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A PRIORI ESTIMATES OF SOLUTIONS OF SUPERLINEAR PROBLEMS

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Dedicated to Professor J. Nečas on the occasion of his 70th birthday

Abstract. In this survey we consider superlinear parabolic problems which possess both blowing-up and global solutions and we study a priori estimates of global solutions.

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Introduction

In this survey we consider superlinear parabolic problems which possess both blowing-up and global solutions and we study a priori estimates of global solutions. Assume that a given parabolic problem is well-posed in a function space X and denote by G the set of all initial functions in X for which the solution is global. Let $u\colon [0,\infty)\to X$ be a global solution. We are interested in estimates of the types

$$\|u(t)\|_X \leqslant C\big(u(0)\big), \tag{1.1}$$

$$\|u(t)\|_X \leqslant C\big(\|u(0)\|_X\big), \tag{1.2}$$

$$\|u(t)\|_X \leqslant C(\delta) \quad \text{for } t\geqslant\delta>0. \tag{1.3}$$

Estimate (1.1) means that the solution u is bounded. In estimate (1.2), the bound depends only on the norm of the initial condition and this estimate easily implies the closedness of the set G. In estimate (1.3), the constant $C(\delta)$ is supposed to be universal for all global solutions so that this estimate is the strongest one. It implies, in particular, a priori estimates for stationary solutions of the problem.

A typical example is the problem

$$
(1.4) \qquad \begin{cases} u_t = \Delta u + |u|^{p-1}u, & x \in \Omega,\ t \in (0,\infty), \\ u = 0, & x \in \partial\Omega,\ t \in (0,\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}
$$

where Ω is a smoothly bounded domain in $\mathbb{R}^n$, $p > 1$, $u_0 \in X$ and X is an appropriate function space ($C(\overline{\Omega})$ or $W^1_q(\Omega)$ with $q > 1$ large enough, for instance). In this problem, the zero solution is a stable stationary solution while the solutions blow up in finite time in the $L^\infty(\Omega)$-norm for "large" initial values. Consequently, $G \neq X$ and 0 belongs to the interior of G.

Let us briefly mention some results concerning a priori estimates of global solutions of problem (1.4); methods of their proofs and results for other problems will be discussed in the main part of this paper. The first result is due to Ni, Sacks and Tavantzis [14], who derived an a priori estimate of type (1.1) for global solutions of (1.4) (and more general problems) provided Ω is convex, $u_0 \geqslant 0$ and $p < 1 + 2/n$. Slightly later, Cazenave and Lions [2] proved estimate (1.2) for general Ω, u_0 and $(3n-4)p < 3n+8$, and estimate (1.1) (without any explicit dependence of C on $u(0)$) for

$$
(1.5) \qquad (n-2)p < n+2.
$$

Note that condition (1.5) cannot be improved, in general: this follows from [6]. Under assumption (1.5), Giga [8] proved (1.2) for global nonnegative solutions and the author [15] for all global solutions of (1.4). Estimate (1.3) for global nonnegative solutions of (1.4) was obtained recently in [5] under the additional assumption $(n-1)p < n+1$ and in [16] under assumptions (1.5) and $n \leqslant 3$. Notice that estimate (1.3) cannot be true for all global solutions since there exist arbitrarily large stationary (sign-changing) solutions.

Let us also mention that there are several reasons for the study of a priori estimates of global solutions mentioned above. If one can prove (1.2) and the problem admits a Lyapunov functional then in the ω-limit set of any solution starting on the boundary of G one can find interesting stationary solutions (see [17], [18], where this fact was used for nontrivial modifications of problem (1.4)). Moreover, a similar estimate for a modified problem enables one to establish the blow-up rate for blowing-up solutions; see [9].

Many proofs of a priori estimates are based on contradiction and rescaling arguments. Their main idea is the following: assume that there exist arbitrarily "large" solutions, rescale these solutions in such a way that the rescaled solutions converge to a nontrivial solution of a limiting problem and prove (or use the fact) that the limiting problem does not possess nontrivial solutions.

Let us start with the classical result of Gidas and Spruck [7] concerning a priori estimates for nonnegative stationary solutions of (1.4).

Theorem 2.1. *Assume* (1.5). *Then there exists a constant* $C > 0$ *such that any nonnegative stationary (classical) solution* u *of* (1.4) *satisfies* $\sup\limits_{\Omega} u < C$.

Idea of the proof. Assume that there exist nonnegative equilibria u_k of (1.4) such that

$$M_k := \max_{x\in\overline{\Omega}} u_k(x) = u_k(x_k) \to \infty \qquad \text{as } k \to \infty.$$

Put

$$v_k(y) = \frac{1}{M_k} u_k(x_k + \lambda_k y), \qquad \text{where } \lambda_k := M_k^{-(p-1)/2}. \tag{2.1}$$

Then v_k solves the problem

$$\begin{cases} 0 = \Delta v + v^p, & x \in \Omega_k, \\ v = 0, & x \in \partial\Omega_k, \end{cases}$$

where $\Omega_k := \{y\colon\ x_k + \lambda_k y \in \Omega\}$ and v_k attains its maximal value 1 at $y = 0$. Using the standard elliptic regularity theory it is not difficult to show that v_k converge to a positive solution v_∞ of the same problem in $\mathbb{R}^n$ or in H_c for some $c > 0$, where H_c is the halfspace

$$H_c = \{y \in \mathbb{R}^n :\ y_n > -c\}.$$

However, the limiting problem does not possess positive solutions provided (1.5) is true (see [7]). □

The next result is due to Giga [8].

Theorem 2.2. *Assume* (1.5). *Let* u *be a global nonnegative solution of* (1.4) *with* $u_0 \in C(\overline{\Omega})$. *Then there exists a constant* C *depending only on* $\sup\limits_{\Omega} u_0$ *such that* $u(x,t) \leqslant C$ *for any* $x \in \Omega$ *and* $t \geqslant 0$.

Idea of the proof. Assume that there exist global nonnegative solutions u_k of (1.4) with equibounded initial values $u_k(0)$ and $t_k > 0$ such that

$$M_k := \sup\{u_k(x,t)\colon\ x \in \Omega,\ t \in (0,t_k]\} = u_k(x_k,t_k) \to \infty \qquad \text{as } k \to \infty.$$

Put

$$v_k(y,s) = \frac{1}{M_k} u_k(x_k + \lambda_k y, t_k + \lambda_k^2 s),$$

where λ_k is as in (2.1). Then v_k solves the problem (1.4) in a rescaled region, and using parabolic regularity theory one obtains that the limit v_∞ solves the same problem in $\mathbb{R}^n \times (-\infty,0]$ or $H_c \times (-\infty,0]$. Moreover, $0 \leqslant v_\infty \leqslant 1 = v_\infty(0,0)$.

Let

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{1}{p+1}\int_\Omega |u|^{p+1}\,\mathrm{d}x \tag{2.2}$$

be the energy functional corresponding to (1.4). It is well known that the energy E is nonincreasing along any solution u of (1.4) and

$$\int_{t_0}^{t_1}\int_\Omega u_t^2 \,\mathrm{d}x\,\mathrm{d}t = E\bigl(u(t_0)\bigr) - E\bigl(u(t_1)\bigr), \tag{2.3}$$

where $u(t) := u(\cdot,t)$. Testing the equation in (1.4) by u yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2\,\mathrm{d}x = -2E\bigl(u(t)\bigr) + \frac{p-1}{p+1}\int_\Omega |u|^{p+1}\,\mathrm{d}x. \tag{2.4}$$

This identity implies both $E\bigl(u(t)\bigr) \geqslant 0$ and the boundedness of the $L^2(\Omega)$-norm of $u(t)$ for any global solution of (1.4) (otherwise the $L^2(\Omega)$-norm of $u(t)$ blows up in finite time).

There exist constants $\delta > 0$ and $C_\delta > 0$ such that $E\bigl(u_k(\delta)\bigr) < C_\delta$ and $t_k > 2\delta$ for k large. Consequently, (2.3) implies a uniform bound for $\int_\delta^{t_k}\int_\Omega |u_{kt}|^2\,\mathrm{d}x\,\mathrm{d}t$. For the rescaled functions v_k, this bound and (1.5) imply that the time derivative of the limiting function v_∞ is zero, hence we get the same contradiction as in the preceding theorem. □

The elliptic scaling from Theorem 2.1 was used also by the author in [16] in order to derive estimate (1.3).

Theorem 2.3. *Assume* (1.5) *and* $n \leqslant 3$. *Let* $\delta > 0$. *Then there exists a constant* $C(\delta)$ *such that* $u(x,t) \leqslant C(\delta)$ *for any* $x \in \Omega$, $t \geqslant \delta$ *and any global nonnegative (classical) solution* u *of* (1.4).

Idea of the proof. Due to Theorem 2.2, it is sufficient to show that $u(x,t) \leqslant C(\delta)$ for any $x \in \Omega$ and some $t \in [0,\delta]$. Assume the contrary and let $\sup_\Omega u_k(\cdot,t) \geqslant k$ for some global solutions u_k and any $t \in [0,\delta]$. Using the energy identity (2.3) (and several other ingredients including Hardy's inequality) one can find $t_k \in [0,\delta]$ such that the $L^2(\Omega)$-norm of the time derivative $u_{kt}(\cdot,t_k)$ can be bounded above by $CM_k^{p/2+1+\varepsilon}$ (where $M_k = \sup_\Omega u_k(\cdot,t_k)$ and $\varepsilon > 0$ is small). This estimate guarantees that the time-derivative u_{kt} vanishes after rescaling and passing to the limit so that we obtain the same contradiction as in Theorem 2.1. The restriction $n \leqslant 3$ is substantial in the limiting procedure (in order to get Hölder estimates for the rescaled solutions). $\square$

One of the main ingredients of the proofs of Theorems 2.2 and 2.3 is the Lyapunov functional E and the energy identity (2.3). Let us consider a problem where the existence of a Lyapunov functional is not known, for example

$$
\left\{
\begin{aligned}
u_t &= \Delta u + |v|^{p-1}v, && x \in \Omega,\ t \in (0,\infty),\\
v_t &= \Delta v + |u|^{q-1}u, && x \in \Omega,\ t \in (0,\infty),\\
u &= v = 0, && x \in \partial\Omega,\ t \in (0,\infty),\\
u(x,0) &= u_0(x),\ v(x,0) = v_0(x), && x \in \Omega,
\end{aligned}
\right.
\tag{2.5}
$$

where Ω is as above, $p, q > 1$. Assuming that (1.1) or (1.2) fail one can still repeat the rescaling procedure from Theorem 2.2 and use an idea of Hu [11] in order to obtain a bounded positive solution of (2.5) in $\mathbb{R}^n \times \mathbb{R}$ or $H_c \times \mathbb{R}$ (see [20] for details and other problems). Anyhow, Fujita's type nonexistence results (see [4] and [13]) guarantee that this problem has no positive solutions even in $\mathbb{R}^n \times \mathbb{R}^+$ and $H_c \times \mathbb{R}^+$ provided

$$
\frac{\max(p,q)+1}{pq-1} \geqslant \frac{n}{2} \quad \text{and} \quad \frac{\max(p,q)+1}{pq-1} \geqslant \frac{n+1}{2},
\tag{2.6}
$$

respectively. Unfortunately, this procedure (and condition (2.6)) do not seem to be optimal: in the case of (1.4), this approach would require

$$
p \leqslant 1 + \frac{2}{n+1}
\tag{2.7}
$$

instead of the optimal condition (1.5). One could still hope for an optimal (or better) result by proving a better Liouville's type nonexistence result concerning global positive bounded solutions in $\mathbb{R}^n \times \mathbb{R}$ or in $H_c \times \mathbb{R}$. However, this seems to be an open problem.

Let us also mention that an analogue to Theorem 2.1 (and its proof) can be obtained also for problems without Lyapunov functional. For example, a priori estimates for nonnegative stationary solutions of (a generalization of) problem (2.5) were derived using the approach of Theorem 2.1 in [21] under the (optimal) condition

$$\frac{1}{p+1}+\frac{1}{q+1}>\frac{n-2}{n}$$

if $n=3$.

Methods using energy and interpolation

Unlike methods in the preceding and in the next sections, the method described in this section does not require the positivity of solutions. It was first used by Cazenave and Lions [2] in order to derive estimate (1.2) for global solutions of (1.4) under the assumption $(3n-4)p<3n+8$ and then improved by the author in [15] (so that the condition $(3n-4)p<3n+8$ could be replaced by the optimal condition (1.5)).

Theorem 3.1. *Assume* (1.5) *and let* $X=H_0^1(\Omega)$ *(or* $X=C(\bar\Omega)$*). Then estimate* (1.2) *is true for any global (classical) solution of* (1.4).

I d e a o f t h e p r o o f. The proof is based on the energy identity (2.3). This identity together with the boundedness of $E\big(u(0)\big)$ and $E\big(u(t)\big)\geqslant 0$ implies

$$\int_0^\infty |u(t)|_2^2\,\mathrm{d}t<C, \tag{3.1}$$

where $|v|_q$ denotes the $L^q(\Omega)$-norm of v. Moreover, identity (2.4) implies the boundedness of $|u(t)|_2$. Squaring and integrating this identity, using

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|_2^2\leqslant|u(t)u_t(t)|_1\leqslant|u(t)|_2|u_t(t)|_2, \tag{3.2}$$

the boundedness of $|u(t)|_2$ and (3.1) yields

$$\sup_t\int_t^{t+1}|u(s)|_{p+1}^{(p+1)q}\,\mathrm{d}s<C, \tag{3.3}$$

where $q=2$. Now (3.1), (3.3) and an interpolation argument show the boundedness of $|u(t)|_\lambda$ provided

$$\lambda<\lambda(q):=p+1-\frac{p-1}{q+1}.$$

Since it is known that the bound for $u(t)$ in $L^\lambda(\Omega)$ implies a bound in $H^1_0(\Omega)$ provided

$$p < 1 + \frac{2\lambda}{n}, \tag{3.4}$$

the above estimate implies our assertion provided $p < 1 + \frac{2\lambda(2)}{n}$, that is $(3n-4)p < 3n+8$. In the general case, one can use Hölder's inequality

$$|u(t)u_t(t)| \leqslant |u(t)|_\lambda |u_t(t)|_{\lambda'}, \qquad \lambda' = \frac{\lambda}{\lambda - 1},$$

instead of (3.2), an interpolation estimate for $|u_t(t)|_{\lambda'}$ and Sobolev maximal regularity estimates to show that the boundedness of $|u(t)|_\lambda$ with $\lambda < \lambda(q)$ implies estimate (3.3) for some $\tilde q > q$ (see [15] for details). An obvious bootstrap procedure completes the proof. □

Although the result of Theorem 3.1 is optimal, its generalization to the Dirichlet problem for the equation $u_t = \Delta u - \frac{1}{2} x \cdot \nabla u - \beta u + |u|^{p-1} u$ in a varying domain as in [9] (which is needed for the corresponding proof of the blow-up rate for nonglobal solutions of (1.4)) seems to be open if $(3n-4)p \geqslant 3n+8$.

Kaplan's method and Hardy's inequality

Let $\varphi_1 > 0$ be the first eigenfunction of the operator $(-\Delta)$ with homogeneous Dirichlet boundary conditions on Ω and let $\lambda_1 > 0$ be the corresponding eigenvalue. Let u be a nonnegative solution of (1.4). If we multiply the equation in (1.4) by φ_1 and integrate over Ω, we obtain

$$\begin{aligned}\Big(\int_\Omega u(t)\varphi_1 \,\mathrm{d}x\Big)_t &= -\lambda_1 \int_\Omega u(t)\varphi_1 \,\mathrm{d}x + \int_\Omega u^p(t)\varphi_1 \,\mathrm{d}x \\ &\geqslant -\lambda_1 \int_\Omega u(t)\varphi_1 \,\mathrm{d}x + c\Big(\int_\Omega u(t)\varphi_1 \,\mathrm{d}x\Big)^p,\end{aligned}$$

which implies blow-up of the solution u in finite time provided $\int_\Omega u(t)\varphi_1 \,\mathrm{d}x$ is large enough for some $t \geqslant 0$. Consequently, if u is any global nonnegative solution of (1.4), then

$$\int_\Omega u(t)\varphi_1 \,\mathrm{d}x < C, \tag{4.1}$$

where C is a universal constant.

Let $\delta(x) := \operatorname{dist}(x, \partial\Omega)$. Then Hardy's type inequality

$$\Big|\frac{v}{\delta}\Big|_2 \leqslant C|\nabla v|_2, \qquad v \in H^1_0(\Omega),$$

standard imbedding theorems and

$$c_1\delta(x) \leqslant \varphi_1(x) \leqslant c_2\delta(x)$$

imply

$$\left|\frac{v}{\varphi_1^r}\right|_q \leqslant C\|v\| \qquad \text{if} \quad q(n-2+2r) \leqslant 2n,\ r \in [0,1], \tag{4.2}$$

where $\|\cdot\|$ denotes the norm in $H_0^1(\Omega)$ and a strict inequality is required if $n < 3$ (see [1], [16]).

Estimates (4.1), (4.2) and their modifications can be successfully used for a priori estimates of nonnegative stationary and global solutions of (1.4) and its modifications. Moreover, the proofs do not require any variational structure. Unfortunately, the corresponding results obtained in this way do not yield optimal results.

Let us start with the classical result of Brezis and Turner [1].

Theorem 4.1. *Let* $(n-1)p < n+1$ *and* $n \geqslant 3$. *Then there exists a constant* $C > 0$ *such that any nonnegative stationary (classical) solution* u *of* (1.4) *satisfies* $\sup\limits_{\Omega} u < C$.

Idea of the proof. Testing the equation in (1.4) with φ_1 and using (4.1) yields

$$\int_\Omega u^p\varphi_1\,\mathrm{d}x = \lambda_1\int_\Omega u\varphi_1\,\mathrm{d}x < C.$$

This estimate and (4.2) (with $q := p+1/(1-\alpha)$, $r := \alpha/(p(1-\alpha)+1)$, $\alpha := 2/(n+1)$) imply

$$\begin{aligned}\int_\Omega u^{p+1}\,\mathrm{d}x = \int_\Omega (u^p\varphi_1)^\alpha\Bigl(\frac{u^q}{\varphi_1^{qr}}\Bigr)^{1-\alpha}\,\mathrm{d}x &\leqslant \Bigl(\int_\Omega (u^p\varphi_1)\,\mathrm{d}x\Bigr)^\alpha\Bigl(\int_\Omega\Bigl(\frac{u}{\varphi_1^r}\Bigr)^q\,\mathrm{d}x\Bigr)^{1-\alpha}\\ &\leqslant C\|u\|^{q(1-\alpha)},\end{aligned}$$

where the exponent $q(1-\alpha)$ is less than 2. On the other hand, testing the equation in (1.4) with u yields

$$\int_\Omega u^{p+1}\,\mathrm{d}x = \int_\Omega |\nabla u|^2\,\mathrm{d}x \geqslant C\|u\|^2.$$

Comparing the last two estimates we obtain the assertion. □

The method of Brezis and Turner was used and generalized for many problems, see [3], for example.

In the parabolic case, Ni, Sacks and Tavantzis [14] used estimate (4.1) and the convexity of the domain in order to get an estimate for $|u(t)|_1$ which implies an $H^1(\Omega)$-bound provided $p < 1 + \frac{2}{n}$ (cf. (3.4)).

Estimate (4.1) plays a key role also in the paper by Fila, Souplet and Weissler [5]. These authors developed the linear and nonlinear theory for parabolic problems with initial values in the weighted spaces $L^q(\Omega, \delta(x)\,\mathrm{d}x)$. Using this theory and Theorem 2.2, they were able to show universal bounds for global nonnegative solutions from Theorem 2.3 under the assumption $(n-1)p < n+1$ for any n.

An approach based on (4.1)–(4.2) was used by Gu and Wang [10] or the author [19] for a priori estimates of nonnegative stationary or global solutions of the problem

$$\begin{cases} u_t = \Delta u + uv - bu, & x \in \Omega,\ t \in (0,\infty), \\ v_t = \Delta v + au, & x \in \Omega,\ t \in (0,\infty), \\ u = v = 0, & x \in \partial\Omega,\ t \in (0,\infty), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \end{cases}$$

respectively. Here Ω is a smoothly bounded domain in $\mathbb{R}^n$, $n \leqslant 3$ in the elliptic case and $n \leqslant 2$ in the parabolic case.

Finally, let us mention that estimates (4.1) and (4.2) can be used in order to get universal bounds for global nonnegative solutions of (2.5) (see [20]) and that these estimates were used also in the detailed proof of Theorem 2.3 (see [16]).

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