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ON THE INCOMPLETE GAMMA FUNCTION AND THE NEUTRIX CONVOLUTION

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Abstract. The incomplete Gamma function $\gamma(\alpha, x)$ and its associated functions $\gamma(\alpha, x_+)$ and $\gamma(\alpha, x_-)$ are defined as locally summable functions on the real line and some convolutions and neutrix convolutions of these functions and the functions x^r and x_-^r are then found.

Keywords: Gamma function, incomplete Gamma function, convolution, neutrix convolution

MSC 2000: 33B10, 46F10

The *incomplete Gamma function* $\gamma(\alpha, x)$ is defined for $\alpha > 0$ and $x \geqslant 0$ by

$$\gamma(\alpha, x) = \int_0^x u^{\alpha-1} \mathrm{e}^{-u} \,\mathrm{d}u, \tag{1}$$

see [5], the integral diverging for $\alpha \leqslant 0$.

Alternatively, we can define the incomplete Gamma function by

$$\gamma(\alpha, x) = \int_0^x |u|^{\alpha-1} \mathrm{e}^{-u} \,\mathrm{d}u, \tag{2}$$

and equation (2) defines $\gamma(\alpha, x)$ for all x, the integral again diverging for $\alpha \leqslant 0$.

We note that if $x > 0$ and $\alpha > 0$, then by integration by parts we see that

$$\gamma(\alpha + 1, x) = \alpha\gamma(\alpha, x) - x^\alpha \mathrm{e}^{-x} \tag{3}$$

and so we can use equation (3) to extend the definition of $\gamma(\alpha, x)$ to negative, non-integer values of α. In particular, it follows that if $-1 < \alpha < 0$ and $x > 0$, then

$$\begin{aligned}\gamma(\alpha, x) &= \alpha^{-1}\gamma(\alpha+1, x) + \alpha^{-1}x^{\alpha}\mathrm{e}^{-x} \\ &= -\alpha^{-1}\int_0^x u^{\alpha}\,\mathrm{d}(\mathrm{e}^{-u}-1) + \alpha^{-1}x^{\alpha}\mathrm{e}^{-x}\end{aligned}$$

and by integration by parts we see that

$$\gamma(\alpha, x) = \int_0^x u^{\alpha-1}(\mathrm{e}^{-u}-1)\,\mathrm{d}u + \alpha^{-1}x^{\alpha}.$$

More generally, it is easily proved by induction that if $-r < \alpha < -r+1$ and $x > 0$, then

$$\gamma(\alpha, x) = \int_0^x u^{\alpha-1}\left[\mathrm{e}^{-u} - \sum_{i=0}^{r-1}\frac{(-u)^i}{i!}\right]\mathrm{d}u + \sum_{i=0}^{r-1}\frac{(-1)^i x^{\alpha+i}}{(\alpha+i)i!}. \tag{4}$$

It follows that

$$\lim_{x\to\infty}\gamma(\alpha, x) = \Gamma(\alpha) \tag{5}$$

for $\alpha \neq 0, -1, -2, \ldots$, where Γ denotes the Gamma function.

We now define locally summable function $\gamma(\alpha, x_+)$ by

$$\gamma(\alpha, x_+) = \begin{cases}\displaystyle\int_0^x u^{\alpha-1}\mathrm{e}^{-u}\,\mathrm{d}u, & x \geqslant 0,\\ 0, & x < 0\end{cases}$$

if $\alpha > 0$ and we define the distribution $\gamma(\alpha, x_+)$ inductively by the equation

$$\gamma(\alpha, x_+) = \alpha^{-1}\gamma(\alpha+1, x_+) + \alpha^{-1}x_+^{\alpha}\mathrm{e}^{-x} \tag{6}$$

for $\alpha < 0$ and $\alpha \neq -1, -2, \ldots$.

If now $x < 0$ and $\alpha > 1$, then by integration by parts we see that

$$\gamma(\alpha+1, x) = -\alpha\gamma(\alpha, x) - |x|^{\alpha}\mathrm{e}^{-x} \tag{7}$$

and so we can use equation (7) to extend the definition of $\gamma(\alpha, x)$ to negative, non-integer values of α.

We now define locally summable function $\gamma(\alpha, x_-)$ by

$$\gamma(\alpha, x_-) = \begin{cases}\displaystyle\int_0^x |u|^{\alpha-1}\mathrm{e}^{-u}\,\mathrm{d}u, & x \leqslant 0,\\ 0, & x > 0\end{cases}$$

if $\alpha > 0$ and we define the distribution $\gamma(\alpha, x_-)$ inductively by the equation

$$\gamma(\alpha, x_-) = -\alpha^{-1}\gamma(\alpha+1, x_-) - \alpha^{-1}x_-^{\alpha}\mathrm{e}^{-x} \tag{8}$$

for $\alpha < 0$ and $\alpha \neq -1, -2, \ldots$. It follows that

$$\lim_{x \to -\infty} \gamma(\alpha, x_-) = \infty.$$

The classical definition of the convolution of two functions f and g is as follows:

Definition 1. Let f and g be functions. Then the *convolution* $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,\mathrm{d}t$$

for all points x for which the integral exists.

It follows easily from the definition that if $f * g$ exists then $g * f$ exists and

$$f * g = g * f, \tag{9}$$

and if $(f * g)'$ and $f * g'$ (or $f' * g$) exists, then

$$(f * g)' = f * g' \quad (\text{or } f' * g). \tag{10}$$

Definition 1 can be extended to define the convolution $f * g$ of two distributions f and g in $\mathcal{D}'$ by the following definition, see Gel'fand and Shilov [4].

Definition 2. Let f and g be distributions in $\mathcal{D}'$. Then the *convolution* $f * g$ is defined by the equation

$$\langle (f * g)(x), \varphi \rangle = \langle f(y), \langle g(x), \varphi(x+y) \rangle \rangle$$

for arbitrary φ in $\mathcal{D}$, provided f and g satisfy at least one of the conditions

(a) either f or g has bounded support,
(b) the supports of f and g are bounded on the same side.

It follows that if the convolution $f * g$ exists by this definition then equations (9) and (10) are satisfied.

The following convolutions were proved in [3]:

$$(x_+^{\alpha}\mathrm{e}^{-x}) * x_+^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^i \gamma(\alpha+i+1, x_+) x^{r-i}, \tag{11}$$

$$\gamma(\alpha, x_+) * x_+^r = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^i \gamma(\alpha+i, x_+) x^{r-i+1}, \tag{12}$$

$$(x_+^{\alpha}\mathrm{e}^{-x}) * x^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^i \Gamma(\alpha+i+1) x^{r-i} \tag{13}$$

for $r = 0, 1, 2, \ldots$ and $\alpha \neq 0, -1, -2, \ldots$.

We now prove some further results involving the convolution.

Theorem 1.

$$(x_-^{\alpha}\mathrm{e}^{-x}) * x_-^r = (-1)^{r-1} \sum_{i=0}^{r} \binom{r}{i} \gamma(\alpha+i+1, x_-) x^{r-i} \tag{14}$$

for $r = 0, 1, 2, \ldots$ *and* $\alpha \neq 0, -1, -2, \ldots$.

Proof. We first of all prove equation (14) when $\alpha > 0$. It is obvious that $(x_-^{\alpha}\mathrm{e}^{-x}) * x_-^r = 0$ if $x > 0$. When $x < 0$ we have

$$\begin{aligned}(x_-^{\alpha}\mathrm{e}^{-x}) * x_-^r &= \int_x^0 |x-u|^r |u|^{\alpha} \mathrm{e}^{-u} \,\mathrm{d}u \\ &= (-1)^r \sum_{i=0}^{r} \binom{r}{i} x^{r-i} \int_x^0 |u|^{\alpha+i} \mathrm{e}^{-u} \,\mathrm{d}u\end{aligned}$$

and equation (14) follows for the case $\alpha > 0$.

Now suppose that equation (14) holds when $-s < \alpha < -s+1$. This is true when $s = 0$. Then taking into account $-s < \alpha < -s+1$ and differentiating $(x_-^{\alpha}\mathrm{e}^{-x}) * x_-^r$, we get

$$(-\alpha x_-^{\alpha-1}\mathrm{e}^{-x} - x_-^{\alpha}\mathrm{e}^{-x}) * x_-^r = -r(x_-^{\alpha}\mathrm{e}^{-x}) * x_-^{r-1}.$$

It follows from our assumption and equation (8) that

$$\begin{aligned}\alpha(x_-^{\alpha-1}\mathrm{e}^{-x}) * x_-^r &= -(x_-^{\alpha}\mathrm{e}^{-x}) * x_-^r + r(x_-^{\alpha}\mathrm{e}^{-x}) * x_-^{r-1} \\ &= (-1)^r \sum_{i=0}^{r} \binom{r}{i} \gamma(\alpha+i+1, x_-) x^{r-i} \\ &\quad + (-1)^r r \sum_{i=0}^{r-1} \binom{r-1}{i} \gamma(\alpha+i+1, x_-) x^{r-i-1}\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{r-1} \sum_{i=0}^{r} \binom{r}{i} [(\alpha+i)\gamma(\alpha+i, x_-)x^{r-i} + x_-^{\alpha+i}\mathrm{e}^{-x}] \\
&\quad + (-1)^r r \sum_{i=1}^{r} \binom{r-1}{i-1} \gamma(\alpha+i, x_-)x^{r-i} \\
&= (-1)^{r-1}\alpha \sum_{i=0}^{r} \binom{r}{i} \gamma(\alpha+i, x_-)x^{r-i} \\
&\quad + (-1)^{r-1} \sum_{i=1}^{r} \left[i\binom{r}{i} - r\binom{r-1}{i-1} \right] \gamma(\alpha+i, x_-)x^{r-i} \\
&\quad + (-1)^{r-1} \sum_{i=0}^{r} \binom{r}{i} x_-^{r+\alpha}\mathrm{e}^{-x} \\
&= (-1)^{r-1}\alpha \sum_{i=0}^{r} \binom{r}{i} \gamma(\alpha+i, x_-)x^{r-i}
\end{aligned}
$$

and so equation (14) holds when $-s-1 < \alpha < -s$. It therefore follows by induction that equation (14) holds for all $\alpha \neq 0, -1, -2, \ldots$, which completes the proof of the theorem. □

Theorem 2.

$$
\gamma(\alpha, x_-) * x_-^r = \frac{(-1)^{r+1}}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} \gamma(\alpha+i, x_-)x^{r-i+1} \tag{15}
$$

for $r = 0, 1, 2, \ldots$ *and* $\alpha \neq 0, -1, -2, \ldots$.

Proof. We first of all prove equation (15) when $\alpha > 0$. It is obvious that $\gamma(\alpha, x_-) * x_-^r = 0$ if $x > 0$. When $x < 0$ we have

$$
\begin{aligned}
\gamma(\alpha, x_-) * x_-^r &= \int_x^0 |x-t|^r \int_0^t |u|^{\alpha-1}\mathrm{e}^{-u}\,\mathrm{d}u\,\mathrm{d}t \\
&= (-1)^r \int_x^0 |u|^{\alpha-1}\mathrm{e}^{-u} \int_u^x (x-t)^r\,\mathrm{d}t\,\mathrm{d}u \\
&= \frac{(-1)^r}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} x^{r-i+1} \int_x^0 |u|^{\alpha+i-1}\mathrm{e}^{-u}\,\mathrm{d}u
\end{aligned}
$$

and equation (15) follows for the case $\alpha > 0$.

Now suppose that equation (15) holds when $-s < \alpha < -s+1$. This is true when $s = 0$. Then noting that $-s-1 < \alpha < -s$ and using equations (8) and (14), we get

$$
\alpha\gamma(\alpha, x_-) * x_-^r = -\gamma(\alpha+1, x_-) * x_-^r - (x_-^{\alpha}\mathrm{e}^{-x}) * x_-^r
$$

$$
\begin{aligned}
&= \frac{(-1)^{r+1}}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} \gamma(\alpha+i+1, x_-) x^{r-i+1} \\
&\quad + (-1)^r \sum_{i=0}^{r} \binom{r}{i} \gamma(\alpha+i+1, x_-) x^{r-i} \\
&= \frac{(-1)^{r+1}}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} [(\alpha+i)\gamma(\alpha+i, x_-) + x_-^{\alpha+i} \mathrm{e}^{-x}] x^{r-i+1} \\
&\quad + (-1)^r \sum_{i=1}^{r+1} \binom{r}{i-1} \gamma(\alpha+i, x_-) x^{r-i+1} \\
&= \frac{(-1)^{r+1}\alpha}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} \gamma(\alpha+i, x_-) x^{r-i+1} \\
&\quad + (-1)^{r+1} \sum_{i=1}^{r+1} \left[\frac{i}{r+1} \binom{r+1}{i} - \binom{r}{i-1} \right] \gamma(\alpha+i, x_-) x^{r-i+1} \\
&\quad + \frac{(-1)^r}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^i x_-^{r+\alpha+1} \mathrm{e}^{-x} \\
&= \frac{(-1)^{r+1}\alpha}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} \gamma(\alpha+i, x_-) x^{r-i+1}
\end{aligned}
$$

and so equation (15) holds when $-s-1 < \alpha < -s$. It therefore follows by induction that equation (15) holds for all $\alpha \neq 0, -1, -2, \ldots$, which completes the proof of the theorem. □

In order to extend Definition 2 to distributions which do not satisfy conditions (a) or (b), we let τ be a function in $\mathcal{D}$ satisfying the conditions

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \leqslant \tau(x) \leqslant 1$,
(iii) $\tau(x) = 1$ for $|x| \leqslant \frac{1}{2}$,
(iv) $\tau(x) = 0$ for $|x| \geqslant 1$.

The function τ_n is then defined by

$$
\tau_n(x) = \begin{cases} 1, & |x| \leqslant n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n \end{cases}
$$

for $n = 1, 2, \ldots$.

The next definition was given in [2].

Definition 3. Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ for $n = 1, 2, \ldots$. Then the *neutrix convolution* $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, provided the limit h exists in the sense that

$$\underset{n\to\infty}{\text{N-lim}}\langle f_n * g, \varphi\rangle = \langle h, \varphi\rangle$$

for all φ in $\mathcal{D}$, where N is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range N'', the real numbers, with negligible functions being finite linear sums of the functions

$$n^\alpha \ln^{r-1} n, \quad \ln^r n \quad (\alpha > 0, \quad r = 1, 2, \ldots)$$

and all functions which converge to zero in the normal sense as n tends to infinity. In particular, if

$$\lim_{n\to\infty}\langle f_n * g, \varphi\rangle = \langle h, \varphi\rangle$$

for all φ in $\mathcal{D}$, we say that the *convolution* $f * g$ exists and equals h.

Note that in this definition the convolution $f_n * g$ is defined in Gel'fand and Shilov's sense, the distribution f_n having compact support. Note also that because of the lack of symmetry in the definition of $f \circledast g$, the neutrix convolution is in general non-commutative.

The following theorem was proved in [2], showing that the neutrix convolution is a generalization of the convolution.

Theorem 3. *Let f and g be distributions in $\mathcal{D}'$ satisfying either condition* (a) *or condition* (b) *of Gel'fand and Shilov's definition. Then the neutrix convolution $f \circledast g$ exists and*

$$f \circledast g = f * g.$$

The next theorem was also proved in [2].

Theorem 4. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix convolution $f \circledast g$ exists. Then the neutrix convolution $f \circledast g$ exists and*

$$(f \circledast g)' = f \circledast g'.$$

Note however that $(f \circledast g)'$ is not necessarily equal to $f' \circledast g$ but we do have the following theorem, which was proved in [3].

Theorem 5. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix convolution $f \circledast g$ exists. If* $\operatorname*{N-lim}_{n\to\infty}\langle (f\tau_n') * g, \varphi\rangle$ *exists and equals $\langle h, \varphi\rangle$ for all φ in $\mathcal{D}$, then the neutrix convolution $f' \circledast g$ exists and*

$$(f \circledast g)' = f' \circledast g + h.$$

For our next results, we need to extend our set of negligible functions to include finite linear sums of

$$n^{\alpha}\mathrm{e}^{n}, \quad \gamma(\alpha, -n_{-}): \quad \alpha \neq 0, -1, -2, \ldots .$$

The following neutrix convolution was proved in [3]:

$$\gamma(\alpha, x_{+}) \circledast x^{r} = \frac{1}{r+1}\sum_{i=1}^{r+1}\binom{r+1}{i}(-1)^{i}\Gamma(\alpha+i)x^{r-i+1} \tag{16}$$

for $r = 0, 1, 2, \ldots$ and $\alpha \neq 0, -1, -2, \ldots$.

We now prove

Theorem 6. *The neutrix convolution $(x_{-}^{\alpha}\mathrm{e}^{-x}) \circledast x^{r}$ exists and*

$$(x_{-}^{\alpha}\mathrm{e}^{-x}) \circledast x^{r} = 0 \tag{17}$$

for $r = 0, 1, 2, \ldots$ and $\alpha \neq 0, -1, -2, \ldots$.

Proof. We first of all prove equation (17) when $\alpha > 0$ and put $(x_{-}^{\alpha}\mathrm{e}^{-x})_n = x_{-}^{\alpha}\mathrm{e}^{-x}\tau_n(x)$. Since $(x_{-}^{\alpha}\mathrm{e}^{-x})_n$ has compact support, it follows that the convolution $(x_{-}^{\alpha}\mathrm{e}^{-x})_n * x^{r}$ exists and

$$\begin{aligned}(x_{-}^{\alpha}\mathrm{e}^{-x})_n * x^{r} &= \int_{-n}^{0}(x-u)^{r}|u|^{\alpha}\mathrm{e}^{-u}\,\mathrm{d}u + \int_{-n-n^{-n}}^{-n}(x-u)^{r}|u|^{\alpha}\mathrm{e}^{-u}\tau_n(u)\,\mathrm{d}u\\ &= I_1 + I_2.\end{aligned} \tag{18}$$

Now

$$I_1 = \sum_{i=0}^{r}\binom{r}{i}x^{r-i}\int_{-n}^{0}|u|^{\alpha+i}\mathrm{e}^{-u}\,\mathrm{d}u = -\sum_{i=0}^{r}\binom{r}{i}x^{r-i}\gamma(\alpha+i+1, -n_{-})$$

and it follows that

$$\operatorname*{N-lim}_{n\to\infty} I_1 = 0. \tag{19}$$

Further, it is easily seen that

$$\lim_{n\to\infty} I_2 = 0 \tag{20}$$

and equation (17) follows from equations (18), (19) and (20) for the case $\alpha > 0$.

Now suppose that equation (17) holds when $-s < \alpha < -s+1$. This is true when $s = 0$. Then by virtue of $-s < \alpha < -s+1$, we have

$$[x_-^\alpha \mathrm{e}^{-x}\tau_n'(x)] * x^r = \int_{-n-n^{-n}}^{-n} |t|^\alpha \mathrm{e}^{-t}\tau_n'(t)(x-t)^r \,\mathrm{d}t$$

and

$$\begin{aligned}\langle [x_-^\alpha \mathrm{e}^{-x}\tau_n'(x)] * x^r, \varphi(x)\rangle &= \int_a^b \int_{-n-n^{-n}}^{-n} |t|^\alpha \mathrm{e}^{-t}\tau_n'(t)(x-t)^r\varphi(x)\,\mathrm{d}t\,\mathrm{d}x \\ &= \int_a^b \int_{-n-n^{-n}}^{-n} [\alpha(x-t)+t(x-t)+rt]|t|^{\alpha-1}\mathrm{e}^{-t}(x-t)^{r-1}\tau_n(t)\varphi(x)\,\mathrm{d}t\,\mathrm{d}x \\ &\quad - n^\alpha \mathrm{e}^n \int_a^b (x+n)^r\varphi(x)\,\mathrm{d}x,\end{aligned}$$

where $[a,b]$ contains the support of φ. It follows easily that

$$\operatorname*{N-lim}_{n\to\infty}\langle [x_-^\alpha \mathrm{e}^{-x}\tau_n'(x)] * x^r, \varphi(x)\rangle = 0. \tag{21}$$

It now follows from Theorems 4 and 5 and equation (21) that

$$(-\alpha x_-^{\alpha-1}\mathrm{e}^{-x} - x_-^\alpha \mathrm{e}^{-x}) \circledast x^r + 0 = r(x_+^\alpha \mathrm{e}^{-x}) \circledast x^{r-1}.$$

Using our assumption, it follows that

$$\alpha(x_-^{\alpha-1}\mathrm{e}^{-x}) \circledast x^r = 0$$

and so equation (17) holds when $-s-1 < \alpha < -s$. It therefore follows by induction that equation (17) holds for all $\alpha \neq 0, -1, -2, \ldots$, which completes the proof of the theorem. □

Corollary 6.1. *The neutrix convolution* $(x_-^\alpha \mathrm{e}^{-x}) \circledast x_+^r$ *exists and*

$$(x_-^\alpha \mathrm{e}^{-x}) \circledast x_+^r = \sum_{i=0}^{r} \binom{r}{i} \gamma(\alpha+i+1, x_-)x^{r-i} \tag{22}$$

for $r = 0, 1, 2, \ldots$ *and* $\alpha \neq 0, -1, -2, \ldots$.

P r o o f. Equation (22) follows from equations (14) and (17) by noting that

$$(x_-^\alpha \mathrm{e}^{-x}) \circledast x^r = (x_-^\alpha \mathrm{e}^{-x}) \circledast x_+^r + (-1)^r (x_-^\alpha \mathrm{e}^{-x}) \circledast x_-^r.$$

□

Corollary 6.2. *The neutrix convolution* $(|x|^\alpha \mathrm{e}^{-x}) \circledast x_+^r$ *exists and*

$$(|x|^\alpha \mathrm{e}^{-x}) \circledast x_+^r = \sum_{i=0}^{r} \binom{r}{i} [(-1)^i \gamma(\alpha+i+1, x_+) + \gamma(\alpha+i+1, x_-)] x^{r-i} \tag{23}$$

for $r = 0, 1, 2, \ldots$ *and* $\alpha \neq 0, -1, -2, \ldots$.

Proof. Equation (23) follows immediately from equations (1) and (22). □

Theorem 7. *The neutrix convolution* $\gamma(\alpha, x_-) \circledast x^r$ *exists and*

$$\gamma(\alpha, x_-) \circledast x^r = 0 \tag{24}$$

for $r = 0, 1, 2, \ldots$ *and* $\alpha \neq 0, -1, -2, \ldots$.

Proof. We first of all prove equation (24) when $\alpha > 0$ and put $\gamma_n(\alpha, x_-) = \gamma(\alpha, x_-)\tau_n(x)$. The convolution $\gamma_n(\alpha, x_-) * x^r$ then exists by Definition 1 and

$$\begin{aligned} \gamma_n(\alpha, x_-) * x^r &= \int_{-n}^{0} (x-t)^r \int_0^t |u|^{\alpha-1} \mathrm{e}^{-u} \, \mathrm{d}u \, \mathrm{d}t \\ &\quad + \int_{-n-n^{-n}}^{-n} (x-t)^r \int_0^t |u|^{\alpha-1} \mathrm{e}^{-u} \, \mathrm{d}u \, \mathrm{d}t = J_1 + J_2. \end{aligned} \tag{25}$$

Now

$$\begin{aligned} J_1 &= \int_{-n}^{0} (x-t)^r \int_0^t |u|^{\alpha-1} \mathrm{e}^{-u} \, \mathrm{d}u \, \mathrm{d}t \\ &= \int_{-n}^{0} |u|^{\alpha-1} \mathrm{e}^{-u} \int_u^{-n} (x-t)^r \, \mathrm{d}t \, \mathrm{d}u \\ &= \frac{1}{r+1} \sum_{i=1}^{r+1} \binom{r+1}{i} x^{r-i+1} \int_{-n}^{0} |u|^{\alpha+i-1} \mathrm{e}^{-u} \, \mathrm{d}u \\ &\quad - \frac{1}{r+1} \sum_{i=1}^{r+1} \binom{r+1}{i} n^i x^{r-i+1} \int_{-n}^{0} |u|^{\alpha-1} \mathrm{e}^{-u} \, \mathrm{d}u \\ &= -\frac{1}{r+1} \sum_{i=1}^{r+1} \binom{r+1}{i} \gamma(\alpha+i, -n_-) x^{r-i+1} \\ &\quad + \frac{1}{r+1} \sum_{i=1}^{r+1} \binom{r+1}{i} n^i \gamma(\alpha, -n_-) x^{r-i+1}. \end{aligned}$$

It follows that

$$\underset{n \to \infty}{\text{N-lim}} \, J_1 = 0. \tag{26}$$

Further, it is easily seen that

$$\lim_{n\to\infty} J_2 = 0 \tag{27}$$

and equation (24) follows from equations (25), (26) and (27) for the case $\alpha > 0$.

Now suppose that equation (24) holds when $-s < \alpha < -s+1$. This is true when $s = 0$. Then by virtue of $-s-1 < \alpha < -s$ and using theorem 6 we get

$$\alpha\gamma(\alpha, x_-) \circledast x^r = \gamma(\alpha+1, x_-) \circledast x^r + (x_-^\alpha \mathrm{e}^{-x}) \circledast x^r = 0 \tag{28}$$

and so (24) holds when $-s-1 < \alpha < -s$. It therefore follows by induction that (24) holds for all $\alpha \neq 0, -1, -2, \dots$ which completes the proof of the theorem. □

Corollary 7.1. *The neutrix convolution $\gamma(\alpha, x_-) \circledast x_+^r$ exists and*

$$\gamma(\alpha, x_-) \circledast x_+^r = -\frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}\gamma(\alpha+i, x_-)x^{r-i+1} \tag{29}$$

for $r = 0, 1, 2, \dots$ and $\alpha \neq 0, -1, -2, \dots$.

Proof. Equation (29) follows from equations (15) and (24) by noting that

$$\gamma(\alpha, x_-) \circledast x^r = \gamma(\alpha, x_-) \circledast x_+^r + (-1)^r\gamma(\alpha, x_-) \circledast x_-^r.$$

□

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