THE 3-PATH-STEP OPERATOR ON TREES AND UNICYCLIC GRAPHS

Bohdan Zelinka, Liberec

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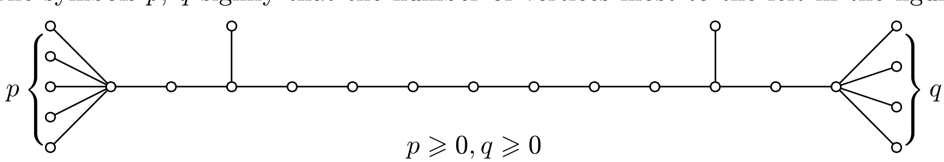
Abstract. E. Prisner in his book Graph Dynamics defines the k-path-step operator on the class of finite graphs. The k-path-step operator (for a positive integer k) is the operator S'_k which to every finite graph G assigns the graph $S'_k(G)$ which has the same vertex set as G and in which two vertices are adjacent if and only if there exists a path of length k in G connecting them. In the paper the trees and the unicyclic graphs fixed in the operator S'_3 are studied.

Keywords: 3-path-step graph operator, tree, unicyclic graph

MSC 2000: 05C38, 05C05

In [2], page 168, the k-path-step graph operator S'_k for a positive integer k is defined. Let G be a finite graph. The graph $S'_k(G)$ has the same vertex set as G and two vertices are adjacent in it if and only if there exists a path of length k in G connecting them.

Further, in [2] the abstract of [1] is quoted; it is said that the paper [1] never appeared. In the abstract it was claimed that the finite connected graphs which are periodic in S'_3 are just the complete graphs, the complete bipartite graphs, the circuits of lengths not divisible by 3, the graphs of one more infinite family and four exceptional graphs (they were not specified). But in [2] some further graphs were presented which are fixed in S'_3. Among them there is a family of trees in Fig. 1. The symbols p, q signify that the number of vertices most to the left in the figure



$p \geqslant 0, q \geqslant 0$

Fig. 1

is p and the number of vertices most to the right is q; these numbers p, q may be arbitrary non-negative integers (including zero). Therefore Fig. 1 represents a whole family of trees. Among the graphs mentioned there is a similar family of unicyclic graphs and two other unicyclic graphs. They are shown in Fig. 2.

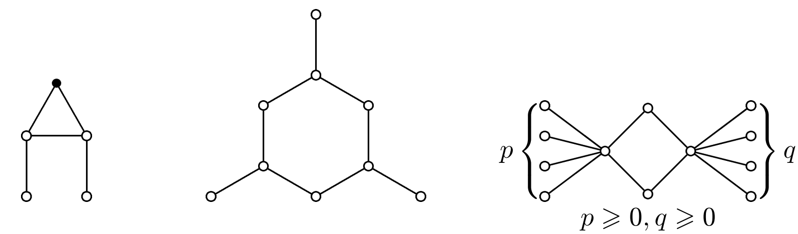


Fig. 2

In this paper we shall prove hat the graphs in Fig. 1 and Fig. 2 are all trees and all unicyclic graphs which are fixed in the operator S_3', i.e. graphs G such that $S_3'(G) \cong G$. We start with trees. All graphs considered are without loops and multiple edges.

Lemma 1. *Let T be a tree such that $S_3'(T) \cong T$. Then T contains no subtree isomorphic to any one of the trees F_1, F_2, F_3 in Fig. 3.*

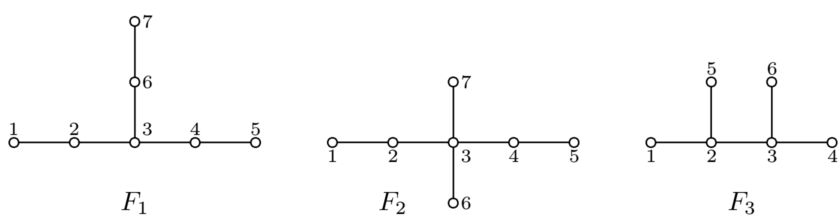


Fig. 3

P r o o f. If T contains a subtree isomorphic to F_1, to F_2 or to F_3, then the image $S_3'(T)$ contains a subgraph isomorphic to $S_3'(F_1)$, to $S_3'(F_2)$, or to $S_3'(F_3)$. These graphs are in Fig. 4. Evidently each of them contains a circuit: therefore $S_3'(T)$ is then not a tree. □

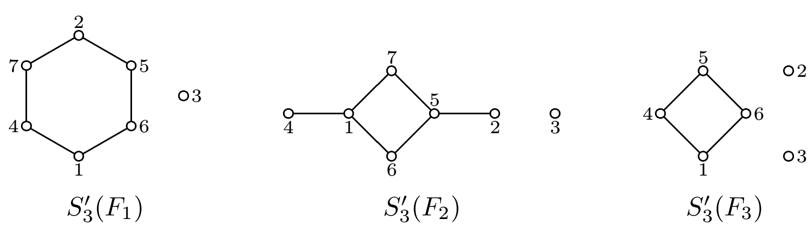


Fig. 4

The absence of a subtree isomorphic to F_1 implies that a tree fixed in S_3' must be a caterpillar. Indeed, caterpillars are characterized among trees by this property. A caterpillar is defined as a tree with the property that by deleting all pendant edges and vertices from it a path is obtained. (A pendant vertex of a tree is its vertex of degree 1, a pendant edge is an edge incident with a pendant vertex.)

Thus, let us have a tree T fixed in S_3'. We describe it as a caterpillar. Let the diameter of T be d. Let D be a diametral path of T. Denote its vertices by $u_0, u_1, \ldots, u_d$ so that the edges of D are $u_i u_{i+1}$ for $i = 0, 1, \ldots, d-1$. The number of vertices adjacent to u_1 (or u_{d-1}) and not belonging to D will be denoted by $p-1$ (or $q-1$, respectively). We admit that these numbers may be zero. Further, by k we denote the number of vertices S which are adjacent to the vertices u_i for $2 \leqslant i \leqslant d-2$.

Lemma 2. *Let k be the above defined number. Then $k = 2$.*

Proof. The number of edges of T is $d + p + q + k - 2$. As $S_3'(T) \cong T$, so must be the number of edges of $S_3'(T)$, i.e. the number of pairs of vertices whose distance in T is 3. On D there are $d-2$ such pairs, namely the pairs $\{u_i, u_{i+3}\}$ for $i = 0, \ldots, d-3$. If a vertex is adjacent to u_1 or to u_{d-1} and does not belong to D, then there exists exactly one vertex at the distance 3 from it, namely u_3 or u_{d-3}. If a vertex is adjacent to u_i for $2 \leqslant i \leqslant d-2$ then there are exactly two vertices at the distance 3 from it, namely u_{i-2} and u_{i+2} (we must suppose the absence of F_3). Hence the number of edges of $S_3'(T)$ is $d + p + q + 2k - 4$. It is equal to the number of edges of T if and only if $k = 2$. $\square$

Thus we may suppose that there exist integers r, s such that $2 \leqslant r < r + 2 \leqslant s \leqslant d-2$ and there exists a vertex v_r adjacent to u_r and a vertex v_s adjacent to u_5 which do not belong to D. Note that the case $r = 2$ is possible only if $p = 1$ and the case $s = d-2$ is possible only if $q = 1$; otherwise a subtree isomorphic to F_3 would occur.

Lemma 3. *Let T be a caterpillar, let* $\operatorname{diam} T \leqslant 6$. *Then T is not fixed in the operator S_3'.*

Proof. If $\operatorname{diam} T \leqslant 2$, then there is no path of length 3 in T. If $3 \leqslant \operatorname{diam} T \leqslant 5$, then the above mentioned numbers r, s do not exist. If $\operatorname{diam} T = 6$, then the unique possibility is $r = 2$, $s = 4$ but then $S_3'(T)$ is a path. $\square$

Therefore in the sequel we will suppose $d = \operatorname{diam} T \geqslant 7$. The image $S_3'(D)$ consists of three connected components which are paths D_0, D_1, D_2; for $j \in \{0, 1, 2\}$ we denote by D_j the path having the vertices u_i with $i \equiv j \pmod 3$. In $S_3'(T)$ there are two paths R, S among these components of length 2 with v_r and v_s as inner vertices. One of the paths D_0, D_1, D_2 must have the property that both its terminal vertices are terminal vertices of the paths R and S; denote this property as $\mathcal{V}$. We shall treat the possible cases. If i, j are from $\{0,1,2\}$, then $C(i,j)$ denotes the case when $d \equiv i \pmod 3$ and D_j has the property $\mathcal{V}$.

Case $C(0,0)$. The path R connects u_0 with u_4, the path S connects u_d with u_{d-4}. Hence $r = 2$, $s = d-2$. The images of pendant vertices v_r, v_s in an isomorphism of T onto $S_3'(T)$ are again pendant vertices u_1 and u_{d-1} and the images of u_r, u_5 are u_4 and u_{d-4}. The images of u_0 and u_d are u_{d-2} and u_2. The distance between u_4 and u_{d-2} in $S_3'(T)$ is $\frac{1}{3}d-2$, the distance between u_{d-4} and u_2 is the same. If $S_3'(T) \cong T$, then necessarily $r = d-s = \frac{1}{3}d-2$ and $d = 12$.

Case $C(0,1)$. If $d \geqslant 12$, then there exists a subtree of $S_3'(T)$ isomorphic to F_1. It consists of three paths of length 2 with the common terminal u_{d-6}; the first has the edges $u_{d-12}u_{d-9}$, $u_{d-4}u_{d-6}$, the second $u_d u_{d-3}$, $u_{d-3}u_{d-3}$, the third $u_{d-2}v_{d-4}$, $v_{d-4}u_{d-6}$. This is a contradiction. The unique case for $d < 12$ is $d = 9$; it is easy to try the corresponding tree and to recognize that it is not fixed in S_3'.

Case $C(0,2)$ may be transferred to $C(0,1)$ by changing the notation u_i to u_{d-i} for each i.

Case $C(1,0)$. The path R connects u_0 with u_4, the path S connects u_{d-1} with u_{d-5}. Hence $r = 2$, $s = d-3$. The images of v_r, v_s are u_1 and u_{d-2} and the images of u_r, u_s are u_4, u_{d-5}. The images of u_0 and u_d are u_2 and u_d. The distance between u_4 and u_d is $\frac{1}{3}(d-1)-1$, the distance between u_2 and u_{d-5} is $\frac{1}{3}(d-1)-2$. If $S_3'(T) \cong T$, then one of these distances must be equal to r and the other to $d-s$. This is possible only for $d = 13$.

Case $C(1,1)$ may be transferred to $C(1,0)$ by changing the notation u_i to u_{d-i} for each i.

Case $C(1,2)$. If $d \geqslant 13$, then there exists a subtree of $S_3'(T)$ isomorphic to F_1. It consists of three paths of length 2 with a common terminal vertex u_6; the first has the edges u_0u_3, u_3u_6, the second $u_{12}u_9$, u_9u_6, the third u_2v_4, v_4u_6. This is a contradiction. The unique cases for $d < 13$ are $d = 7$ and $d = 10$; it is easy to try the corresponding trees and to recognize that they are not fixed in S_3'.

Case $C(2,0)$. If $d \geqslant 14$, then there exists a subtree of $S_3'(T)$ isomorphic to F_1. It consists of three paths of length 2 with a common terminal vertex u_{d-6}: the first has the edges $u_{d-12}u_{d-9}$, $u_{d-9}u_{d-6}$, the second $u_d u_{d-3}$, $u_{d-3}u_{d-6}$, the third $u_{d-2}v_{d-4}$, $v_{d-4}u_{d-6}$. This is a contradiction. The unique cases for $d < 14$ are $d = 8$ and $d = 11$; it is easy to try the corresponding trees and to recognize that they are not fixed in S_3'.

Case $C(2,1)$. The path R connects u_1 with u_5, the path S connects u_{d-1} with u_{d-5}. Hence $r = 3$, $s = d-3$. The images of v_r, v_s are u_2 and u_{d-2} and the images of u_r, u_5 are u_5, u_{d-5}. The images of u_0 and u_d are again u_0 and u_d. The distance between u_5 and u_d is $\frac{1}{3}(d-2)-1$ and the distance between u_0 and u_{d-5} is also $\frac{1}{3}(d-2)-1$. If $S'_1(T) \cong T$, then $r = d - s - \frac{1}{3}(d-2) - 1$ and $d = 14$.

Case $C(2,2)$ may be transferred to $C(2,0)$ by changing the notation u_i to u_{d-i} for each i.

By these considerations we have proved the following lemma.

Lemma 4. *Let T be a tree such that $S'_3(T) \cong T$. Then T is a caterpillar, $12 \leqslant \operatorname{diam} T \leqslant 14$ and in T there exist exactly two vertices of degree 3 with the distance from both the terminal vertices of a diametral path greater than or equal to 2.*

From our lemmas and from the considerations which precede Lemma 4 we obtain a theorem.

Theorem 1. *Let T be a finite tree such that $S'_3(T) \cong T$. Then T belongs to the family of trees depicted in Fig. 1.*

The family from Fig. 1 is again depicted in Fig. 5 (diameter 12), Fig. 6 (diameter 13) and Fig. 7 (diameter 14). For $d = 12$ there is only one tree; to u_1 and u_{11} no vertices not belonging to D may be adjacent, because then a subtree isomorphic to F_3 would occur. For $d = 13$ it is possible for only one of the vertices u_1, u_{12}: In Fig. 6 it is u_{12}. The second possibility would be a mirror image of the former. For $d = 14$ vertices not belonging to D may be adjacent to both u_1 and u_{13}.

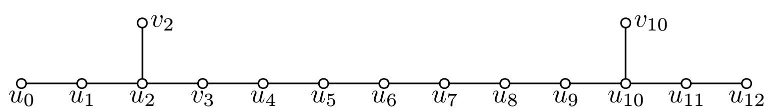


Fig. 5

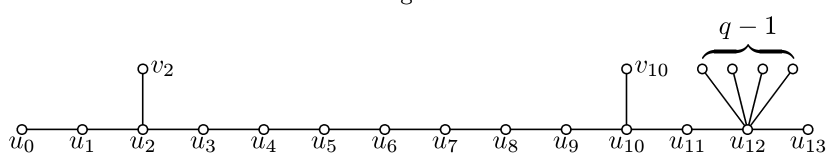


Fig. 6

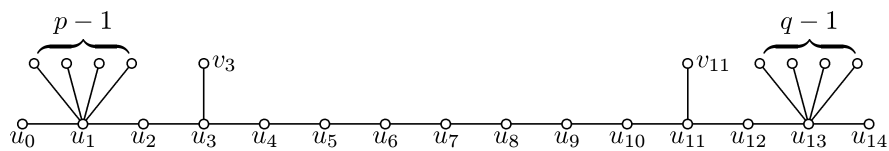
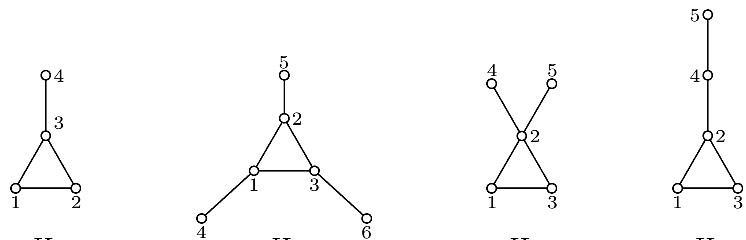


Fig. 7

Now we turn to unicyclic graphs.

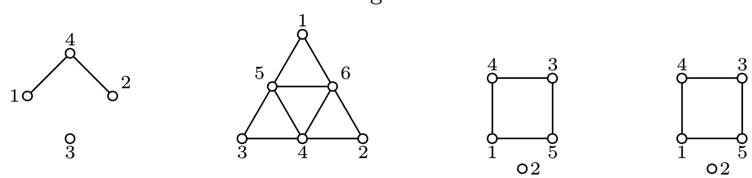
Theorem 2. *Let G be a finite unicyclic graph such that $S_3'(G) \cong G$. Then either G is a circuit of length not divisible by 3, or it is some of the graphs depicted in Fig. 2.*

P r o o f. Let G be an acyclic graph, let $\ell(G)$ be the length of its circuit. If $\ell(G) = 3$ and G is not isomorphic to the graph of $\ell(G) = 3$ from Fig. 2, then either it is isomorphic to a subgraph of H_1, or contains a subgraph isomorphic to H_2, H_3 or H_4 in Fig. 8. The images of those graphs are in Fig. 9. In the first case $S_3'(G)$ is isomorphic to a subgraph of $S_3'(H_1)$ and it is a forest, in the second case $S_3'(G)$ has a subgraph isomorphic to $S_3'(H_2)$, $S_3'(H_3)$ or $S_3'(H_4)$ and thus it has a circuit of length 4 or of length 6, therefore it is not isomorphic to G. If $\ell(G) = 4$ and G is not isomorphic to any graph of the family of graphs with $\ell(G) = 4$ depicted in Fig. 2, then it has a subgraph isomorphic to H_5 or H_6 in Fig. 10. The graph $S_3'(G)$ has then a subgraph isomorphic to $S_3'(H_5)$ or to $S_3'(H_6)$ in Fig. 11; in both the cases



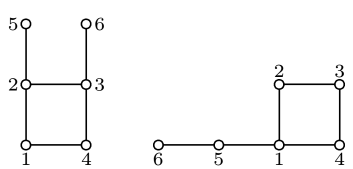
H_1 H_2 H_3 H_4

Fig. 8



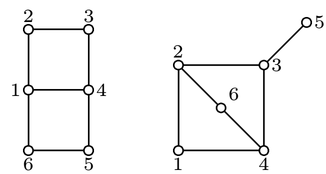
$S_3'(H_1)$ $S_3'(H_2)$ $S_3'(H_3)$ $S_3'(H_4)$

Fig. 9



H_5 H_6

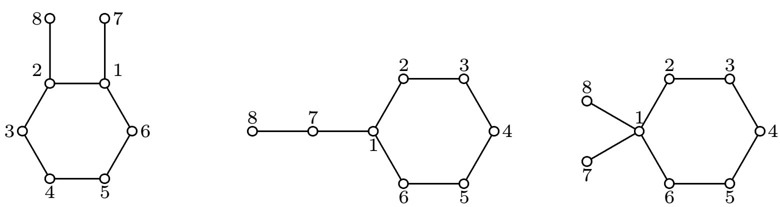
Fig. 10



$S_3'(H_5)$ $S_3'(H_6)$

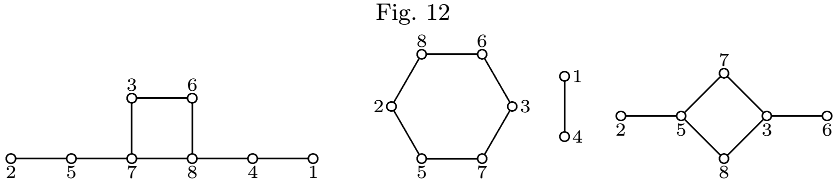
Fig. 11

it contains two circuits of length 4 and cannot be isomorphic to G. If $\ell(G) = 6$ and G is not isomorphic to the graph with this $\ell(G)$ in Fig. 2, then it is either a circuit of length 6, or it is isomorphic to H_1, H_8 or H_9 in Fig. 12. In the first case $S_3'(H)$ consists of three connected components being complete graphs with two vertices, in the second case it contains a subgraph isomorphic to $S_3'(H_1)$, $S_3'(H_8)$ or $S_3'(H_9)$ in Fig. 13 and thus it is not isomorphic to G. Finally, let $\ell(G) = 5$ or $\ell(G) \geqslant 7$. If $\ell(G)$ is divisible by 3, then $S_3'(G)$ contains three circuits of length $\frac{1}{3}\ell(G)$ and is not isomorphic to G. Thus suppose that $\ell(G)$ is not divisible by 3. If G is not only a circuit, then G contains a vertex not belonging to its circuit, but adjacent to one of its vertices. Such a vertex has distance 3 from the vertices of the circuit. The graph $S_3'(G)$ contains a circuit of length $\ell(G)$ and a vertex adjacent to two vertices of that circuit (in Fig. 14 for $\ell(G) = 5$), and thus it is not isomorphic to G.



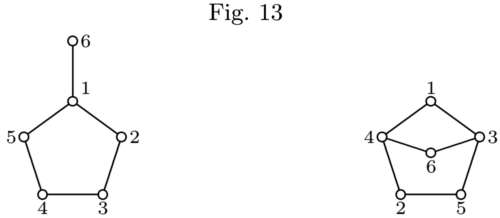
H_7 H_8 H_9

Fig. 12



$S_3'(H_7)$ $S_3'(H_8)$ $S_3'(H_9)$

Fig. 13



H_{10} $S_3'(H_{10})$

Fig. 14

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Author's address: *Bohdan Zelinka*, Department of Applied Mathematics, Technical University of Liberec, Voroněžská 13, 460 01 Liberec 1, e-mail: `bohdan.zelinka@vslib.cz`.