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Zdeněk Halas

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CONTINUOUS DEPENDENCE OF SOLUTIONS OF GENERALIZED LINEAR ORDINARY DIFFERENTIAL EQUATIONS ON A PARAMETER

Zdeněk Halas, Olomouc

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Abstract. We present here the problem of continuous dependence for generalized linear ordinary differential equations in the case when uniform convergence is violated. This work continues research of M. Ashordia (1993) and M. Tvrdý (2002).

Keywords: generalized linear ordinary differential equations, fundamental matrix, continuous dependence on a parameter, emphatic convergence

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0. Preface

This paper is devoted to generalized linear ordinary differential equations. Their solutions are functions of bounded variation. For such equations the problem of continuous dependence of solutions of the initial value problem on a parameter $k \in \mathbb{N}$ is considered.

All integrals which occur in this paper are the Kurzweil-Stieltjes integrals (KS-integrals). The definition of the KS-integral and a brief description of its basic properties are given in Subsection 1.2.

In Subsection 1.3 basic facts about systems of generalized linear ordinary differential equations are introduced and a survey of known results concerning the continuous dependence of solutions of generalized differential equations on a parameter is given. In particular, results by M. Ashordia and M. Tvrdý concerning respectively the case of uniform convergence and the case when uniform convergence is violated are recalled here (see Lemma 1.5 and Theorem 1.7).

This work continues research from [1] and from [7, chapter 3] which concerns the continuous dependence of solutions of generalized differential equations on a parameter. Our main result is Theorem 2.2.

1. Preliminaries

1.1 Basic notation. The following notation and definitions will be used throughout this text: $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{R}$ is the set of real numbers; $\mathbb{R}^{m\times n}$ is the space of real $m \times n$ matrices $B = (b_{ij})_{\substack{i=1,\ldots,m\\ j=1,\ldots,n}}$ with the norm

$$|B| = \max_{j=1,\ldots,n} \sum_{i=1}^{m} |b_{ij}|;$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ stands for the set of real column n-vectors $b = (b_i)_{i=1}^n$.

For a matrix $B \in \mathbb{R}^{n\times n}$, $\det B$ denotes the determinant of B. If $\det B \neq 0$, then the matrix inverse to B is denoted by B^{-1}. Further, B^T is the matrix transposed to B. The symbol I stands for the identity matrix and 0 for the zero matrix.

If $a, b \in \mathbb{R}$ are such that $-\infty < a < b < +\infty$, then $[a, b]$ stands for the closed interval $\{x \in \mathbb{R};\ a \leqslant x \leqslant b\}$, (a, b) is its interior and $(a, b]$, $[a, b)$ are the corresponding half-closed intervals.

The sets $D = \{t_0, t_1, t_2, \ldots, t_m\}$ of points in the closed interval $[a, b]$ such that $a = t_0 < t_1 < t_2 < \ldots < t_m = b$ are called divisions of $[a, b]$. The set of all divisions of the interval $[a, b]$ is denoted by $\mathscr{D}[a, b]$.

For a matrix valued function $B\colon\ [a, b] \to \mathbb{R}^{m\times n}$, its variation $\operatorname{var}_a^b B$ on the interval $[a, b]$ is defined by

$$\operatorname{var}_a^b B = \sup_{D\in\mathscr{D}[a,b]} \sum_{i=1}^{m} |B(t_i) - B(t_{i-1})|.$$

If $\operatorname{var}_a^b B < +\infty$, we say that the function B is of bounded variation on the interval $[a, b]$. $\mathbf{BV}^{m\times n}[a, b]$ denotes the set of all $m \times n$ matrix valued functions of bounded variation on $[a, b]$. We will write $\mathbf{BV}^n[a, b]$ instead of $\mathbf{BV}^{n\times 1}[a, b]$ and $\mathbf{BV}[a, b]$ instead of $\mathbf{BV}^{1\times 1}[a, b]$.

We will write briefly $B(t+) = \lim\limits_{\tau\to t+} B(\tau)$, $B(s-) = \lim\limits_{\tau\to s-} B(\tau)$ and

$$\Delta^+ B(t) = B(t+) - B(t),\ \Delta^- B(s) = B(s) - B(s-),\ \Delta B(r) = B(r+) - B(r-)$$

for $t \in [a, b)$, $s \in (a, b]$, $r \in (a, b)$.

If a sequence of $m \times n$ matrix valued functions $\{B_k(t)\}_{k=1}^\infty$ converges uniformly to a matrix valued function $B_0(t)$ on $[c, d] \subset [a, b]$, i.e. $\lim\limits_{k\to\infty} \sup\limits_{t\in[c,d]} |B_k(t) - B_0(t)| = 0$, we write $B_k \rightrightarrows B_0$ on $[c,d]$. We say that $\{B_k(t)\}_{k=1}^{\infty}$ converges locally uniformly to $B_0(t)$ on a set $M \subset [a,b]$, if $B_k \rightrightarrows B_0$ on each closed subinterval $J \subset M$.

For further details concerning the space $\mathbf{BV}^{m\times n}[a,b]$, see e.g. [8].

We say that a proposition $P(n)$ holds for almost all (briefly a.a.) $n \in \mathbb{N}$ if it is true for all $n \in \mathbb{N} \setminus K$ where K is a finite set.

1.2 Kurzweil-Stieltjes integral. In this subsection we will recall the definition of the KS-integral.

Let $-\infty < a < b < +\infty$. For given $m \in \mathbb{N}$, a division $D = \{t_0, t_1, \dots, t_m\} \in \mathscr{D}[a,b]$ and $\xi = (\xi_1, \xi_2, \dots, \xi_m) \in \mathbb{R}^m$, the couple $P = (D, \xi)$ is called a partition of $[a,b]$ if

$$t_{j-1} \leqslant \xi_j \leqslant t_j \quad \text{for all} \quad j = 1, 2, \dots, m.$$

The set of all partitions of the interval $[a,b]$ is denoted by $\mathscr{P}[a,b]$.

An arbitrary positive valued function $\delta\colon [a,b] \to (0,+\infty)$ is called a gauge on $[a,b]$. Given a gauge δ on $[a,b]$, the partition $P = (D,\xi) = \big(\{t_0, t_1, \dots, t_m\}, (\xi_1, \xi_2, \dots, \xi_m)\big) \in \mathscr{P}[a,b]$ is said to be δ-fine, if

$$[t_{j-1}, t_j] \subset \big(\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)\big) \quad \text{for all} \quad j = 1, 2, \dots, m.$$

The set of all δ-fine partitions of the interval $[a,b]$ is denoted by $\mathscr{A}(\delta; [a,b])$.

For functions $f, g\colon [a,b] \to \mathbb{R}$ and a partition $P \in \mathscr{P}[a,b]$, $P = \big(\{t_0, t_1, \dots, t_m\}, (\xi_1, \xi_2, \dots, \xi_m)\big)$ we define

$$S_P(f\Delta g) = \sum_{i=1}^{m} f(\xi_i)[g(t_i) - g(t_{i-1})].$$

We say that $I \in \mathbb{R}$ is the KS-integral of f with respect to g from a to b if

$$\forall \varepsilon > 0 \ \exists \delta\colon [a,b] \to (0,+\infty) \ \forall P \in \mathscr{A}(\delta[a,b])\colon \ |I - S_P(f\Delta g)| < \varepsilon.$$

In such a case we write $I = \int_a^b f \, \mathrm{d}g$ or $I = \int_a^b f(t) \, \mathrm{d}g(t)$.

It is known (cf. [8, 5.20, 5.15]) that the KS-integral $\int_a^b f \, \mathrm{d}g$ exists, e.g., if $f \in \mathbf{BV}[a,b]$ and $g \in \mathbf{BV}[a,b]$. For the basic properties of the KS-integral, see [8] and [5].

If $F\colon [a,b] \to \mathbb{R}^{m\times n}$, $G\colon [a,b] \to \mathbb{R}^{n\times p}$ and $H\colon [a,b] \to \mathbb{R}^{p\times m}$ are matrix valued functions, then the symbols

$$\int_a^b F \, \mathrm{d}[G] \quad \text{and} \quad \int_a^b \mathrm{d}[H] F$$

stand for the matrices

$$\left(\sum_{j=1}^{n}\int_a^b f_{ij}\,\mathrm{d}[g_{jk}]\right)_{\substack{i=1,\dots,m\\k=1,\dots,p}} \quad\text{and}\quad \left(\sum_{i=1}^{m}\int_a^b f_{ki}\,\mathrm{d}[h_{ij}]\right)_{\substack{k=1,\dots,p\\j=1,\dots,n}},$$

whenever all integrals appearing in the sums exist. Since the integral of a matrix valued function with respect to a matrix valued function is a matrix whose elements are sums of KS-integrals of real functions with respect to real functions, it is easy to reformulate all the statements from Section 5 in [8] for matrix valued functions (cf. [5, I.4]).

1.3 Generalized linear differential equations—fundamental properties. Here we describe some fundamental properties of generalized linear differential equations. More detailed information can be found in [5]. We restrict ourselves to the interval $[0,1]$. The modification to the case of an arbitrary closed interval $[a,b]\subset\mathbb{R}$ in place of $[0,1]$ is evident.

Definition and basic properties. Assume that

$$A\in\mathbf{BV}^{n\times n}[0,1],\ f\in\mathbf{BV}^{n}[0,1] \tag{1.1}$$

and consider the equation

$$x(t)=x(s)+\int_s^t \mathrm{d}[A]x+f(t)-f(s). \tag{1.2}$$

Let $[a,b]\subset[0,1]$. We say that a function $x\colon\ [a,b]\to\mathbb{R}^n$ is a solution of (1.2) on $[a,b]$ if there exists $\int_a^b \mathrm{d}[A]x\in\mathbb{R}^n$ and (1.2) holds for all $t,s\in[a,b]$.

Moreover, if $t_0\in[a,b]$ and $\tilde{x}\in\mathbb{R}^n$ are given, we say that $x\colon\ [a,b]\to\mathbb{R}^n$ is a solution of the initial value problem (1.2), $x(t_0)=\tilde{x}$ on $[a,b]$ if it is a solution of (1.2) on $[a,b]$ and $x(t_0)=\tilde{x}$, i.e. if

$$x(t)=\tilde{x}+\int_{t_0}^t \mathrm{d}[A]x+f(t)-f(t_0) \tag{1.3}$$

for all $t\in[a,b]$.

Notice that under the assumptions (1.1) each solution of the equation (1.2) on $[0,1]$ is of bounded variation on $[0,1]$ (see [5, III.1.3]).

Theorem 1.1 ([5, III.1.4]). *Let* $A \in \mathbf{BV}^{n\times n}[0,1]$. *If* $t_0 \in [0,1]$, *then the initial value problem* (1.3) *possesses for any* $f \in \mathbf{BV}^n[0,1]$, $\tilde{x} \in \mathbb{R}^n$ *a unique solution* $x(t)$ *defined on* $[0,1]$ *if and only if* $\det[\mathrm{I} - \Delta^- A(t)] \neq 0$ *on* $(t_0, 1]$ *and* $\det[\mathrm{I} + \Delta^+ A(t)] \neq 0$ *on* $[0, t_0)$.

Lemma 1.2 ([5, III.1.6]). *Assume that* $A \in \mathbf{BV}^{n\times n}[0,1]$, $f \in \mathbf{BV}^n[0,1]$. *Let* $x(t)$ *be a solution of the equation* (1.2) *on an interval* $[a,b] \subset [0,1]$. *Then the onesided limits* $x(a+)$, $x(t+)$, $x(t-)$, $x(b-)$, $t \in (a,b)$ *exist and the relations*

$$\begin{cases} x(t+) = [\mathrm{I} + \Delta^+ A(t)]x(t) + \Delta^+ f(t) & \text{for all} \quad t \in [a,b), \\ x(t-) = [\mathrm{I} - \Delta^- A(t)]x(t) - \Delta^- f(t) & \text{for all} \quad t \in (a,b] \end{cases} \tag{1.4}$$

hold.

Fundamental matrix.

Lemma 1.3 ([5, III.2.10, III.2.11]). *For a given* $A \in \mathbf{BV}^{n\times n}[0,1]$ *such that* $\det[\mathrm{I} - \Delta^- A(t)] \neq 0$ *on* $(0,1]$ *and* $\det[\mathrm{I} + \Delta^+ A(t)] \neq 0$ *on* $[0,1)$ *there exists a unique* $U\colon [0,1] \times [0,1] \to \mathbb{R}^{n\times n}$ *such that*

$$U(t,s) = \mathrm{I} + \int_s^t \mathrm{d}[A(r)]U(r,s) \tag{1.5}$$

for all $t, s \in [0,1]$.

Moreover, there exists a unique matrix valued function $X(t)$ *such that*

$$U(t,s) = X(t)X^{-1}(s) \tag{1.6}$$

for every $s,t \in [0,1]$, *where* $X\colon [0,1] \to \mathbb{R}^{n\times n}$ *satisfies the matrix equation* $X(t) = \mathrm{I} + \int_0^t \mathrm{d}[A]X$ *for* $t \in [0,1]$. *The matrix* $X(t)$ *is regular for each* $t \in [0,1]$.

Let $\tau \in [0,1]$. The matrix $U(t,s)$ can be used to express the unique solution of the homogeneous initial value problem

$$x(t) = \tilde{x} + \int_\tau^t \mathrm{d}[A]x$$

on $[\tau, 1]$. This solution is given by the relation

$$x(t) = U(t,\tau)\tilde{x}. \tag{1.7}$$

Definition 1.4. The matrix $X\colon [0,1] \to \mathbb{R}^{n\times n}$ given by Lemma 1.3 is called the fundamental matrix for the homogeneous generalized linear differential equation

$$x(t) = x(s) + \int_s^t \mathrm{d}[A]x \quad \text{for each } s,t \in [0,1]. \tag{1.8}$$

1.4 Continuous dependence of solutions on a parameter. Let us start with the following Lemma 1.5 for the case of uniform convergence of $A_k(t)$ to $A_0(t)$ on a compact interval. This lemma was stated in Theorem 1 in [1]. We present it in the form which we will need for the proof of our main convergence result (see Theorem 2.2). We will work with initial value problems

$$x(t) = \tilde{x}_k + \int_a^t \mathrm{d}[A_k]x + f_k(t) - f_k(a), \quad k \in \mathbb{N}_0 \text{ and } t \in [a,b]. \tag{1.9}$$

Lemma 1.5 ([1, Theorem 1]). *Let $0 \leqslant a < b \leqslant 1$, $A_k \in \mathbf{BV}^{n\times n}[a,b]$ for $k \in \mathbb{N}_0$ and let $\det[\mathrm{I} - \Delta^- A_0(t)] \neq 0$ on $(a,b]$. Assume that the sequence $\{A_k\}_{k=1}^\infty$ satisfies the following two conditions:*

(I) $\sup\limits_{k\in\mathbb{N}} \operatorname{var}_a^b A_k < +\infty$,

(II) $\big(A_k(t) - A_k(a)\big) \rightrightarrows \big(A_0(t) - A_0(a)\big)$ *on* $[a,b]$.

Then for $k = 0$ and for a.a. $k \in \mathbb{N}$ there exists a fundamental matrix X_k corresponding to A_k and $X_k \rightrightarrows X_0$ on $[a,b]$.

Moreover, let $\det[\mathrm{I} + \Delta^+ A_0(t)] \neq 0$ on $[a,b)$, $\tilde{x}_k \to \tilde{x}_0$, $f_k \in \mathbf{BV}^n[a,b]$ for $k \in \mathbb{N}_0$ and let the sequence $\{f_k\}_{k=1}^\infty$ satisfy the condition $\big(f_k(t) - f_k(a)\big) \rightrightarrows \big(f_0(t) - f_0(a)\big)$ on $[a,b]$.

Then for $k = 0$ and a.a. $k \in \mathbb{N}$ there exists a unique solution x_k of (1.9) on $[a,b]$ and

$$x_k \rightrightarrows x_0 \quad \text{on} \quad [a,b],$$

where $x_0(t)$ is solution of (1.9) for $k = 0$.

Below we will formulate a result concerning the case when the assumption (II) from Lemma 1.5 is not satisfied. To this aim, let us introduce the following notation.

Notation 1.6. Assume that a sequence $\{A_k\}_{k=1}^\infty \subset \mathbf{BV}^{n\times n}[0,1]$ is given and $A_0 \in \mathbf{BV}^{n\times n}[0,1]$. For $k \in \mathbb{N}$ and an arbitrary closed interval $J = [\alpha,\beta] \subset [0,1]$, define

$$A_k^J(t) = A_k(t) - A_k(\alpha) \quad \text{for} \quad k \in \mathbb{N}_0,\ t \in J.$$

Theorem 1.7 ([7, Theorem 3.3.2]). *Let $A_k \in \mathbf{BV}^{n\times n}[0,1]$ for $k \in \mathbb{N}_0$ and $\det[\mathrm{I} - \Delta^- A_0(t)] \neq 0$ on $(0,1]$. Furthermore, assume that there is a finite set $D \subset [0,1]$ such that*

$$A_k^J(s) \rightrightarrows A_0^J(s) \text{ on } J \text{ for any closed interval } J \subset [0,1] \setminus D, \tag{1.10}$$

$$\sup_{k\in\mathbb{N}} \operatorname*{var}_0^1 A_k < +\infty \text{ and } \det[\mathrm{I} - \Delta^- A_k(t)] \neq 0 \text{ for all } t \in D \text{ and for a.a. } k \in \mathbb{N}, \tag{1.11}$$

and

$$\begin{cases} \text{if } \tau \in D, \text{ then } \forall\, \xi \in \mathbb{R}^n \text{ and } \forall\, \varepsilon > 0\ \exists\, \delta > 0 \\ \text{such that } \forall\, \delta' \in (0,\delta)\ \exists\, k_0 \in \mathbb{N} \text{ such that the relations} \\ \qquad |u_k(\tau) - u_k(\tau - \delta') - \Delta^- A_0(\tau)[\mathrm{I} - \Delta^- A_0(\tau)]^{-1}\xi| < \varepsilon, \\ \qquad |v_k(\tau + \delta') - v_k(\tau) - \Delta^+ A_0(\tau)\xi| < \varepsilon \\ \text{are satisfied } \forall\, k \geqslant k_0 \text{ and } \forall\, u_k,\, v_k \text{ such that } |\xi - u_k(\tau - \delta')| \leqslant \delta, \\ |\xi - v_k(\tau)| \leqslant \delta \text{ and} \\ \qquad u_k(t) = u_k(\tau - \delta') + \displaystyle\int_{\tau-\delta'}^t \mathrm{d}[A_k]\, u_k(s) \quad \text{on } [\tau - \delta', \tau], \\ \qquad v_k(t) = v_k(\tau) + \displaystyle\int_\tau^t \mathrm{d}[A_k] v_k(s) \quad \text{on } [\tau, \tau + \delta']. \end{cases} \tag{1.12}$$

Then for a.a. $k \in \mathbb{N}$ the fundamental matrix X_k corresponding to A_k is defined on $[0,1]$ and

$$\lim_{k\to\infty} X_k(t) = X_0(t) \quad \text{on} \quad [0,1]. \tag{1.13}$$

R e m a r k 1.8. Theorem 1.7 is a slightly modified version of [7, Theorem 3.3.2]. Notation is simplified and, in particular, from the proof given in [7, Theorem 3.3.2] it follows that the assumption $\det[\mathrm{I} - \Delta^- A_k(t)] \neq 0$ on $(0,1]$ for all $k \in \mathbb{N}$ used in [7] is not necessary and can be replaced by a weaker one, i.e. $\det[\mathrm{I} - \Delta^- A_k(t)] \neq 0$ for all $t \in D$ and for a.a. $k \in \mathbb{N}$.

Conditions (1.10)–(1.12) characterize the concept of emphatic convergence introduced by J. Kurzweil (c.f. [3, Definition 4.1]). For more details see [7, Definition 3.2.8] or [4].

2. Main result

Let us consider a sequence of nonhomogeneous initial value problems

$$x(t) = \tilde{x}_k + \int_0^t \mathrm{d}[A_k]x + f_k(t) - f_k(0), \quad t \in [0,1],\ k \in \mathbb{N}_0 \tag{2.1}$$

where

$$A_k \in \mathbf{BV}^{n\times n}[0,1],\ f_k \in \mathbf{BV}^n[0,1] \quad \text{and} \quad \tilde{x}_k \in \mathbb{R}^n \quad \text{for} \quad k \in \mathbb{N}_0. \tag{2.2}$$

Notation 2.1. For a $k \in \mathbb{N}$ and an arbitrary closed interval $J = [\alpha, \beta] \subset [0,1]$, define

$$f_k^J(t) = f_k(t) - f_k(\alpha) \quad \text{for } k \in \mathbb{N}_0, \quad t \in J.$$

Our main result is the following assertion which extends Theorem 1.7 and provides conditions ensuring the continuous dependence of solutions of (2.1) on a parameter k. In comparison with Theorem 1.7, condition (1.12) has to be somewhat modified.

Theorem 2.2. *Assume that* (2.2) *and* (1.11) *hold and let*

$$\det[\mathrm{I} - \Delta^- A_0(t)] \neq 0 \ \textit{on}\ (0,1] \quad \textit{and} \quad \det[\mathrm{I} + \Delta^+ A_0(t)] \neq 0 \ \textit{on}\ [0,1).$$

Let $\tilde{x}_k \to \tilde{x}_0$ *and let* $x_0(t)$ *denote a solution of*

$$x(t) = \tilde{x}_0 + \int_0^t \mathrm{d}[A_0]x + f_0(t) - f_0(0), \quad t \in [0,1]. \tag{2.3}$$

Moreover, assume that

(2.4) *there is a finite set* $D \subset [0,1]$ *such that* $A_k^J \rightrightarrows A_0^J$ *and* $f_k^J \rightrightarrows f_0^J$ *on* J *for any closed interval* $J \subset [0,1] \setminus D$

and

$$\left\{\begin{array}{l} \textit{if } \tau \in D, \textit{ then } \forall\, \xi \in \mathbb{R}^n \textit{ and } \forall\, \varepsilon > 0\ \exists\, \delta > 0 \textit{ such that} \\ \forall\, \delta' \in (0,\delta)\ \exists\, k_0 \in \mathbb{N} \textit{ such that the relations} \\ |u_k(\tau) - u_k(\tau - \delta') - \Delta^- A_0(\tau)[\mathrm{I} - \Delta^- A_0(\tau)]^{-1}\xi \\ \qquad - [\mathrm{I} - \Delta^- A_0(\tau)]^{-1}\Delta^- f_0(\tau)| < \varepsilon, \\ |v_k(\tau + \delta') - v_k(\tau) - \Delta^+ A_0(\tau)\xi - \Delta^+ f_0(\tau)| < \varepsilon \\ \textit{are satisfied } \forall\, k \geqslant k_0 \textit{ and } \forall\, u_k,\ v_k \textit{ fulfilling (2.6), (2.7) and such that} \\ |\xi - u_k(\tau - \delta')| \leqslant \delta,\ |\xi - v_k(\tau)| \leqslant \delta, \textit{ where} \end{array}\right. \tag{2.5}$$

$$(2.6)\quad u_k(t) = u_k(\tau-\delta') + \int_{\tau-\delta'}^{t} \mathrm{d}[A_k]u_k + f_k(t) - f_k(\tau-\delta') \quad on \quad [\tau-\delta',\tau],$$

$$(2.7)\quad v_k(t) = v_k(\tau) + \int_{\tau}^{t} \mathrm{d}[A_k]v_k + f_k(t) - f_k(\tau) \quad on \quad [\tau,\tau+\delta'].$$

Then for a.a. $k \in \mathbb{N}$ *the solution* x_k *of* (2.1) *exists on* $[0,1]$ *and*

$$\lim_{k\to\infty} x_k(t) = x_0(t) \tag{2.8}$$

for any $t \in [0,1]$*, where* $x_0(t)$ *is the solution of* (2.3)*. Moreover,* (2.8) *holds locally uniformly on* $[0,1]\setminus D$*.*

Proof. First, notice that Lemma 1.5 implies that (2.8) holds locally uniformly on $[0,1]\setminus D$.

Assume that $D=\{\tau\}$, where $\tau\in(0,1)$.

Due to Theorems 1.1 and 1.7, we can see that the problem (2.1) has a solution x_k on $[0,1]$ for a.a. $k\in\mathbb{N}$. Indeed, by Theorem 1.7 there exists a fundamental matrix X_k corresponding to A_k for a.a. $k\in\mathbb{N}$. Moreover, $\det[\mathrm{I}-\Delta^{-}A_k(t)]\neq 0$ on $[0,1]$ for a.a. $k\in\mathbb{N}$ wherefrom, by Theorem 1.1, our claim follows.

The rest of the proof is divided into three steps. First, we prove that (2.8) is true for $t\in[0,\tau)$, then for $t=\tau$ and finally for $t\in(\tau,1]$.

Step 1. Let $\alpha\in(0,\tau)$ be given. Then by Lemma 1.5 the relation (2.8) holds uniformly on $[0,\alpha]$. Therefore (2.8) is true for any $t\in[0,\tau)$.

Step 2. Now we will prove that (2.8) is true also for $t=\tau$. Note that according to (1.4) we have

$$x_0(\tau-) = x_0(\tau) - \Delta^{-}A_0(\tau)x_0(\tau) - \Delta^{-}f_0(\tau).$$

Given an arbitrary $\delta'\in(0,\tau)$ and $k\in\mathbb{N}$, we obtain

$$\begin{aligned}
|x_0(\tau)-x_k(\tau)| &\leqslant |x_0(\tau)-\Delta^{-}A_0(\tau)x_0(\tau)-\Delta^{-}f_0(\tau)-x_0(\tau-\delta')| \\
&\quad + |x_0(\tau-\delta')-x_k(\tau-\delta')| + |x_k(\tau-\delta')+\Delta^{-}A_0(\tau)x_0(\tau)+\Delta^{-}f_0(\tau)-x_k(\tau)| \\
&= |x_0(\tau-)-x_0(\tau-\delta')| + |x_0(\tau-\delta')-x_k(\tau-\delta')| \\
&\quad + |x_k(\tau)-x_k(\tau-\delta')-\Delta^{-}A_0(\tau)[\mathrm{I}-\Delta^{-}A_0(\tau)]^{-1}x_0(\tau-) \\
&\quad - [\mathrm{I}-\Delta^{-}A_0(\tau)]^{-1}\Delta^{-}f_0(\tau)|,
\end{aligned}$$

where we made use of the fact that the relation

$$\mathrm{I}+B[\mathrm{I}-B]^{-1} = [\mathrm{I}-B]^{-1}$$

is true whenever the matrix $[\mathrm{I}-B]$ is regular.

Choose $\varepsilon > 0$. According to (2.5) we can choose $\delta \in (0, \varepsilon)$ in such a way that for each $\delta' \in (0, \delta)$ there exists $k_1 = k_1(\delta') \in \mathbb{N}$ such that

$$|u_k(\tau) - u_k(\tau - \delta') - \Delta^- A_0(\tau)[\mathrm{I} - \Delta^- A_0(\tau)]^{-1} x_0(\tau-) - [\mathrm{I} - \Delta^- A_0(\tau)]^{-1} \Delta^- f_0(\tau)| < \varepsilon$$

holds for any $k \geqslant k_1$ and for each solution $u_k(t)$ of equation (2.6) with $|x_0(\tau-) - u_k(\tau - \delta')| \leqslant \delta$.

Set $u_k(t) = x_k(t)$ for $t \in [\tau - \delta', \tau]$. Choose $\delta' \in (0, \delta)$ such that $|x_0(\tau-) - x_0(\tau - \delta')| < \delta/2$. Taking into account that $x_k(t) \to x_0(t)$ on $[0, \tau)$ as $k \to \infty$ we get the existence of a $k_0 \in \mathbb{N}$, $k_0 \geqslant k_1$, such that $|x_0(\tau - \delta') - x_k(\tau - \delta')| < \delta/2$ for all $k \geqslant k_0$. Therefore the estimate

$$|x_0(\tau-) - x_k(\tau - \delta')| \leqslant |x_0(\tau-) - x_0(\tau - \delta')| + |x_0(\tau - \delta') - x_k(\tau - \delta')| < \delta$$

is true for $k \geqslant k_0$. Moreover, we have

$$|x_k(\tau) - x_k(\tau - \delta') - \Delta^- A_0(\tau)[\mathrm{I} - \Delta^- A_0(\tau)]^{-1} x_0(\tau-) - [\mathrm{I} - \Delta^- A_0(\tau)]^{-1} \Delta^- f_0(\tau)| < \varepsilon.$$

To summarize, we have

$$|x_0(\tau) - x_k(\tau)| < \frac{\delta}{2} + \frac{\delta}{2} + \varepsilon < 2\varepsilon \quad \text{for all } k \geqslant k_0,$$

i.e. $x_k(\tau) \to x_0(\tau)$ for $k \to \infty$.

Step 3. Proof of the convergence on $(\tau, 1]$ consists of two parts. First we show that there is a $\delta > 0$ such that $x_k(t) \to x_0(t)$ converges on $(\tau, \tau + \delta)$ as $k \to \infty$. Then this pointwise convergence is extended to $(\tau, 1]$.

Let $\varepsilon > 0$ be given and let $\delta_0 \in (0, \varepsilon)$ be such that

$$|x_0(s) - x_0(\tau+)| < \varepsilon \quad \text{for all } s \in (\tau, \tau + \delta_0).$$

By the assumption (2.5), there is a $\delta \in (0, \delta_0)$ such that

$$\forall \delta' \in (0, \delta)\, \exists\, k_1 = k_1(\delta') \in \mathbb{N}$$

and such that

$$|v_k(\tau + \delta') - v_k(\tau) - \Delta^+ A_0(\tau) x_0(\tau) - \Delta^+ f_0(\tau)| < \varepsilon$$

is true for each $k \geqslant k_1$ and for each solution $v_k(t)$ of (2.7) satisfying

$$|x_0(\tau) - v_k(\tau)| \leqslant \delta.$$

Now, for each $\delta' \in (0, \delta)$ the distance between $x_0(\tau + \delta')$ and $x_k(\tau + \delta')$ can be estimated. In view of the fact that

$$x_0(\tau+) - x_0(\tau) = \Delta^+ A_0(\tau) x_0(\tau) + \Delta^+ f_0(\tau)$$

we have

$$\begin{aligned} &|x_0(\tau+\delta') - x_k(\tau+\delta')| \leqslant |x_0(\tau+\delta') - x_0(\tau+)| \\ &\quad + |x_0(\tau+) - x_0(\tau) + x_k(\tau) - x_k(\tau+\delta')| + |x_0(\tau) - x_k(\tau)| \\ &= |x_0(\tau+\delta') - x_0(\tau+)| \\ &\quad + |\Delta^+ A_0(\tau) x_0(\tau) + \Delta^+ f_0(\tau) + x_k(\tau) - x_k(\tau+\delta')| + |x_0(\tau) - x_k(\tau)|. \end{aligned}$$

Since $x_k(\tau) \to x_0(\tau)$ for $k \to \infty$, we get an existence of $k_0 \in \mathbb{N}$, $k_0 \geqslant k_1$ such that $|x_0(\tau) - x_k(\tau)| < \delta$ for all $k \geqslant k_0$. Since $\tau + \delta' \in (\tau, \tau + \delta_0)$, we have $|x_0(\tau + \delta') - x_0(\tau+)| < \varepsilon$. Setting $v_k(t) = x_k(t)$ on $[\tau, \tau + \delta']$, we get

$$|x_k(\tau+\delta') - x_k(\tau) - \Delta^+ A_0(\tau) x_0(\tau) - \Delta^+ f_0(\tau)| < \varepsilon \quad \text{for all } k \geqslant k_0.$$

Altogether, for any $k \geqslant k_0$ the estimate

$$|x_0(\tau+\delta') - x_k(\tau+\delta')| \leqslant \varepsilon + \delta + \varepsilon < 3\varepsilon$$

is valid. Consequently, we have $x_k(t) \to x_0(t)$ for $k \to \infty$ on $(\tau, \tau + \delta)$.

Now, choose an arbitrary σ in $(\tau, \tau + \delta)$. Making use of Lemma 1.5 with $[a, b] = [\sigma, 1]$ the proof of the validity of (2.8) for any $t \in [0, 1]$ can be completed. The extension to the case $D = \{\tau_1, \tau_2, \ldots, \tau_m\}$, $m > 1$ is obvious. □

Corollary 2.3. *Let A_k, f_k be left-continuous for all $k \in \mathbb{N}$. Assume that* (2.2) *holds and let $\sup_{k \in \mathbb{N}} \operatorname{var}_0^1 A_k < +\infty$ and $\det[\mathrm{I} + \Delta^+ A_0(t)] \neq 0$ on $[0, 1)$. Let $\tilde{x}_k \to \tilde{x}_0$ and let $x_0(t)$ denote a solution of* (2.3)*. Moreover, assume* (2.4) *and*

$$\text{(2.9)} \quad \begin{cases} \text{if } \tau \in D, \text{ then } \forall \xi \in \mathbb{R}^n \text{ and } \forall \varepsilon > 0 \ \exists\, \delta > 0 \text{ such that } \forall\, \delta' \in (0, \delta) \ \exists\, k_0 \in \mathbb{N} \\ \text{such that the relation } |v_k(\tau + \delta') - v_k(\tau) - \Delta^+ A_0(\tau)\xi - \Delta^+ f_0(\tau)| < \varepsilon \\ \text{is satisfied } \forall\, k \geqslant k_0 \text{ and } \forall\, v_k \text{ fulfilling (2.7) and such that } |\xi - v_k(\tau)| \leqslant \delta. \end{cases}$$

Then for a.a. $k \in \mathbb{N}$ the solution x_k of (2.1) *exists on $[0, 1]$ and* (2.8) *holds for any $t \in [0, 1]$, where $x_0(t)$ is a solution of* (2.3)*. Moreover,* (2.8) *holds locally uniformly on $[0, 1] \setminus D$.*

P r o o f. Assertion of Corollary 2.3 follows from the proof of Theorem 2.2 and from Lemma 1.5. □

The following example illustrates Theorem 2.2 and its Corollary 2.3.

E x a m p l e 2.4. Let $P\colon \mathbb{R} \to \mathbb{R}^{n\times n}$ be measurable, locally Lebesgue integrable on $\mathbb{R}$ and such that $P(t)\,P(s) = P(s)\,P(t)$, let $g\colon \mathbb{R} \to \mathbb{R}^n$ be Lebesgue integrable on $[0,1]$, $r \in \mathbb{R}^n$, $D = \{\tau\}$ and $0 < \tau < 1$. For $k \in \mathbb{N}$ and $t \in \mathbb{R}$ define

$$B_k(t) = P + k\chi_{(\tau,\tau+\frac{1}{k})}(t)\,\mathrm{I} \quad \text{and} \quad g_k(t) = g(t) + k\chi_{(\tau,\tau+\frac{1}{k})}(t)r$$

and

$$A_k(t) = \int_0^t B_k(s)\,\mathrm{d}s \quad \text{and} \quad f_k(t) = \int_0^t g_k(s)\,\mathrm{d}s$$

and consider initial value problems for ordinary differential equations

$$x' = B_k(t)x + g_k(t), \quad x(0) = \tilde{x}_k, \quad k \in \mathbb{N}, \tag{2.10}$$

where $\tilde{x}_k$, $\tilde{x}_0 \in \mathbb{R}^n$ and $\tilde{x}_k \to \tilde{x}_0$. The problems (2.10) can be equivalently reformulated in the form

$$x(t) = \tilde{x}_k + \int_0^t \mathrm{d}[A_k]x + f_k(t) - f_k(0), \quad k \in \mathbb{N},\ t \in \mathbb{R}.$$

Clearly, for each $k \in \mathbb{N}$ the problem (2.10) has a unique solution x_k on $\mathbb{R}$.

We can see that

$$\operatorname*{var}_0^1 A_k \leqslant |P| + k\int_\tau^{\tau+1/k} \mathrm{d}s = |P| + 1 < \infty \quad \text{for all } k \in \mathbb{N}. \tag{2.11}$$

Furthermore, for each $t \in \mathbb{R}$,

$$\lim_{k\to\infty} A_k(t) = Pt + \chi_{(\tau,\infty)}(t)\,\mathrm{I} \quad \text{and} \quad \lim_{k\to\infty} f_k(t) = g(t) + \chi_{(\tau,\infty)}(t)r. \tag{2.12}$$

Defining

$$A_0(t) = Pt + \chi_{(\tau,\infty)}(t)D^+ \quad \text{and} \quad f_0(t) = \int_0^t g(s)\,\mathrm{d}s + \chi_{(\tau,\infty)}(t)d^+$$

for $t \in \mathbb{R}$, where $D^+ \in \mathbb{R}^{n\times n}$ and $d^+ \in \mathbb{R}^n$ are to be determined later, we can see that

$$A_k^J \rightrightarrows A_0^J \quad \text{and} \quad f_k^J \rightrightarrows f_0^J \quad \text{holds for each compact } J \subset [0,1]\setminus\{\tau\}. \tag{2.13}$$

Moreover, $A_k \rightrightarrows A_0$ and $f_k \rightrightarrows f_0$ on $[0,\tau]$ and therefore, due to Lemma 1.5, $x_k \rightrightarrows x_0$ on $[0,\tau]$, where x_0 is a solution of

$$x(t) = \tilde{x}_0 + \int_0^t \mathrm{d}[A_0]x + f_0(t) - f_0(0), \quad t \in [0,1]. \tag{2.14}$$

Since $B_k(t)B_k(s) = B_k(s)B_k(t)$ holds for each $t, s \in \mathbb{R}$ and each $k \in \mathbb{N}$, the fundamental matrices V_k for $v' = B_k(t)v$ fulfilling the condition $V_k(\tau) = \mathrm{I}$ are for $t \in \mathbb{R}$ and $k \in \mathbb{N}$ given by

$$V_k(t) = \exp\left(\int_\tau^t B_k(s)\,\mathrm{d}s\right) = \Phi(t)\Phi^{-1}(\tau)\begin{cases} 1 & \text{if } t < \tau, \\ \mathrm{e}^{k(t-\tau)} & \text{if } t \in (\tau, \tau + 1/k), \\ \mathrm{e} & \text{if } t \geqslant \tau + 1/k, \end{cases}$$

where $\Phi(t) = \mathrm{e}^{Pt}$, $t \in \mathbb{R}$, is the fundamental matrix for the ordinary differential equation $x' = Px$ fulfilling $\Phi(0) = \mathrm{I}$.

Now, let $k \in \mathbb{N}$, $\varepsilon > 0$, $\delta' > 0$, $\xi \in \mathbb{R}^n$, $\tilde{v}_k \in \mathbb{R}^n$ and let v_k be a solution of

$$v' = B_k(t)v + g_k(t), \quad v(\tau) = \tilde{v}_k \tag{2.15}$$

on $[\tau, 1]$. Then

$$v_k(\tau+\delta') = \Phi(\tau+\delta')\Phi^{-1}(\tau+\tfrac{1}{k})v_k(\tau+\tfrac{1}{k}) + \Phi(\tau+\delta')\int_{\tau+1/k}^{\tau+\delta'} \Phi^{-1}(s)g(s)\,\mathrm{d}s$$

and

$$\begin{aligned} v_k(\tau+\tfrac{1}{k}) = {} & \mathrm{e}\Phi(\tau+\tfrac{1}{k})\Phi^{-1}(\tau)\tilde{v}_k + \Phi(\tau+\tfrac{1}{k})\int_\tau^{\tau+1/k} \mathrm{e}^{1-k(s-\tau)}\Phi^{-1}(s)g(s)\,\mathrm{d}s \\ & + k\Phi(\tau+\tfrac{1}{k})\left(\int_\tau^{\tau+1/k} \mathrm{e}^{1-k(s-\tau)}\Phi^{-1}(s)\,\mathrm{d}s\right)r, \end{aligned}$$

which yields

$$\begin{aligned} & |v_k(\tau+\delta') - \tilde{v}_k - \Delta^+A_0(\tau)\xi - \Delta^+f_0(\tau)| \\ & \leqslant |[\mathrm{e}\Phi(\tau+\delta')\Phi^{-1}(\tau) - \mathrm{I}]\tilde{v}_k - D^+\xi| + |\Phi(\tau+\delta')|\left|\int_\tau^{\tau+1/k} \mathrm{e}^{1-k(s-\tau)}\Phi^{-1}(s)g(s)\,\mathrm{d}s\right| \\ & \quad + \left|\Phi(\tau+\delta')\int_{\tau+1/k}^{\tau+\delta'} \Phi^{-1}(s)g(s)\,\mathrm{d}s\right| \\ & \quad + \left|k\Phi(\tau+\delta')\int_\tau^{\tau+1/k} \mathrm{e}^{1-k(s-\tau)}\Phi^{-1}(s)\,\mathrm{d}s - d^+\,\mathrm{I}\right||r|. \end{aligned}$$

It is easy to see that we can choose $\delta_1 > 0$ and $k_1 \in \mathbb{N}$ so that the second and third terms on the right hand side of the above inequality are less than $\varepsilon/5$ for all $\delta' \in (0,\delta_1)$ and $k \in \mathbb{N} \cap (k_0,\infty)$. Furthermore, since $\lim\limits_{\delta'\to 0} \Phi(\tau+\delta')\Phi^{-1}(\tau) = \mathrm{I}$, we can see that when choosing $D^+ = (\mathrm{e}-1)\,\mathrm{I}$, we can find $\delta_2 \in (0,\delta_1)$ such that the first term becomes smaller than $\varepsilon/5$ whenever $\delta' \in (0,\delta_2)$ and $|\tilde{v}_k - \xi| < \delta_2$. Finally, observing that

$$\lim_{k\to\infty} k \int_\tau^{\tau+1/k} \mathrm{e}^{1-k(s-\tau)}\Phi^{-1}(s)\,\mathrm{d}s = \mathrm{e}\Phi^{-1}(\tau),$$

we can conclude that, when setting $d^+ = \mathrm{e}$, we can choose $k_0 \in \mathbb{N} \cap (k_1,\infty)$ and $\delta \in (0,\delta_0)$ so that also the fourth term becomes smaller than $\varepsilon/5$ whenever $k \in \mathbb{N} \cap (k_0,\infty)$ and $\delta' \in (0,\delta)$. To summarize, all assumptions of Corollary 2.3 are satisfied if we define

$$A_0(t) = Pt + \chi_{(\tau,\infty)}(t)(\mathrm{e}-1)\,\mathrm{I} \quad \text{and} \quad f_0(t) = \int_0^t g(s)\,\mathrm{d}s + \chi_{(\tau,\infty)}(t)\mathrm{e}r.$$

Therefore $\lim\limits_{k\to\infty} x_k(t) = x_0(t)$ holds for each $t \in [0,1]$.

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Author's address: *Zdeněk Halas*, Department of Mathematical Analysis and Applications of Mathematics, Faculty of Science, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic, e-mail: `zdenekhalas@seznam.cz`.