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MEAN VALUES AND ASSOCIATED MEASURES OF
 δ -SUBHARMONIC FUNCTIONS

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Abstract. Let u be a δ -subharmonic function with associated measure μ , and let v be a superharmonic function with associated measure ν , on an open set E . For any closed ball $B(x, r)$, of centre x and radius r , contained in E , let $\mathcal{M}(u, x, r)$ denote the mean value of u over the surface of the ball. We prove that the upper and lower limits as $s, t \rightarrow 0$ with $0 < s < t$ of the quotient $(\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)) / (\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t))$, lie between the upper and lower limits as $r \rightarrow 0+$ of the quotient $\mu(B(x, r)) / \nu(B(x, r))$. This enables us to use some well-known measure-theoretic results to prove new variants and generalizations of several theorems about δ -subharmonic functions.

Keywords: superharmonic, δ -subharmonic, Riesz measure, spherical mean values

MSC 2000: 31B05

1. INTRODUCTION

Let E be an open subset of \mathbb{R}^n , let u be δ -subharmonic on E , and let v be superharmonic on E . Let μ and ν be the Borel measures associated with u and v by the Riesz Decomposition Theorem, so that μ is signed and ν is positive. Let $B(x, r)$ denote the closed ball with centre x and radius r contained in E , and let $\mathcal{M}(u, x, r)$ denote the spherical mean value of u over $\partial B(x, r)$. We shall prove that the upper and lower limits as $s, t \rightarrow 0$ with $0 < s < t$ of

$$(1) \quad \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)}$$

lie between the upper and lower limits as $r \rightarrow 0+$ of

$$(2) \quad \frac{\mu(B(x, r))}{\nu(B(x, r))}.$$

This enables us to use the measure-theoretic results of Besicovitch [3], [4] to study the behaviour of δ -subharmonic functions.

This work was inspired by a recent paper of Sodin [12]. However, the techniques we devised have much wider ramifications, so that Sodin's results appear only as fairly minor details. We generalize not only Sodin's results, but also some due to Armitage [2] and Watson [14]. We also present new analogues of some theorems about Poisson integrals which appeared in [1], [5], and [16], a new form of the Domination Principle, and variants of recent results of Fuglede [10].

Our starting point is the well-known formula

$$\mathcal{M}(u, x, s) = \mathcal{M}(u, x, t) + p_n \int_s^t r^{1-n} \mu(B(x, r)) dr,$$

in which $0 < s < t$, $B(x, t) \subseteq E$, and $p_n = \max\{1, n - 2\}$. See, for example, [2] Lemma 3. We shall put

$$I_\mu(x; s, t) = p_n \int_s^t r^{1-n} \mu(B(x, r)) dr.$$

Then the quotient (1) can be written as either

$$(3) \quad \frac{I_\mu(x; s, t)}{I_\nu(x; s, t)}$$

or

$$(4) \quad \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{I_\nu(x; s, t)}.$$

From (3) it is easy to see the connection with (2).

If we take ν to be the Lebesgue measure λ , we have

$$I_\lambda(x; s, t) = p_n v_n (t^2 - s^2)/2,$$

where $v_n = \lambda(B(0, 1))$. Then, up to a multiplicative constant, (4) becomes

$$\frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{t^2 - s^2}.$$

Theorem 3 below gives conditions on this quotient which ensure that μ can be written in the form

$$\mu = \omega - \sum_j c_j \delta_j$$

where ω is a positive measure, each c_j is a specific positive constant, each δ_j is a unit mass at a given point x_j , and there are countably many indices. This is analogous to decomposition formulas for the boundary measures of Poisson integrals given in [5], [1], and [16].

Theorem 4 shows how the quotient (4) can be used to determine which sets are positive for μ . Roughly, if

$$\limsup_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{I_\nu(x; s, t)} \geq 0$$

for all $x \in S$, then S is positive for μ . The condition can be weakened on a ν -null subset of S . This result contains as special cases those due to Sodin [12], which include the one known as Grishin's lemma [11].

Theorems 5 and 6 generalize results of Armitage [2] by extending them to points where his conditions that an infinity occur no longer hold. Theorems 10 and 11 similarly extend results of Watson [14].

By analogy with results on half-space Poisson integrals given in [1] and [16], Theorem 7 gives conditions on the quotient (1) which ensure that $u - Av$ is superharmonic for some real number A . For example, the condition

$$\limsup_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)} \geq A$$

for all $x \in E$, is sufficient. A minor modification of the proof, in the special case where u is a positive superharmonic function, E is Greenian, and $v = G_E \nu$ is a Green potential, yields the following domination principle as Theorem 8: If

$$\limsup_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)}$$

is never $-\infty$, and is greater than or equal to 1 for ν -almost all x , then $u \geq v$.

In Theorem 9, we use (1) to determine the μ -null subsets of E . One of its corollaries is an extension to δ -subharmonic functions of the fact that polar sets are null for the restriction of ν to the set where v is finite. A different such extension was established by Fuglede [10].

Given a Borel subset B of E , we denote by μ_B the restriction of μ to B .

2. THE MEASURE-THEORETIC CONNECTION

Theorem 1 contains the necessary measure theory. It is implicit in [15], but may not have been stated explicitly before. References are given to the original papers of Besicovitch; an alternative source is [9].

Theorem 1. *Let μ be a signed measure and ν a positive measure on E . Let*

$$f(x) = \lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}$$

whenever the limit exists, let $Z^+ = \{x \in E: f(x) = \infty\}$, and let $Z^- = \{x \in E: f(x) = -\infty\}$. Then f is defined and finite ν -a.e. on E , and there are positive ν -singular measures σ^+ and σ^- , concentrated on Z^+ and Z^- respectively, such that

$$(5) \quad d\mu = f d\nu + d\sigma^+ - d\sigma^-.$$

Proof. By [3] Theorem 2, f is defined and finite ν -a.e. By [4] Theorem 6, f is the Radon-Nikodým derivative of μ with respect to ν , so that (5) holds with σ^+ and σ^- the positive and negative variations of the ν -singular part of μ .

To show that σ^+ is concentrated on Z^+ , we put $d\omega = f d\nu - d\sigma^-$. Then both ν and ω are σ^+ -singular, so that by [3] Theorem 3,

$$\lim_{r \rightarrow 0} \frac{\nu(B(x, r))}{\sigma^+(B(x, r))} = 0$$

and

$$\lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{\sigma^+(B(x, r))} = \lim_{r \rightarrow 0} \frac{\omega(B(x, r))}{\sigma^+(B(x, r))} + 1 = 1,$$

for σ^+ -almost all x . Hence $f(x) = \infty$ for σ^+ -almost all x , so that σ^+ is concentrated on Z^+ . Similarly, σ^- is concentrated on Z^- .

Corollary 1. *Let μ be a signed measure and ν a positive measure on E , and let S be a Borel subset of E . If*

$$(6) \quad \limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{\nu(B(x, r))} > -\infty$$

for all $x \in S$ at which the upper limit is defined, and

$$(7) \quad \limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{\nu(B(x, r))} \geq A$$

for ν -almost all $x \in S$, then $(\mu - A\nu)_S \geq 0$.

Proof. By Theorem 1, $d\mu_S = f d\nu_S + d\sigma_S^+ - d\sigma_S^-$ with $f(x)$ equal to the upper limit in (7) and σ_S^- concentrated on $\{x \in S: f(x) = -\infty\}$. By (6) this set is empty, and by (7) $f \geq A$. Hence $d\mu_S - A d\nu_S \geq d\sigma_S^+ \geq 0$. \square

Corollary 2. Let μ be a signed measure and ν a positive measure on E . Let S be a Borel subset of E such that, for each $x \in S$, either

$$(8) \quad \lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{\nu(B(x, r))} = 0$$

or the limit does not exist. Then $\mu_S = 0$.

P r o o f. By Theorem 1, $\{x \in E : (8) \text{ holds}\}$ is μ -null, and the set of points where f is undefined is also μ -null. \square

We include for completeness the definition of

$$\limsup_{0 < s < t \rightarrow 0} f(s, t),$$

although it is the natural one. Those of the corresponding \liminf and \lim are then obvious.

Definition. Suppose that $f(s, t)$ is defined as an extended-real number whenever $0 < s < t < a$, and that $\ell \in \mathbb{R}$. We write

$$\limsup_{0 < s < t \rightarrow 0} f(s, t) = \ell$$

if to each $\varepsilon > 0$ there corresponds $\delta > 0$ such that $f(s, t) < \ell + \varepsilon$ whenever $0 < s < t < \delta$, and there is a sequence $\{(s_k, t_k)\}$ such that $0 < s_k < t_k \rightarrow 0$ and $f(s_k, t_k) \rightarrow \ell$ as $k \rightarrow \infty$. We also write

$$\limsup_{0 < s < t \rightarrow 0} f(s, t) = \infty$$

if there is a sequence $\{(s_k, t_k)\}$ such that $0 < s_k < t_k \rightarrow 0$ and $f(s_k, t_k) \rightarrow \infty$. Finally, we write

$$\limsup_{0 < s < t \rightarrow 0} f(s, t) = -\infty$$

if to each $A \in \mathbb{R}$ there corresponds $\delta > 0$ such that $f(s, t) < A$ whenever $0 < s < t < \delta$.

We can now establish the connection on which all our results are based.

Theorem 2. If u is δ -subharmonic on E with associated measure μ , and ν is a positive measure on E , then

$$(9) \quad \limsup_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{I_\nu(x; s, t)} \leq \limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}$$

whenever the latter exists. The reverse inequality holds for lower limits, and

$$(10) \quad \lim_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{I_\nu(x; s, t)} = \lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}$$

for ν -almost all $x \in E$.

P r o o f. Given x for which the upper limit on the right-hand side of (9) exists, denote that upper limit by ℓ . If $\ell = \infty$ there is nothing to prove. Otherwise, given a real number $A > \ell$ we can find $\delta > 0$ such that

$$\frac{\mu(B(x, r))}{\nu(B(x, r))} < A \quad \text{whenever } 0 < r < \delta.$$

If $\nu(B(x, r)) = 0$ for all $r < \eta$ ($\leq \delta$), then the above inequality can hold only if $\mu(B(x, r)) < 0$ for all such r . Then $I_\nu(x; s, t) = 0$ whenever $t < \eta$, and

$$\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t) = I_\mu(x; s, t) < 0,$$

so that (9) holds with both sides $-\infty$. On the other hand, if $\nu(B(x, r)) > 0$ for all r , then

$$\frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{I_\nu(x; s, t)} = \frac{p_n}{I_\nu(x; s, t)} \int_s^t r^{1-n} \nu(B(x, r)) \frac{\mu(B(x, r))}{\nu(B(x, r))} dr < A$$

whenever $0 < s < t < \delta$, and again (9) holds.

Obviously (9) implies the reverse inequality for lower limits. Now (10) follows from [3] Theorem 2. \square

The particular cases of Theorem 2, in which ν is the Lebesgue measure λ or the unit mass δ_x at x , are of special importance.

Corollary 1. *If u is δ -subharmonic with associated measure μ on E , then*

$$\lim_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{t^2 - s^2} = \frac{p_n}{2} \lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^n}$$

whenever the latter exists.

P r o o f. Whenever $B(x, t) \subseteq E$, we have

$$I_\lambda(x; s, t) = p_n \int_s^t r^{1-n} (v_n r^n) dr = p_n v_n (t^2 - s^2)/2,$$

so that the result follows from Theorem 2. \square

Corollary 2. *If u is δ -subharmonic with associated measure μ on E , then for each $x \in E$ we have*

$$\mu(\{x\}) = \lim_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\log(t/s)} \quad \text{if } n = 2,$$

and

$$\mu(\{x\}) = \lim_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{s^{2-n} - t^{2-n}} \quad \text{if } n \geq 3.$$

P r o o f. Writing $\delta = \delta_x$, we have

$$I_\delta(x; s, t) = p_n \int_s^t r^{1-n} dr = \begin{cases} \log(t/s) & \text{if } n = 2, \\ s^{2-n} - t^{2-n} & \text{if } n \geq 3, \end{cases}$$

so that Theorem 2 gives the result. □

3. A REPRESENTATION THEOREM

Theorem 2 and its corollaries enable us to prove a new representation theorem for δ -subharmonic functions, which is analogous to known results about Poisson integrals on a ball due to Bruckner, Lohwater and Ryan [5], and on a half-space due to Armitage [1] and Watson [16].

Theorem 3. *Let u be δ -subharmonic with associated measure μ on E . If*

$$(11) \quad \lim_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{t^2 - s^2} \geq 0$$

for λ -almost all $x \in E$, and

$$(12) \quad \lim_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x_j, s) - \mathcal{M}(u, x_j, t)}{t^2 - s^2} = -\infty$$

for only the points x_j in a countable set C , then μ can be written in the form

$$(13) \quad \mu = \omega + \sum_j \left(\lim_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x_j, s) - \mathcal{M}(u, x_j, t)}{\log(t/s)} \right) \delta_j$$

if $n = 2$,

$$(14) \quad \mu = \omega + \sum_j \left(\lim_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x_j, s) - \mathcal{M}(u, x_j, t)}{s^{2-n} - t^{2-n}} \right) \delta_j$$

if $n \geq 3$, where ω is a positive measure such that $\omega(C) = 0$, and δ_j is the unit mass at x_j .

P r o o f. In view of Theorem 2 Corollary 1, condition (11) implies that

$$\lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{\lambda(B(x, r))} \geq 0$$

for λ -almost all $x \in E$, and condition (12) implies that

$$\lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{\lambda(B(x, r))} = -\infty$$

only if $x \in C$. Therefore, by Theorem 1,

$$d\mu = f d\lambda + d\sigma^+ - d\sigma^-$$

with $f \geq 0$ and σ^- concentrated on C . Furthermore, for each j , Theorem 2 Corollary 2 shows that the limits in (13) and (14) are equal to $\mu(\{x_j\})$. Thus

$$d\mu = (f d\lambda + d\sigma^+) + \sum_j \mu(\{x_j\})\delta_j$$

yields the required representation. □

In particular, Theorem 3 allows the following characterization of a point mass.

Corollary. *Let u be δ -subharmonic with associated measure μ on E . If*

$$\lim_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{t^2 - s^2}$$

is 0 for λ -almost all $x \in E$, is finite except at x_0 , and is ∞ at x_0 , then μ is a positive constant multiple of the unit mass at x_0 .

P r o o f. Applying Theorem 3 to $-u$, we obtain

$$-\mu = \omega - \left(\lim_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x_0, s) - \mathcal{M}(u, x_0, t)}{\log(t/s)} \right) \delta_0$$

if $n = 2$,

$$-\mu = \omega - \left(\lim_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x_0, s) - \mathcal{M}(u, x_0, t)}{s^{2-n} - t^{2-n}} \right) \delta_0$$

if $n \geq 3$, where ω is a positive measure such that $\omega(\{x_0\}) = 0$. By Theorem 2 Corollary 2, $-\mu = \omega - \mu(\{x_0\})\delta_0$ in either case. Applying Theorem 3 to u itself, we find that μ is positive, so that ω is null. □

4. POSITIVE SETS FOR ASSOCIATED MEASURES

The proof of its corollary illustrates how Theorem 3 can sometimes be used to show that the measure associated with a δ -subharmonic function is positive. Theorem 4 below is a refinement that allows us to determine which are the positive sets for the measure. It is similar in essence to the case $Y = \emptyset$ of [15] Theorem 6.

Recall that μ_S denotes the restriction of μ to the set S .

Theorem 4. *Let u be δ -subharmonic with associated measure μ on E , let S be a Borel subset of E , and let ν be a positive measure on E . If*

$$\limsup_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{I_\nu(x; s, t)}$$

is not $-\infty$ for any $x \in S$, and is nonnegative for ν -almost all $x \in S$, then $\mu_S \geq 0$.

Proof. By Theorem 2,

$$\limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}$$

is not $-\infty$ for any $x \in S$, and is nonnegative for ν -almost all $x \in S$. Therefore $\mu_S \geq 0$, by Theorem 1 Corollary 1. □

Theorem 4 contains the results of Sodin [12] which, in turn, are extensions of Grishin's lemma [11]. Other extensions of Grishin's lemma were obtained by Fuglede [10]. Our next corollary extends Sodin's theorem to n -dimensions.

Corollary 1. *Let u be δ -subharmonic with associated measure μ on E , and let S be the set of points in E with the following property: There are sequences $\{s_k\}$ and $\{t_k\}$, which depend on the point x , such that $0 < s_k < t_k \rightarrow 0$ and*

$$(15) \quad \mathcal{M}(u, x, s_k) \leq \mathcal{M}(u, x, t_k)$$

for all k . Then $\mu_S \leq 0$.

Proof. Sodin proved that S is a Borel set. If $x \in S$, then

$$\liminf_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{I_\lambda(x; s, t)} \leq 0.$$

Therefore $\mu_S \leq 0$, by Theorem 4. □

Theorem 4 is much stronger than its first corollary. To see this, consider the case where $d\mu(y) = f(y) dy$ with f continuous and nonnegative, and with the zero set Z of f nonempty but with empty interior. Then, whenever $B(x, t) \subseteq E$ and $0 < s < t$, we have $\mathcal{M}(u, x, s) > \mathcal{M}(u, x, t)$, so that the corollary can only be applied to $-u$ and not to u , and it yields only the inequality $\mu \geq 0$. However, for any $x \in Z$ we have

$$\lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{\lambda(B(x, r))} = 0,$$

so that

$$\lim_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{I_\lambda(x; s, t)} = 0$$

by Theorem 2. Now Theorem 4 can be applied to both u and $-u$ (with $S = Z$), and confirms that μ_Z is null.

The next corollary generalizes both of Sodin's "remarks" to n -dimensions, with weaker hypotheses.

Corollary 2. *Let u be δ -subharmonic with associated measure μ on E , and let S be a Borel subset of E on which there is defined a positive measure ν such that for some constant $\beta \geq 0$*

$$\nu(B(x, r)) \geq \kappa r^\beta$$

whenever $x \in S$ and $0 < r < r_x$, where $\kappa = \kappa_x > 0$. Let $\alpha > 0$, and let h be an absolutely continuous function on $[0, \alpha]$ such that $h'(r) = o(r^{\beta-n+1})$ as $r \rightarrow 0$. If, to each $x \in S$, there correspond sequences $\{s_k\}$ and $\{t_k\}$ such that $0 < s_k < t_k \rightarrow 0$ and

$$\mathcal{M}(u, x, s_k) - \mathcal{M}(u, x, t_k) \geq h(s_k) - h(t_k) \quad \forall k,$$

then $\mu_S \geq 0$.

P r o o f. Given $x \in S$ and $\varepsilon > 0$, for all sufficiently large k we have

$$\begin{aligned} \mathcal{M}(u, x, s_k) - \mathcal{M}(u, x, t_k) &\geq - \int_{s_k}^{t_k} h'(r) dr \geq -\varepsilon \kappa p_n \int_{s_k}^{t_k} r^{\beta-n+1} dr \\ &\geq -\varepsilon p_n \int_{s_k}^{t_k} r^{1-n} \nu(B(x, r)) dr = -\varepsilon I_\nu(x; s_k, t_k), \end{aligned}$$

so that

$$\limsup_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{I_\nu(x; s, t)} \geq 0.$$

By Theorem 4, $\mu_S \geq 0$. □

In the above corollary, the case $n = \beta = 2$ is [12] Remark 1, which does not mention a measure ν . The choice $\nu = \lambda$ gives the result. The case $n = 2, \beta > 0$, is [12] Remark 2. With regard to the existence of ν , Sodin mentioned only the work of Tricot [13]. However, there are many other results in this direction. For example, if S is a q -set for some $q \in [0, n]$ (as, for example, in [8]), then the q -dimensional Hausdorff measure ν on S satisfies $\nu(B(x, r)) \sim (2r)^q$ as $r \rightarrow 0$, at every regular point of S .

5. SPECIFIC RATES

The next theorem generalizes one due to Armitage [2], which we deduce as a corollary.

Theorem 5. *Let u be δ -subharmonic with associated measure μ on E . Let $\alpha > 0$, let f be a positive, increasing, absolutely continuous function on $[0, \alpha]$, and let*

$$\hat{f}(s, t) = p_n \int_s^t r^{1-n} f(r) \, dr$$

whenever $0 \leq s < t \leq \alpha$. Then

$$(16) \quad \limsup_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\hat{f}(s, t)} \leq \limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{f(r)}$$

for every $x \in E$.

P r o o f. Given x , define a positive measure ν on $B(x, \alpha)$ by putting

$$d\nu(y) = \|x - y\|^{1-n} f'(\|x - y\|) \, dy + \sigma_n f(0) \, d\delta_x(y),$$

where σ_n is the surface area of the unit sphere in \mathbb{R}^n . Then

$$\nu(B(x, r)) = \sigma_n \int_0^r f'(s) \, ds + \sigma_n f(0) = \sigma_n f(r)$$

if $0 < r \leq \alpha$, so that

$$I_\nu(x; s, t) = p_n \int_s^t r^{1-n} \sigma_n f(r) \, dr = \sigma_n \hat{f}(s, t)$$

whenever $0 < s < t \leq \alpha$. The result now follows from Theorem 2. □

Armitage's result did not involve differences of spherical mean values, and so required an additional hypothesis on \hat{f} , as follows.

Corollary 1. *Let u be δ -subharmonic with associated measure μ on E . Let $\alpha > 0$, let f be a positive, increasing, absolutely continuous function on $[0, \alpha]$, and let*

$$\hat{f}(s, t) = p_n \int_s^t r^{1-n} f(r) \, dr$$

whenever $0 \leq s < t \leq \alpha$. If $\hat{f}(0, \alpha) = \infty$, then

$$\limsup_{s \rightarrow 0} \frac{\mathcal{M}(u, x, s)}{\hat{f}(s, \alpha)} \leq \limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{f(r)}$$

for all $x \in E$.

P r o o f. Given $x \in E$, let ℓ denote the left-hand side of (16). In view of (16), it suffices to prove that

$$(17) \quad \limsup_{s \rightarrow 0} \frac{\mathcal{M}(u, x, s)}{\hat{f}(s, \alpha)} \leq \ell.$$

We may assume that $\ell < \infty$. Given a real number $A > \ell$, choose $\delta > 0$ such that

$$\frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\hat{f}(s, t)} < A \quad \text{whenever } 0 < s < t < \delta.$$

Fix $t < \delta$. Given $\varepsilon > 0$, choose $\eta < t$ such that both

$$\frac{\mathcal{M}(u, x, t)}{\hat{f}(s, \alpha)} < \varepsilon \quad \text{and} \quad \frac{\hat{f}(t, \alpha)}{\hat{f}(s, \alpha)} < \varepsilon$$

whenever $0 < s < \eta$. Then

$$\begin{aligned} \frac{\mathcal{M}(u, x, s)}{\hat{f}(s, \alpha)} &= \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\hat{f}(s, t)} \cdot \frac{\hat{f}(s, t)}{\hat{f}(s, \alpha)} + \frac{\mathcal{M}(u, x, t)}{\hat{f}(s, \alpha)} \\ &< A \left(1 - \frac{\hat{f}(t, \alpha)}{\hat{f}(s, \alpha)} \right) + \varepsilon < \max\{A, (1 - \varepsilon)A\} + \varepsilon \end{aligned}$$

if $0 < s < \eta$, and (17) follows. □

The extra generality of Theorem 5 over Corollary 1 allows us to generalize the corollary of Armitage's theorem and remove its restrictions on q .

Corollary 2. *Let u be δ -subharmonic with associated measure μ on E , and let $x \in E$. Then*

$$\limsup_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{s^{q+2-n} - t^{q+2-n}} \leq \left(\frac{n-2}{n-q-2} \right) \limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^q}$$

if $0 \leq q < n-2$,

$$\limsup_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\log(t/s)} \leq p_n \limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^{n-2}},$$

and

$$\limsup_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{t^{q+2-n} - s^{q+2-n}} \leq \left(\frac{p_n}{q+2-n} \right) \limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^q}$$

if $q > n-2$.

Proof. If we take $f(r) = r^q$ ($q \geq 0$) in Theorem 5, so that $\hat{f}(s, t) = p_n \int_s^t r^{q+1-n} dr$, then $\hat{f}(s, t)$ is equal to p_n times

$$\begin{aligned} & \frac{s^{q+2-n} - t^{q+2-n}}{n-q-2} && \text{if } q < n-2, \\ & \log(t/s) && \text{if } q = n-2, \\ & \frac{t^{q+2-n} - s^{q+2-n}}{q+2-n} && \text{if } q > n-2, \end{aligned}$$

which gives the result. □

If S is a regular q -set [8] contained in E , and μ is the q -dimensional Hausdorff measure on S , then

$$\lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^q} = 2^q$$

for μ -almost all $x \in S$. Therefore, for such x ,

$$\lim_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{s^{q+2-n} - t^{q+2-n}} = \frac{p_n 2^q}{n-q-2}$$

if $q \neq n-2$, and

$$\lim_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\log(t/s)} = p_n 2^q$$

if $q = n-2$, for any superharmonic function u whose associated measure is μ . These identities follow easily from Theorem 5 Corollary 2.

6. CONDITIONS FOR SUPERHARMONICITY

Theorem 2 can easily be re-written in a form that generalizes [2] Theorem 1, which we deduce as a corollary. This formulation is then used to provide conditions under which $u - Av$ is superharmonic for some real number A , as well as a new version of the domination principle.

Theorem 6. *Let u be δ -subharmonic and v superharmonic on E , with associated measures μ and ν respectively. Then*

$$(18) \quad \limsup_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)} \leq \limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}$$

whenever the latter exists. The reverse inequality holds for lower limits, and

$$\lim_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)} = \lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}$$

for ν -almost all $x \in E$.

P r o o f. The result follows from Theorem 2, because

$$\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t) = I_\nu(x; s, t)$$

by [2] Lemma 3. □

Corollary. *Let u be δ -subharmonic and v superharmonic on E , with associated measures μ and ν respectively. If $x \in E$ and $v(x) = \infty$, then*

$$\limsup_{s \rightarrow 0} \frac{\mathcal{M}(u, x, s)}{\mathcal{M}(v, x, s)} \leq \limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}$$

and the reverse inequality holds for lower limits.

P r o o f. Given $x \in E$ such that $v(x) = \infty$, let ℓ denote the left-hand side of (18). In view of (18), it suffices to prove that

$$(19) \quad \limsup_{s \rightarrow 0} \frac{\mathcal{M}(u, x, s)}{\mathcal{M}(v, x, s)} \leq \ell.$$

We may assume that $\ell < \infty$. Given a real number $A > \ell$, choose $\delta > 0$ such that

$$\frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)} < A \quad \text{whenever } 0 < s < t < \delta.$$

Since $\mathcal{M}(v, x, r) \rightarrow \infty$ as $r \rightarrow 0$, we may suppose that $\mathcal{M}(v, x, r) > 0$ for all $r < \delta$. Fix $t < \delta$. Given $\varepsilon > 0$, choose $\eta < t$ such that both

$$\frac{\mathcal{M}(v, x, t)}{\mathcal{M}(v, x, s)} < \varepsilon \quad \text{and} \quad \frac{\mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s)} < \varepsilon$$

whenever $0 < s < \eta$. Then

$$\begin{aligned} \frac{\mathcal{M}(u, x, s)}{\mathcal{M}(v, x, s)} &= \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)} \left(1 - \frac{\mathcal{M}(v, x, t)}{\mathcal{M}(v, x, s)}\right) + \frac{\mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s)} \\ &< \max\{A, (1 - \varepsilon)A\} + \varepsilon \end{aligned}$$

if $0 < s < \eta$. This proves (19). \square

We now use Theorem 6 to prove analogues of a domination theorem and a uniqueness theorem about Poisson integrals on half-spaces given in [1] and [16]. Conditions for the measure to be positive or null in that context translate into conditions for superharmonicity or harmonicity here.

Theorem 7. *Let u be δ -subharmonic on E , and let v be superharmonic on E with associated measure ν . If*

$$\limsup_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)}$$

is never $-\infty$, and is greater than or equal to A for ν -almost all x , then $u - Av$ is superharmonic on E .

Proof. Let μ be the measure associated to u . By Theorem 6, our hypotheses imply that

$$\limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}$$

is never $-\infty$, and is greater than or equal to A for ν -almost all x . Therefore, we can use Theorem 1 Corollary 1 to show that $\mu - A\nu \geq 0$. Hence $u - Av$ is superharmonic on E . \square

Note that the case $A = 0$ of Theorem 7 gives a condition for u itself to be superharmonic. Theorem 7 is analogous to both [16] Theorem 2 and an earlier result about Poisson integrals on a disc, [5] Theorem 2. It also implies the following condition for u to be harmonic; compare [1] Theorem 4 and the comment on that result in [16] (p. 470).

Corollary. *Let u be δ -subharmonic on E , and let v be superharmonic on E with associated measure ν . If*

$$\lim_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)}$$

is finite whenever it exists, and is 0 for ν -almost all x , then u is harmonic on E .

P r o o f. Applying Theorem 7 to both u and $-u$, we see that both functions are superharmonic on E . \square

A minor variation in the proof of Theorem 7 yields a new form of the Domination Principle ([7], pp. 67, 194).

Theorem 8. *Let E be Greenian, let $v = G_E\nu$ be the Green potential of a positive measure ν on E , and let u be a positive superharmonic function on E . If*

$$\limsup_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)}$$

is never $-\infty$, and is greater than or equal to 1 for ν -almost all x , then $u \geq G_E\nu$.

P r o o f. Let μ be the measure associated to u . As in the proof of Theorem 7, our hypotheses imply that $\mu \geq \nu$, so that $G_E\mu \geq G_E\nu$. Since u is positive, its greatest harmonic minorant is nonnegative, and so $u \geq G_E\nu$. \square

7. NULL SETS FOR ASSOCIATED MEASURES

Theorem 4 obviously implies a condition for a set to be null for the associated measure μ . In this section we state the result explicitly and relate it to known theorems. For example, if u is superharmonic on E and $S = \{x \in E : u(x) < \infty\}$, it is well-known that any polar subset of E is μ_S -null ([7], p. 68). That result was generalized to δ -subharmonic functions, with S replaced by $\{x \in E : \text{fine } \liminf_{y \rightarrow x} |u(y)| < \infty\}$, by Fuglede [10] Theorem 2.1. Theorem 9 Corollary 2 gives a different generalization.

Theorem 9. *Let u be δ -subharmonic and v superharmonic on E , with associated measures μ and ν respectively, and let S be a Borel subset of E . If*

$$\lim_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)}$$

is not infinite for any $x \in S$, and is zero for ν -almost all $x \in S$, then μ_S is null.

P r o o f. Write $\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)$ as $I_\nu(x; s, t)$, and apply Theorem 4 to both u and $-u$. \square

The first corollary gives a restricted version of the theorem which involves quotients of the form $\mathcal{M}(u, x, s)/\mathcal{M}(v, x, s)$, and thus parallels Theorem 6 Corollary.

Corollary 1. *Let u be δ -subharmonic with associated measure μ on E , let v be superharmonic on E , and let S be a Borel subset of E . If, for each $x \in S$, $v(x) = \infty$ and there is a null sequence $\{r_k\}$ such that*

$$(20) \quad \lim_{k \rightarrow \infty} \frac{\mathcal{M}(u, x, r_k)}{\mathcal{M}(v, x, r_k)} = 0,$$

then μ_S is null.

Proof. For any $x \in S$ we have $\mathcal{M}(v, x, r) \rightarrow \infty$ as $r \rightarrow 0$. Therefore, for any fixed t such that $B(x, t) \subseteq E$,

$$\lim_{k \rightarrow \infty} \frac{\mathcal{M}(u, x, r_k) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, r_k) - \mathcal{M}(v, x, t)} = 0$$

in view of (20). Therefore

$$\lim_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)}$$

is zero if it exists, and it exists for ν -almost all x (where ν is the measure associated to v) by Theorem 6. Now Theorem 9 shows that μ_S is null. \square

Corollary 2. *Let u be δ -subharmonic with associated measure μ on E . If S is a Borel subset of E such that for each $x \in S$*

$$(21) \quad \liminf_{r \rightarrow 0} |\mathcal{M}(u, x, r)| < \infty,$$

then any polar subset of E is μ_S -null.

Proof. Let N be a polar subset of E , and let v be a superharmonic function on E such that $v(x) = \infty$ for every $x \in N$. Then, for any $x \in S \cap N$, the condition (21) implies the existence of a null sequence $\{r_k\}$ such that (20) holds. By Corollary 1, $\mu_{S \cap N}$ is null. \square

Corollary 1 is considerably stronger than Corollary 2. To illustrate this, we consider an open ball B with a G_δ polar subset N . We construct two positive superharmonic functions u, v on B , with $u(x) = v(x) = \infty$ for all $x \in N$, such that $\mu(N) = 0$

(see [2], p. 61) and $\nu(B \setminus N) = 0$ (see [6]), where μ, ν are the measures associated to u, v respectively. Since μ and ν are mutually singular, we have

$$\lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{\nu(B(x, r))} = 0$$

for ν -almost all x [3], so that

$$\lim_{r \rightarrow 0} \frac{\mathcal{M}(u, x, r)}{\mathcal{M}(v, x, r)} = 0$$

by [2] Theorem 1. So Corollary 1 confirms that there is a ν -null set M such that $\mu_{N \setminus M}$ is null, but Corollary 2 is inapplicable because (21) fails to hold for any $x \in N$.

8. MORE EXTENSIONS OF KNOWN RESULTS

We conclude with two extensions of results in [14].

Theorem 10. *Let u be δ -subharmonic and v superharmonic on E , with associated measures μ and ν respectively. Let*

$$f(x) = \lim_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)}$$

whenever the limit exists, let $Z^+ = \{x \in E: f(x) = \infty\}$, and let $Z^- = \{x \in E: f(x) = -\infty\}$. Then f is defined and finite ν -a.e. on E , and there are positive ν -singular measures σ^+ and σ^- , concentrated on Z^+ and Z^- respectively, such that $d\mu = f d\nu + d\sigma^+ - d\sigma^-$.

Proof. By Theorem 6,

$$f(x) = \lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}$$

whenever this limit exists. The result now follows from Theorem 1. □

Note that, if $f(x)$ is finite whenever it exists, then μ is absolutely continuous with respect to ν .

Theorem 10 generalizes [14] Theorem 6, which we now deduce as a corollary.

Corollary. *Let u be δ -subharmonic and v superharmonic on E , with associated measures μ and ν respectively, let $X = \{x \in E: v(x) = \infty\}$, let*

$$g(x) = \lim_{r \rightarrow 0} \frac{\mathcal{M}(u, x, r)}{\mathcal{M}(v, x, r)}$$

whenever the limit exists, let $Z^+ = \{x \in X: g(x) = \infty\}$, and let $Z^- = \{x \in X: g(x) = -\infty\}$. Then g is defined and finite ν -a.e. on X , and there are positive ν -singular measures σ^+ and σ^- , concentrated on Z^+ and Z^- respectively, such that $d\mu_X = g d\nu_X + d\sigma^+ - d\sigma^-$.

Proof. If $x \in X$, then

$$\limsup_{r \rightarrow 0} \frac{\mathcal{M}(u, x, r)}{\mathcal{M}(v, x, r)} \leq \limsup_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)}$$

by (19), and the reverse inequality holds for lower limits. Therefore $g(x)$ is equal to the $f(x)$ in Theorem 10, whenever $f(x)$ exists. The result follows. \square

Theorem 10 enables us to prove a corresponding generalization of [14] Theorem 8, as follows. This generalization provides conditions under which a Borel set is a positive set for the Riesz measure of a δ -subharmonic function, whereas [14] Theorem 8 applied only to a Borel *polar* set.

Theorem 11. *Let u be δ -subharmonic and v superharmonic on E , with associated measures μ and ν respectively. Let $q \in [0, n - 2]$, and let S be a Borel subset of E . If*

$$\limsup_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)} > -\infty$$

for all $x \in S \setminus Y$, where Y is an m_q -null Borel set, if

$$\limsup_{0 < s < t \rightarrow 0} \frac{\mathcal{M}(u, x, s) - \mathcal{M}(u, x, t)}{\mathcal{M}(v, x, s) - \mathcal{M}(v, x, t)} \geq 0$$

for ν -almost all $x \in S \setminus Y$, and if

$$(22) \quad \liminf_{r \rightarrow 0} r^{n-q-2} \mathcal{M}(u, x, r) > -\infty$$

for $|\mu|$ -almost all $x \in Y$, then $\mu_S \geq 0$. If (22) is replaced by

$$\liminf_{r \rightarrow 0} r^{n-q-2} \mathcal{M}(u, x, r) \geq 0,$$

then the result remains valid if $0 < m_q(Y) < \infty$.

Proof. Follow the proof of [14] Theorem 8, but use Theorem 10 above instead of [14] Theorem 6. \square

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