilde{u}_h(\Gamma_h) - \tilde{u}_h(\Gamma)\|_0 \leqslant \tilde{C} + 1$$

for $h < h_0$.

Using (4.37), (4.38) and (4.39) in (4.36), we obtain

$$(4.40) \qquad \int_0^1 G_2 \,\mathrm{d}y \to 0.$$

Consequently,

$$(4.41) \qquad M_{11} \to 0 \quad \text{as } h \to 0$$

follows from (4.33), (4.34), (4.35) and (4.40).

Employing the particular property of the extension, and (4.28), we arrive at

$$(4.42) \qquad M_{12} + M_{13} \leqslant \int_{\partial\Omega_h} u_h^2 \,\mathrm{d}s \to 0.$$

From (4.32), (4.41) and (4.42) we obtain that

$$(4.43) \qquad M_1 \to 0.$$

Then (4.29), (4.31) and (4.43) imply

$$\|u_h\|^2_{0,\partial\Omega_h} \to \|u\|^2_{0,\partial\Omega},$$

so that

$$\|u\|_{0,\partial\Omega} = 0$$

follows from (4.28). Consequently, $u|_\Omega \in H^1_0(\Omega)$ and due to the condition (4.27), $u|_\Omega$ coincides with the solution of the Dirichlet problem on Ω.

Since (4.16) implies that

$$\tilde{u}_{\hat{h}}|_\Omega \rightharpoonup u|_\Omega \quad \text{in } H^1(\Omega)$$

and the weak limit is unique, the *whole* sequence $\{\tilde{u}_h|_\Omega\}$ tends to $u|_\Omega$ weakly in $H^1(\Omega)$. □

Corollary 4.1. *Let $\alpha_h \to \alpha$ in $\mathbb{R}^{n+1}$, $\alpha_h \in U^{(n)}$, $h \to 0_+$. Let $\tilde{u}_h(\alpha_h)$ be extensions of the approximate solutions by penalty method with extrapolation, $\Omega_\alpha \equiv \Omega(F(\alpha))$.*

Then

$$\tilde{u}_h(\alpha_h)|_{\Omega_\alpha} \rightharpoonup u_0(F(\alpha)) \quad \textit{(weakly) in } H^1(\Omega_\alpha). \tag{4.44}$$

P r o o f. By definition, we have

$$\tilde{u}_h(\alpha_h) = \sum_{i=0}^{k} a_i \tilde{u}^0_h(\alpha_h, \gamma_i).$$

Using Proposition 4.1 and (2.2), we obtain (4.4). □

Proposition 4.2. *Let $\alpha_h \to \alpha$ in $\mathbb{R}^{n+1}$, $\alpha_h \in U^{(n)}$, $h \to 0_+$. Then*

$$j_i(\alpha_h, u_h(\alpha_h)) \to j_i(\alpha, u_0(F(\alpha))), \quad i = 1, 2. \tag{4.45}$$

where $u_0(F(\alpha))$ is the solution of the Dirichlet problem on the domain $\Omega(F(\alpha))$.

P r o o f. *Case $i = 1$.* Corollary 4.1 and the Rellich's theorem yield that

$$\tilde{u}_h(\alpha_h)|_{\Omega_\alpha} \to u_0(F(\alpha)) \quad \text{in } L^2(\Omega_\alpha). \tag{4.46}$$

Denoting $\Omega_\alpha = \Omega$, $\Omega(F(\alpha_h)) = \Omega_h$, $u_h(\alpha_h) = u_h$, $u_0(F(\alpha)) = u$, we have

$$(4.47) \qquad \left| \int_\Omega (\tilde{u}_h - \bar{u})^2 \,\mathrm{d}x - \int_\Omega (u - \bar{u})^2 \,\mathrm{d}x \right| \leqslant \|\tilde{u}_h - u\|_{0,\Omega} \|\tilde{u}_h + u - 2\bar{u}\|_{0,\Omega} \to 0,$$

by virtue of (4.46). It is readily seen that

$$(4.48) \qquad \int_{\Omega_h} (\tilde{u}_h - \bar{u})^2 \,\mathrm{d}x = \int_\Omega + \int_{\Omega_h - \Omega} - \int_{\Omega - \Omega_h} .$$

Let us estimate the last two terms as follows (cf. (4.9))

$$(4.49) \qquad \begin{aligned} \left| \int_{\Omega_h - \Omega} - \int_{\Omega - \Omega_h} \right| &\leqslant \int_{\Delta(\Omega_h, \Omega)} (\tilde{u}_h - \bar{u})^2 \,\mathrm{d}x \\ &\leqslant \left(\operatorname{meas} \Delta(\Omega_h, \Omega) \right)^{1/2} \|\tilde{u}_h - \bar{u}\|^2_{L^4(\Omega_\delta)} \to 0 \end{aligned}$$

since

$$\|\tilde{u}_h - \bar{u}\|_{L^4(\Omega_\delta)} \leqslant C \|\tilde{u}_h - \bar{u}\|_{1,\Omega_\delta} \leqslant \tilde{C}$$

follows from the embedding theorem and (4.15). Combining (4.47), (4.48) and (4.49), we arrive at (4.45) for $i = 1$.

Case $i = 2$. We have

$$\begin{aligned} \left| \int_{\Omega_h} f u_h \,\mathrm{d}x - \int_\Omega f u \,\mathrm{d}x \right| &\leqslant \left| \int_{\Omega_h} f u_h \,\mathrm{d}x - \int_\Omega f \tilde{u}_h \,\mathrm{d}x \right| + \left| \int_\Omega f (\tilde{u}_h - u) \,\mathrm{d}x \right| \\ &\leqslant \int_{\Delta(\Omega_h, \Omega)} |f \tilde{u}_h| \,\mathrm{d}x + \|f\|_{0,\Omega} \cdot \|\tilde{u}_h - u\|_{0,\Omega} \to 0 \end{aligned}$$

since

$$\int_{\Delta(\Omega_h, \Omega)} |f \tilde{u}_h| \,\mathrm{d}x \leqslant \|f\|_{0,\Delta(\Omega_h,\Omega)} \|\tilde{u}_h\|_{0,\Omega_\delta} \to 0,$$

as follows from (4.15), and (4.46) can be employed. □

Theorem 4.2. *Let* $\{\alpha_h^{(i)}\}$, $h \to 0$, $i = 1, 2$, *be a sequence of solutions of the approximate optimization problem* $(3.16)_i$. *Then there exists a subsequence* $\{\alpha_{\hat{h}}^{(i)}\}$, $\hat{h} \to 0$, *such that*

$$(4.50) \qquad \alpha_{\hat{h}}^{(i)} \to \alpha^{(i)} \quad \textit{in } \mathbb{R}^{n+1}$$

and

$$(4.51) \qquad \tilde{u}_{\hat{h}}(\alpha_{\hat{h}}^{(i)})|_{\Omega_\alpha^{(i)}} \rightharpoonup u_0(F(\alpha^i)) \quad \textit{(weakly) in } H^1(\Omega_\alpha^{(i)})$$

holds for the extensions $\tilde{u}_{\hat{h}}$ *of the solutions* $u_{\hat{h}}(\alpha_{\hat{h}}^{(i)})$ *by penalty method with extrapolation;* $\alpha^{(i)}$ *is a solution of the optimization problem* $(3.14)_i$*,* $\Omega_\alpha^{(i)} = \Omega(F(\alpha^{(i)}))$*. Any cluster point of* $\{\alpha_h^{(i)}\}$ *has the same properties, i.e., it coincides with a solution of* $(3.14)_i$ *and* (4.51) *holds.*

Proof. Since $U^{(n)}$ is compact in $\mathbb{R}^{n+1}$, a subsequence $\{\alpha_{\hat{h}}^{(i)}\} \subset \{\alpha_h^{(i)}\}$ exists, such that (4.50) holds and $\alpha^{(i)} \in U^{(n)}$. Let $\alpha \in U^{(n)}$ be given. By definition, we have

$$j_i(\alpha_{\hat{h}}^{(i)}, u_{\hat{h}}(\alpha_{\hat{h}}^{(i)})) \leqslant j_i(\alpha, u_{\hat{h}}(\alpha)) \quad \forall \hat{h}.$$

Passing to the limit with $\hat{h} \to 0$ and using Proposition 4.2 on both sides, we obtain

$$j_i(\alpha^{(i)}, u_0(F(\alpha^{(i)}))) \leqslant j_i(\alpha, u_0(F(\alpha))).$$

Consequently, $\alpha^{(i)}$ is a solution of the problem $(3.14)_i$. The convergence (4.51) follows from Proposition 4.1 and the rest of the theorem is obvious. □

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