Gary Chartrand; Garry L. Johns; Kathleen A. McKeon; Ping Zhang
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RAINBOW CONNECTION IN GRAPHS

Gary Chartrand, Kalamazoo, Garry L. Johns, Saginaw Valley,
Kathleen A. McKeon, New London, Ping Zhang, Kalamazoo

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Abstract. Let G be a nontrivial connected graph on which is defined a coloring $c\colon E(G) \to \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$, of the edges of G, where adjacent edges may be colored the same. A path P in G is a rainbow path if no two edges of P are colored the same. The graph G is rainbow-connected if G contains a rainbow $u-v$ path for every two vertices u and v of G. The minimum k for which there exists such a k-edge coloring is the rainbow connection number $\mathrm{rc}(G)$ of G. If for every pair u, v of distinct vertices, G contains a rainbow $u-v$ geodesic, then G is strongly rainbow-connected. The minimum k for which there exists a k-edge coloring of G that results in a strongly rainbow-connected graph is called the strong rainbow connection number $\mathrm{src}(G)$ of G. Thus $\mathrm{rc}(G) \leqslant \mathrm{src}(G)$ for every nontrivial connected graph G. Both $\mathrm{rc}(G)$ and $\mathrm{src}(G)$ are determined for all complete multipartite graphs G as well as other classes of graphs. For every pair a, b of integers with $a \geqslant 3$ and $b \geqslant (5a-6)/3$, it is shown that there exists a connected graph G such that $\mathrm{rc}(G) = a$ and $\mathrm{src}(G) = b$.

Keywords: edge coloring, rainbow coloring, strong rainbow coloring

MSC 2000: 05C15, 05C38, 05C40

1. Introduction

Let G be a nontrivial connected graph on which is defined a coloring $c\colon E(G) \to \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$, of the edges of G, where adjacent edges may be colored the same. A $u-v$ path P in G is a *rainbow path* if no two edges of P are colored the same. The graph G is *rainbow-connected* (*with respect to* c) if G contains a rainbow $u-v$ path for every two vertices u and v of G. In this case, the coloring c is called a *rainbow coloring* of G. If k colors are used, then c is a *rainbow k-coloring*. The minimum k for which there exists a rainbow k-coloring of the edges of G is the *rainbow connection number* $\mathrm{rc}(G)$ of G. A rainbow coloring of G using $\mathrm{rc}(G)$ colors is called a *minimum rainbow coloring* of G.

Let c be a rainbow coloring of a connected graph G. For two vertices u and v of G, a *rainbow $u-v$ geodesic* in G is a rainbow $u-v$ path of length $d(u,v)$, where $d(u,v)$ is the distance between u and v (the length of a shortest $u-v$ path in G). The graph G is *strongly rainbow-connected* if G contains a rainbow $u-v$ geodesic for every two vertices u and v of G. In this case, the coloring c is called a *strong rainbow coloring* of G. The minimum k for which there exists a coloring $c\colon E(G) \to \{1,2,\ldots,k\}$ of the edges of G such that G is strongly rainbow-connected is the *strong rainbow connection number* $\mathrm{src}(G)$ of G. A strong rainbow coloring of G using $\mathrm{src}(G)$ colors is called a *minimum strong rainbow coloring* of G. Thus $\mathrm{rc}(G) \leqslant \mathrm{src}(G)$ for every connected graph G.

Since every coloring that assigns distinct colors to the edges of a connected graph is both a rainbow coloring and a strong rainbow coloring, every connected graph is rainbow-connected and strongly rainbow-connected with respect to some coloring of the edges of G. Thus the rainbow connection numbers $\mathrm{rc}(G)$ and $\mathrm{src}(G)$ are defined for every connected graph G. Furthermore, if G is a nontrivial connected graph of size m whose diameter (the largest distance between two vertices of G) is $\mathrm{diam}(G)$, then

$$\mathrm{diam}(G) \leqslant \mathrm{rc}(G) \leqslant \mathrm{src}(G) \leqslant m. \tag{1}$$

To illustrate these concepts, consider the Petersen graph P of Figure 1, where a rainbow 3-coloring of P is also shown. Thus $\mathrm{rc}(P) \leqslant 3$. On the other hand, if u and v are two nonadjacent vertices of P, then $d(u,v) = 2$ and so the length of a $u-v$ path is at least 2. Thus any rainbow coloring of P uses at least two colors and so $\mathrm{rc}(P) \geqslant 2$. If P has a rainbow 2-coloring c, then there exist two adjacent edges of G that are colored the same by c, say $e = uv$ and $f = vw$ are colored the same. Since there is exactly one $u-w$ path of length 2 in P, there is no rainbow $u-w$ path in P, which is a contradiction. Therefore, $\mathrm{rc}(P) = 3$.

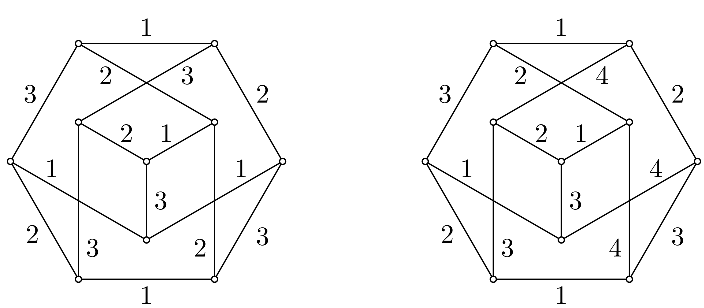

Figure 1. A rainbow 3-coloring and a strong rainbow 4-coloring of the Petersen graph

Since $\text{rc}(P) = 3$, it follows that $\text{src}(P) \geqslant 3$. Furthermore, since the edge chromatic number of the Petersen graph is known to be 4, any 3-coloring c of the edges of P results in two adjacent edges uv and vw being assigned the same color. Since u, v, w is the only $u - w$ geodesic in P, the coloring c is not a strong rainbow coloring. Because the 4-coloring of the edges of P shown in Figure 2 is a strong rainbow coloring, $\text{src}(P) = 4$.

As another example, consider the graph G of Figure 2(a), where a rainbow 4-coloring c of G is also shown. In fact, c is a minimum rainbow coloring of G and so $\text{rc}(G) = 4$, as we now verify.

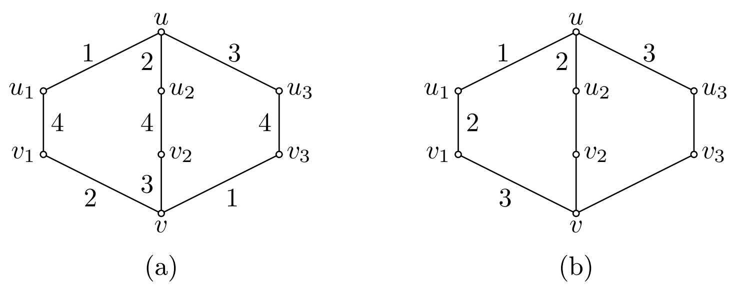

Figure 2. A graph G with $\text{rc}(G) = \text{src}(G) = 4$

Since $\text{diam}(G) \geqslant 3$, it follows that $\text{rc}(G) \geqslant 3$. Assume, to the contrary, $\text{rc}(G) = 3$. Then there exists a rainbow 3-coloring c' of G. Since every $u - v$ path in G has length 3, at least one of the three $u - v$ paths in G is a rainbow $u - v$ path, say u, u_1, v_1, v is a rainbow $u - v$ path. We may assume that $c'(uu_1) = 1$, $c'(u_1v_1) = 2$, and $c'(v_1v) = 3$. (See Figure 2(b).)

If x and y are two vertices in G such that $d(x, y) = 2$, then G contains exactly one $x - y$ path of length 2, while all other $x - y$ paths have length 4 or more. This implies that no two adjacent edges can be colored the same. Thus we may assume, without loss of generality, that $c'(uu_2) = 2$ and $c'(uu_3) = 3$. (See Figure 2(b).) Thus $\{c'(vv_2), c'(vv_3)\} = \{1, 2\}$. If $c'(vv_2) = 1$ and $c'(vv_3) = 2$, then $c'(u_2v_2) = 3$ and $c'(u_3v_3) = 1$. In this case, there is no rainbow $u_1 - v_3$ path in G. On the other hand, if $c'(vv_2) = 2$ and $c'(vv_3) = 1$, then $c'(u_2v_2) \in \{1, 3\}$ and $c'(u_3v_3) = 2$. If $c'(u_2v_2) = 1$, then there is no rainbow $u_2 - v_3$ path in G; while if $c'(u_2v_2) = 3$, there is no rainbow $u_2 - v_1$ path in G, a contradiction. Therefore, as claimed, $\text{rc}(G) = 4$.

Since $4 = \text{rc}(G) \leqslant \text{src}(G)$ for the graph G of Figure 2 and the rainbow 4-coloring of G in Figure 2(a) is also a strong rainbow 4-coloring, $\text{src}(G) = 4$ as well.

If G is a nontrivial connected graph of size m, then we saw in (1) that $\text{diam}(G) \leqslant \text{rc}(G) \leqslant \text{src}(G) \leqslant m$. In the following result, it is determined which connected graphs G attain the extreme values 1, 2 or m.

Proposition 1.1. *Let G be a nontrivial connected graph of size m. Then*

(a) $\mathrm{src}(G) = 1$ *if and only if G is a complete graph,*

(b) $\mathrm{rc}(G) = 2$ *if and only if* $\mathrm{src}(G) = 2$,

(c) $\mathrm{rc}(G) = m$ *if and only if G is a tree.*

Proof. We first verify (a). If G is a complete graph, then the coloring that assigns 1 to every edge of G is a strong rainbow 1-coloring of G and so $\mathrm{src}(G) = 1$. On the other hand, if G is not complete, then G contains two nonadjacent vertices u and v. Thus each $u - v$ geodesic in G has length at least 2 and so $\mathrm{src}(G) \geqslant 2$.

To verify (b), first assume that $\mathrm{rc}(G) = 2$ and so $\mathrm{src}(G) \geqslant 2$ by (1). Since $\mathrm{rc}(G) = 2$, it follows that G has a rainbow 2-coloring, which implies that every two nonadjacent vertices are connected by a rainbow path of length 2. Because such a path is a geodesic, $\mathrm{src}(G) = 2$. On the other hand, if $\mathrm{src}(G) = 2$, then $\mathrm{rc}(G) \leqslant 2$ by (1) again. Furthermore, since $\mathrm{src}(G) = 2$, it follows by (a) that G is not complete and so $\mathrm{rc}(G) \geqslant 2$. Thus $\mathrm{rc}(G) = 2$.

We now verify (c). Suppose first that G is not a tree. Then G contains a cycle $C\colon v_1, v_2, \ldots, v_k, v_1$, where $k \geqslant 3$. Then the $(m-1)$-coloring of the edges of G that assigns 1 to the edges v_1v_2 and v_2v_3 and assigns the $m-2$ distinct colors from $\{2, 3, \ldots, m-1\}$ to the remaining $m-2$ edges of G is a rainbow coloring. Thus $\mathrm{rc}(G) \leqslant m-1$. Next, let G be a tree of size m. Assume, to the contrary, that $\mathrm{rc}(G) \leqslant m-1$. Let c be a minimum rainbow coloring of G. Then there exist edges e and f such that $c(e) = c(f)$. Assume, without loss generality, that $e = uv$ and $f = xy$ and G contains a $u - y$ path $u, v, \ldots, x, y$. Then there is no rainbow $u - y$ path in G, which is a contradiction. □

Proposition 1.1 also implies that the only connected graphs G for which $\mathrm{rc}(G) = 1$ are the complete graphs and that the only connected graphs G of size m for which $\mathrm{src}(G) = m$ are trees.

2. Some rainbow connection numbers of graphs

In this section, we determine the rainbow connection numbers of some well-known graphs. We refer to the book [1] for graph-theoretical notation and terminology not described in this paper. We begin with cycles of order n. Since $\mathrm{diam}(C_n) = \lfloor n/2 \rfloor$, it follows by (1) that $\mathrm{src}(C_n) \geqslant \mathrm{rc}(C_n) \geqslant \lfloor n/2 \rfloor$. This lower bound for $\mathrm{rc}(C_n)$ and $\mathrm{src}(C_n)$ is nearly the exact value of these numbers.

Proposition 2.1. *For each integer $n \geqslant 4$, $\mathrm{rc}(C_n) = \mathrm{src}(C_n) = \lceil n/2 \rceil$.*

Proof. Let $C_n\colon v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ and for each i with $1 \leqslant i \leqslant n$, let $e_i = v_i v_{i+1}$. We consider two cases, according to whether n is even or n is odd.

Case 1. *n is even.* Let $n = 2k$ for some integer $k \geqslant 2$. Thus $\mathrm{src}(C_n) \geqslant \mathrm{rc}(C_n) \geqslant \mathrm{diam}(C_n) = k$. Since the edge coloring c_0 of C_n defined by $c_0(e_i) = i$ for $1 \leqslant i \leqslant k$ and $c_0(e_i) = i - k$ if $k + 1 \leqslant i \leqslant n$ is a strong rainbow k-coloring, it follows that $\mathrm{rc}(C_n) \leqslant \mathrm{src}(C_n) \leqslant k$ and so $\mathrm{rc}(C_n) = \mathrm{src}(C_n) = k$.

Case 2. *n is odd.* Then $n = 2k + 1$ for some integer $k \geqslant 2$. First define an edge coloring c_1 of C_n by $c_1(e_i) = i$ for $1 \leqslant i \leqslant k + 1$ and $c_1(e_i) = i - k - 1$ if $k + 2 \leqslant i \leqslant n$. Since c_1 is a strong rainbow $(k+1)$-coloring of C_n, it follows that $\mathrm{rc}(C_n) \leqslant \mathrm{src}(C_n) \leqslant k + 1$.

Since $\mathrm{rc}(C_n) \geqslant \mathrm{diam}(C_n) = k$, it follows that $\mathrm{rc}(C_n) = k$ or $\mathrm{rc}(C_n) = k + 1$. We claim that $\mathrm{rc}(C_n) = k + 1$. Assume, to the contrary, that $\mathrm{rc}(C_n) = k$. Let c' be a rainbow k-coloring of C_n and let u and v be two antipodal vertices of C_n. Then the $u - v$ geodesic in C_n is a rainbow path and the other $u - v$ path in C_n is not a rainbow path since it has length $k + 1$. Suppose, without loss of generality, that $c'(v_{k+1} v_{k+2}) = k$.

Consider the vertices v_1, v_{k+1}, and v_{k+2}. Since the $v_1 - v_{k+1}$ geodesic $P\colon v_1, v_2, \ldots, v_{k+1}$ is a rainbow path and the $v_1 - v_{k+2}$ geodesic $Q\colon v_1, v_n, v_{n-1}, \ldots, v_{k+2}$ is a rainbow path, some edge on P is colored k as is some edge on Q. Since the $v_2 - v_{k+2}$ geodesic $v_2, v_3, \ldots, v_{k+2}$ is a rainbow path, it follows that $c'(v_1 v_2) = k$. Similarly, the $v_n - v_{k+1}$ geodesic $v_n, v_{n-1}, v_{n-2}, \ldots, v_{k+1}$ is a rainbow path and so $c'(v_n v_1) = k$. Thus $c'(v_1 v_2) = c'(v_n v_1) = k$. This implies that there is no rainbow $v_2 - v_n$ path in G, producing a contradiction. Thus $\mathrm{rc}(C_n) = \mathrm{src}(C_n) = k + 1$. $\square$

A well-known class of graphs constructed from cycles are the wheels. For $n \geqslant 3$, the *wheel* W_n is defined as $C_n + K_1$, the join of C_n and K_1, constructed by joining a new vertex to every vertex of C_n. Thus $W_3 = K_4$. Next, we determine rainbow connection numbers of wheels.

Proposition 2.2. *For $n \geqslant 3$, the rainbow connection number of the wheel W_n is*

$$\mathrm{rc}(W_n) = \begin{cases} 1 & \text{if } n = 3, \\ 2 & \text{if } 4 \leqslant n \leqslant 6, \\ 3 & \text{if } n \geqslant 7. \end{cases}$$

Proof. Suppose that W_n consists of an n-cycle $C_n\colon v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ and another vertex v joined to every vertex of C_n. Since $W_3 = K_4$, it follows by Proposition 1.1 that $\mathrm{rc}(W_3) = 1$. For $4 \leqslant n \leqslant 6$, the wheel W_n is not complete and so $\mathrm{rc}(W_n) \geqslant 2$. Since the 2-coloring $c\colon E(W_n) \to \{1, 2\}$ defined by $c(v_i v) = 1$ if i is odd, $c(v_i v) = 2$ if i is even, and $c(v_i v_{i+1}) = 1$ if i is odd, and $c(v_i v_{i+1}) = 2$ if i is even is a rainbow coloring, it follows that $\mathrm{rc}(W_n) = 2$ for $4 \leqslant n \leqslant 6$.

Finally, suppose that $n \geqslant 7$. Since the 3-coloring $c\colon E(W_n) \to \{1, 2, 3\}$ defined by $c(v_i v) = 1$ if i is odd, $c(v_i v) = 2$ if i is even, and $c(e) = 3$ for each $e \in E(C_n)$ is a rainbow coloring, it follows that $\mathrm{rc}(W_n) \leqslant 3$. It remains to show that $\mathrm{rc}(W_n) \geqslant 3$. Since W_n is not complete, $\mathrm{rc}(W_n) \geqslant 2$. Assume, to the contrary, that $\mathrm{rc}(W_n) = 2$. Let c' be a rainbow 2-coloring of W_n. Without loss of generality, assume that $c'(v_1 v) = 1$. For each i with $4 \leqslant i \leqslant n-2$, v_1, v, v_i is the only $v_1 - v_i$ path of length 2 in W_n and so $c'(v_i v) = 2$ for $4 \leqslant i \leqslant n-2$. Since $c(v_4 v) = 2$, it follows that $c(v_n v) = 1$. This forces $c(v_3 v) = 2$, which in turn forces $c(v_{n-1} v) = 1$. Similarly, $c(v_{n-1} v) = 1$ forces $c(v_2 v) = 2$. Since $c(v_2 v) = 2$ and $c(v_5 v) = 2$, there is no rainbow $v_2 - v_5$ path in W_n, which is a contradiction. Therefore, $\mathrm{rc}(W_n) = 3$ for $n \geqslant 7$. □

Proposition 2.3. *For $n \geqslant 3$, the strong rainbow connection number of the wheel W_n is*

$$\mathrm{src}(W_n) = \lceil n/3 \rceil.$$

P r o o f. Suppose that W_n consists of an n-cycle $C_n\colon v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ and another vertex v joined to every vertex of C_n. Since $W_3 = K_4$, it follows by Proposition 1.1 that $\mathrm{src}(W_3) = 1$. If $4 \leqslant n \leqslant 6$, then $\mathrm{rc}(W_n) = 2$ by Proposition 2.2 and so $\mathrm{src}(W_n) = 2$ by Proposition 1.1. Therefore, $\mathrm{src}(W_n) = \lceil n/3 \rceil$ for $4 \leqslant n \leqslant 6$.

Thus we may assume $n \geqslant 7$. Then there is an integer k such that $3k - 2 \leqslant n \leqslant 3k$. We first show that $\mathrm{src}(W_n) \geqslant k$. Assume, to the contrary, that $\mathrm{src}(W_n) \leqslant k-1$. Let c be a strong rainbow $(k-1)$-coloring of W_n. Since $\deg v = n > 3(k-1)$, there exists $S \subseteq V(C_n)$ such that $|S| = 4$ and all edges in $\{uv\colon u \in S\}$ are colored the same. Thus there exist at least two vertices $u', u'' \in S$ such that $d_{C_n}(u', u'') \geqslant 3$ and $d_{W_n}(u', u'') = 2$. Since u', v, u'' is the only $u' - u''$ geodesic in W_n, it follows that there is no rainbow $u' - u''$ geodesic in W_n, which is a contradiction. Thus $\mathrm{src}(W_n) \geqslant k$.

To show that $\mathrm{src}(W_n) \leqslant k$, we provide a strong rainbow k-coloring $c\colon E(W_n) \to \{1, 2, \ldots, k\}$ of W_n defined by

$$c(e) = \begin{cases} 1 & \text{if } \ e = v_i v_{i+1} \text{ and } i \text{ is odd}, \\ 2 & \text{if } \ e = v_i v_{i+1} \text{ and } i \text{ is even}, \\ j+1 & \text{if } \ e = v_i v \text{ if } i \in \{3j+1, 3j+2, 3j+3\} \text{ for } 0 \leqslant j \leqslant k-1. \end{cases}$$

Therefore, $\mathrm{src}(W_n) = k = \lceil n/3 \rceil$ for $n \geqslant 7$ as well. □

We now determine the rainbow connection numbers of all complete multipartite graphs, beginning with the strong connection number of the complete bipartite graph $K_{s,t}$ with $1 \leqslant s \leqslant t$.

Theorem 2.4. *For integers s and t with $1 \leqslant s \leqslant t$,*

$$\operatorname{src}(K_{s,t}) = \lceil \sqrt[s]{t} \rceil.$$

P r o o f. Since $\operatorname{src}(K_{1,t}) = t$, the result follows for $s = 1$. So we may assume that $s \geqslant 2$. Let $\lceil \sqrt[s]{t} \rceil = k$. Hence

$$1 \leqslant k - 1 < \sqrt[s]{t} \leqslant k.$$

Therefore, $(k-1)^s < t \leqslant k^s$ and so $(k-1)^s + 1 \leqslant t \leqslant k^s$.

First, we show that $\operatorname{src}(K_{s,t}) \geqslant k$. Assume, to the contrary, that $\operatorname{src}(K_{s,t}) \leqslant k-1$. Then there exists a strong rainbow $(k-1)$-coloring of $K_{s,t}$. Let U and W be the partite sets of $K_{s,t}$, where $|U| = s$ and $|W| = t$. Suppose that $U = \{u_1, u_2, \ldots, u_s\}$. Let there be given a strong rainbow $(k-1)$-coloring c of $K_{s,t}$. For each vertex $w \in W$, we can associate an ordered s-tuple $\operatorname{code}(w) = (a_1, a_2, \ldots, a_s)$ called the color code of w, where $a_i = c(u_i w)$ for $1 \leqslant i \leqslant s$. Since $1 \leqslant a_i \leqslant k-1$ for each i $(1 \leqslant i \leqslant s)$, the number of distinct color codes of the vertices of W is at most $(k-1)^s$. However, since $t > (k-1)^s$, there exists at least two distinct vertices w' and w'' of W such that $\operatorname{code}(w') = \operatorname{code}(w'')$. Since $c(u_i w') = c(u_i w'')$ for all i $(1 \leqslant i \leqslant s)$, it follows that $K_{s,t}$ contains no rainbow $w' - w''$ geodesic in $K_{s,t}$, contradicting our assumption that c is a strong rainbow $(k-1)$-coloring of $K_{s,t}$. Thus, as claimed, $\operatorname{src}(K_{s,t}) \geqslant k$.

Next, we show that $\operatorname{src}(K_{s,t}) \leqslant k$, which we establish by providing a strong rainbow k-coloring of $K_{s,t}$. Let $A = \{1, 2, \ldots, k\}$ and $B = \{1, 2, \ldots, k-1\}$. The sets A^s and B^s are Cartesian products of the s sets A and s sets B, respectively. Thus $|A^s| = k^s$ and $|B^s| = (k-1)^s$. Hence $|B^s| < t \leqslant |A^s|$. Let $W = \{w_1, w_2, \ldots, w_t\}$, where the vertices of W are labeled with t elements of A^s and such that the vertices $w_1, w_2, \ldots, w_{(k-1)^s}$ are labeled by the $(k-1)^s$ elements of B^s. For each i with $1 \leqslant i \leqslant t$, denote the label of w_i by

$$\mathbf{w}_i = (w_{i,1}, w_{i,2}, \ldots, w_{i,s}). \tag{2}$$

For each i with $1 \leqslant i \leqslant (k-1)^s$, we have $1 \leqslant w_{i,j} \leqslant k-1$ for $1 \leqslant j \leqslant s$. We now define a coloring $c\colon E(K_{s,t}) \to \{1, 2, \ldots, k\}$ of the edges of $K_{s,t}$ by

$$c(w_i u_j) = w_{i,j} \quad \text{where } 1 \leqslant i \leqslant t \text{ and } 1 \leqslant j \leqslant s.$$

Thus for $1 \leqslant i \leqslant t$, the color code $\operatorname{code}(w_i)$ of w_i provided by the coloring c is in fact $\mathbf{w}_i$, as described in (2). Hence distinct vertices in W have distinct color codes.

We show that c is a strong rainbow k-coloring of $K_{s,t}$. Certainly, for $w_i \in W$ and $u_j \in U$, the $w_i - u_j$ path w_i, u_j is a rainbow geodesic. Let w_a and w_b be two vertices of W. Since these vertices have distinct color codes, there exists some l with $1 \leqslant l \leqslant s$ such that $\operatorname{code}(w_a)$ and $\operatorname{code}(w_b)$ have different l-th coordinates. Thus $c(w_a u_l) \neq c(w_b u_l)$ and w_a, u_l, w_b is a rainbow $w_a - w_b$ geodesic in $K_{s,t}$. We now consider two vertices u_p and u_q in U, where $1 \leqslant p < q \leqslant s$. Since there exists a vertex $w_i \in W$ with $1 \leqslant i \leqslant (k-1)^s$ such that $w_{i,p} \neq w_{i,q}$, it follows that u_p, w_i, u_q is a rainbow $u_p - u_q$ geodesic in $K_{s,t}$. Thus, as claimed, c is a strong rainbow k-coloring of $K_{s,t}$ and so $\operatorname{src}(K_{s,t}) \leqslant k$. □

With the aid of Theorem 2.4, we are now able to determine the strong rainbow connection numbers of all complete multipartite graphs.

Theorem 2.5. *Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph, where $k \geqslant 3$ and $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_k$ such that $s = \sum\limits_{i=1}^{k-1} n_i$ and $t = n_k$. Then*

$$\operatorname{src}(G) = \begin{cases} 1 & \text{if } n_k = 1, \\ 2 & \text{if } n_k \geqslant 2 \text{ and } s > t, \\ \lceil \sqrt[s]{t} \rceil & \text{if } s \leqslant t. \end{cases}$$

Proof. Let $n = \sum\limits_{i=1}^{k} n_i$. If $n_k = 1$, then $G = K_n$ and by Proposition 1.1, $\operatorname{src}(G) = 1$. Suppose next that $n_k \geqslant 2$ and $s > t$. Since $n_k \geqslant 2$, it follows that $G \neq K_n$ and so $\operatorname{src}(G) \geqslant 2$ by Proposition 1.1. It remains to show that $\operatorname{src}(G) \leqslant 2$ in this case.

Partition the multiset $S = \{n_1, n_2, \ldots, n_k\}$ into two submultisets

$$A = \{a_1, a_2, \ldots, a_p\} \text{ and } B = \{b_1, b_2, \ldots, b_q\},$$

where then $p + q = k$, such that

$$a = \sum_{i=1}^{p} a_i \leqslant \sum_{j=1}^{q} b_j = b$$

and $b - a$ is the minimum nonnegative integer among all such partitions of S. Hence $K_{a,b}$ is a spanning subgraph of G. Since $\operatorname{diam}(K_{a,b}) = 2$, for every two nonadjacent vertices u and v of $K_{a,b}$, a path P is a $u-v$ geodesic in $K_{a,b}$ if and only if P is a $u-v$ geodesic in G. Thus, from Theorem 2.4,

$$\operatorname{src}(G) \leqslant \operatorname{src}(K_{a,b}) = \lceil \sqrt[a]{b} \rceil .$$

We claim that $b \leqslant 2^a$. Assume, to the contrary, that $b > 2^a$. Since $s > t$, it follows that $q \geqslant 2$. We consider two cases, according to $a \leqslant 3$ or $a \geqslant 4$. If G is a complete k-partite graph with $a \leqslant 3$, then the only ordered pairs (a, b) for $K_{a,b}$ are: $(2,3)$, $(2,4)$, $(3,3)$, $(3,4)$, $(3,5)$, $(3,6)$. In all cases, $\operatorname{src}(G) \leqslant \operatorname{src}(K_{a,b}) = \lceil \sqrt[a]{b} \rceil = 2$. Hence we may assume that $a \geqslant 4$. Let b_1 be the smallest element of B. Hence $a + b_1 > b - b_1$. Because $a \geqslant 4$, it follows that

$$b_1 > \frac{b-a}{2} > \frac{2^a - a}{2} > \frac{3a - a}{2} = a.$$

Let $A' = \{b_1\}$ and let the multiset $B' = S - \{b_1\}$. Since $b_2 \in B'$, $b_1 \leqslant b_2$, and $a < b_1$, this contradicts the defining properties of the sets A and B. Hence, as claimed, $b \leqslant 2^a$. Thus

$$\operatorname{src}(G) \leqslant \lceil \sqrt[a]{b} \rceil \leqslant \lceil \sqrt[a]{2^a} \rceil = 2,$$

giving us the desired result.

Next, suppose that $s \leqslant t$. Let W be the unique independent set of $n_k = t$ vertices of G. Since $K_{s,t}$ is a connected spanning subgraph of G, it follows again, since $\operatorname{diam}(G) = 2$, that

$$\operatorname{src}(G) \leqslant \operatorname{src}(K_{s,t}) = \lceil \sqrt[s]{t} \rceil .$$

We claim that $\operatorname{src}(G) = \lceil \sqrt[s]{t} \rceil$. Assume, to the contrary, that $\operatorname{src}(G) = l < \lceil \sqrt[s]{t} \rceil$. Then $t > l^s$. This implies that there exists a strong rainbow l-coloring c of G. Since every vertex of G belonging to W has degree s in G, the coloring c produces a color code $\operatorname{code}(w)$ for each vertex w of W consisting of an ordered s-tuple, each entry of which is an element of $\{1, 2, \ldots, l\}$. Since the number of distinct color codes for the vertices of W is at most l^s and $|W| = t > l^s$, there exist two vertices w' and w'' in W having the same color code. This, however, implies that the two edges in each $w' - w''$ geodesic in G have the same color, contradicting the assumption that c is a strong rainbow l-coloring of G. □

According to Theorems 2.4 and 2.5, the strong rainbow connection number of a complete multipartite graph can be arbitrarily large. This is not the case for the rainbow connection number of a complete multipartite graph however, as we show next. We begin with complete bipartite graphs.

Theorem 2.6. *For integers s and t with $2 \leqslant s \leqslant t$,*

$$\operatorname{rc}(K_{s,t}) = \min\{\lceil \sqrt[s]{t} \rceil,\ 4\}.$$

Proof. First, observe that for $2 \leqslant s \leqslant t$, $\lceil \sqrt[s]{t} \rceil \geqslant 2$. Let U and W be the partite sets of $K_{s,t}$, where $|U| = s$ and $|W| = t$. Suppose that $U = \{u_1, u_2, \ldots, u_s\}$. We consider three cases.

Case 1. $\lceil \sqrt[s]{t} \rceil = 2$. Then $s \leqslant t \leqslant 2^s$. Since

$$2 \leqslant \operatorname{rc}(K_{s,t}) \leqslant \operatorname{src}(K_{s,t}) = \lceil \sqrt[s]{t} \rceil = 2,$$

it follows that $\operatorname{rc}(K_{s,t}) = 2$.

Case 2. $\lceil \sqrt[s]{t} \rceil = 3$. Then $2^s + 1 \leqslant t \leqslant 3^s$. Since

$$2 \leqslant \operatorname{rc}(K_{s,t}) \leqslant \operatorname{src}(K_{s,t}) = \lceil \sqrt[s]{t} \rceil = 3,$$

it follows that $\operatorname{rc}(K_{s,t}) = 2$ or $\operatorname{rc}(K_{s,t}) = 3$. We claim that $\operatorname{rc}(K_{s,t}) = 3$. Assume, to the contrary, that there exists a rainbow 2-coloring of $K_{s,t}$. Corresponding to this rainbow 2-coloring of $K_{s,t}$, there is a color code $\operatorname{code}(w)$ assigned to each vertex $w \in W$, consisting of an ordered s-tuple $(a_1, a_2, \ldots, a_s)$, where $a_i = c(u_i w) \in \{1, 2\}$ for $1 \leqslant i \leqslant s$. Since $t > 2^s$, there exist two distinct vertices w' and w'' of W such that $\operatorname{code}(w') = \operatorname{code}(w'')$. Since the edges of every $w' - w''$ path of length 2 are colored the same, there is no rainbow $w' - w''$ path in $K_{s,t}$, a contradiction. Thus, as claimed, $\operatorname{rc}(K_{s,t}) = 3$.

Case 3. $\lceil \sqrt[s]{t} \rceil \geqslant 4$. Then $t \geqslant 3^s + 1$. We claim that $\operatorname{rc}(K_{s,t}) = 4$. First, we show that $\operatorname{rc}(K_{s,t}) \geqslant 4$. Assume, to the contrary, that there exists a rainbow 3-coloring of $K_{s,t}$. In this case, corresponding to this rainbow 3-coloring of $K_{s,t}$, there is a color code, $\operatorname{code}(w)$, assigned to each vertex $w \in W$, consisting of an ordered s-tuple $(a_1, a_2, \ldots, a_s)$, where $a_i = c(u_i w) \in \{1, 2, 3\}$ for $1 \leqslant i \leqslant s$. Since $t > 3^s$, there exist two distinct vertices w' and w'' of W such that $\operatorname{code}(w') = \operatorname{code}(w'')$. Since every $w' - w''$ path in $K_{s,t}$ has even length, the only possible rainbow $w' - w''$ path must have length 2. However, since $\operatorname{code}(w') = \operatorname{code}(w'')$, the colors of the edges of every $w' - w''$ path of length 2 are the same. Hence there is no rainbow $w' - w''$ path in $K_{s,t}$, a contradiction. Thus, as claimed, $\operatorname{rc}(K_{s,t}) \geqslant 4$.

To verify that $\operatorname{rc}(K_{s,t}) \leqslant 4$, we show that there exists a rainbow 4-coloring of $K_{s,t}$. Let $A = \{1, 2, 3\}$, $W = \{w_1, w_2, \ldots, w_t\}$, $W' = \{w_1, w_2, \ldots, w_{3^s}\}$, and $W'' = W - W'$. Assign to the vertices in W' the 3^s distinct elements of A^s and assign to the vertices in W'' the identical code whose first coordinate is 4 and all whose remaining coordinates are 3. Corresponding to this assignment of codes is a coloring of the edges of $K_{s,t}$, where $c(w_i u_j) = k$ if the jth coordinate of $\mathrm{code}(w_i)$ is k. We claim that this coloring is, in fact, a rainbow 4-coloring of $K_{s,t}$. Let x and y be two nonadjacent vertices of $K_{s,t}$. Suppose first that $x, y \in W$. We consider three cases.

Case i. $x, y \in W'$. Since $\mathrm{code}(x) \neq \mathrm{code}(y)$, there exists i with $1 \leqslant i \leqslant s$ such that $\mathrm{code}(x)$ and $\mathrm{code}(y)$ have different ith coordinates. Then the path x, u_i, y is a rainbow $x - y$ path of length 2 in $K_{s,t}$.

Case ii. $x \in W'$ *and* $y \in W''$. Suppose that the first coordinate of $\mathrm{code}(x)$ is a, where $1 \leqslant a \leqslant 3$. Then x, u_1, y is a rainbow $x - y$ path of length 2 in $K_{s,t}$ whose edges are colored a and 4.

Case iii. $x, y \in W''$. Let $z \in W'$ such that the first coordinate of $\mathrm{code}(z)$ is 1 and the second coordinate of $\mathrm{code}(z)$ is 2. Then x, u_1, z, u_2, y is a rainbow $x - y$ path of length 4 in $K_{s,t}$ whose edges are colored 4, 1, 2, 3, respectively.

Finally, suppose that $x, y \in U$. Then $x = u_i$ and $y = u_j$, where $1 \leqslant i < j \leqslant s$. Then there exists a vertex $w \in W'$ whose ith and jth coordinates are distinct. Then x, w, y is a rainbow $x - y$ path in $K_{s,t}$.

Thus this coloring is a rainbow 4-coloring of $K_{s,t}$ and so $\mathrm{rc}(K_{s,t}) = 4$ in this case. □

Next, we determine rainbow connection numbers of all complete multipartite graphs.

Theorem 2.7. *Let $G = K_{n_1, n_2, \ldots, n_k}$ be a complete k-partite graph, where $k \geqslant 3$ and $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_k$ such that $s = \sum_{i=1}^{k-1} n_i$ and $t = n_k$. Then*

$$\mathrm{rc}(G) = \begin{cases} 1 & \text{if } n_k = 1, \\ 2 & \text{if } n_k \geqslant 2 \text{ and } s > t, \\ \min\{\lceil \sqrt[s]{t} \rceil, 3\} & \text{if } s \leqslant t. \end{cases}$$

Proof. Let $n = s + t = \sum_{i=1}^{k} n_i$. If $n_k = 1$, then $G = K_n$ and by Proposition 1.1, $\mathrm{rc}(G) = 1$. Suppose next that $n_k \geqslant 2$ and $s > t$. By Theorem 2.5, $\mathrm{src}(G) = 2$ and so $\mathrm{rc}(G) = 2$ by Proposition 1.1.

Next, suppose that $s \leqslant t$. Since $n_k \geqslant 2$, it follows that $G \neq K_n$ and so $\mathrm{rc}(G) \geqslant 2$. By Theorem 2.5, $\mathrm{src}(G) = \lceil \sqrt[s]{t} \rceil$ and so $\mathrm{rc}(G) \leqslant \lceil \sqrt[s]{t} \rceil$. To show that $\mathrm{rc}(G) \leqslant 3$ as well, we provide a rainbow 3-coloring of G. Let $V_1, V_2, \ldots, V_k$ be the partite sets of G with

$$V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}\}$$

for $1 \leqslant i \leqslant k$. Furthermore, let

$$U = V_1 \cup V_2 \cup \ldots \cup V_{k-1} = \{u_1, u_2, \ldots, u_s\}$$

such that $u_i = v_{k-1,i}$ for $1 \leqslant i \leqslant n_{k-1}$. Thus $|U| = s$. Define a coloring c^* of the edges of G by

$$c^*(e) = \begin{cases} 1 & \text{if } \ e = v_{i,j}v_{i+1,j} \text{ for } 1 \leqslant i \leqslant k-2 \text{ and } 1 \leqslant j \leqslant n_i \text{ or} \\ & \text{if } \ e = u_l v_{k,l} \text{ for } 1 \leqslant l \leqslant s, \\ 2 & \text{if } \ e = v_{1,j}v_{k,l} \text{ for } 1 \leqslant j \leqslant n_1 \text{ and } s+1 \leqslant l \leqslant t, \\ 3 & \text{otherwise.} \end{cases}$$

Let x and y be two nonadjacent vertices of G. Then $x, y \in V_i$ for some i with $1 \leqslant i \leqslant k$. Let $x = v_{i,p}$ and $y = v_{i,q}$, where $1 \leqslant p < q \leqslant n_i$. If $1 \leqslant i \leqslant k-1$, then $x, v_{i+1,p}, y$ is a rainbow $x - y$ path in G whose edges are colored 1 and 3. Thus we may assume that $i = k$. If $1 \leqslant p < q \leqslant s$, then x, u_p, y is a rainbow $x - y$ path in G whose edges are colored 1 and 3. If $s+1 \leqslant p < q \leqslant t$, then $x, v_{1,1}, v_{2,1}, y$ is a rainbow $x - y$ path in G whose edges are colored 2, 1 and 3, respectively. If $1 \leqslant p \leqslant s$ and $s+1 \leqslant q \leqslant t$, then $x, v_{1,1}, y$ is a rainbow $x - y$ path whose edges are colored 3 and 2. Thus $\mathrm{rc}(G) \leqslant 3$. Therefore, as claimed, $\mathrm{rc}(G) \leqslant \min\{\lceil \sqrt[s]{t} \rceil, 3\}$.

Assume, to the contrary, that $\mathrm{rc}(G) < \min\{\lceil \sqrt[s]{t} \rceil, 3\} \leqslant 3$. Since $\mathrm{rc}(G) \geqslant 2$, it follows that $\mathrm{rc}(G) = 2$. Let c' be a rainbow 2-coloring of G. Thus, we can associate a color code $\mathrm{code}(w) = (a_1, a_2, \ldots, a_s)$ to each vertex $w \in W$, where $a_i = c(u_i w) \in \{1, 2\}$ for $1 \leqslant i \leqslant s$. Since $\sqrt[s]{t} > 2$, it follows that $t > 2^s$ and so there exist two distinct vertices w' and w'' of W such that $\mathrm{code}(w') = \mathrm{code}(w'')$. Hence the two edges of each $w' - w''$ path of length 2 are colored the same and so there is no rainbow $w' - w''$ path in $K_{s,t}$, producing a contradiction. Thus, as claimed, $\mathrm{rc}(K_{s,t}) = 3 = \min\{\lceil \sqrt[s]{t} \rceil, 3\}$ in this case. □

3. On rainbow connection numbers with prescribed values

We have seen that $\mathrm{rc}(G) \leqslant \mathrm{src}(G)$ for every nontrivial connected graph G. By Proposition 1.1, it follows that for every positive integer a and for every tree T of size a, $\mathrm{rc}(T) = \mathrm{src}(T) = a$. Furthermore, for $a \in \{1, 2\}$, $\mathrm{rc}(G) = a$ if and only if $\mathrm{src}(G) = a$. If $a = 3$ and $b \geqslant 4$, then by Propositions 2.2 and 2.3, $\mathrm{rc}(W_{3b}) = 3$ and $\mathrm{src}(W_{3b}) = b$. For $a \geqslant 4$, we have the following.

Theorem 3.1. *Let a and b be integers with $a \geqslant 4$ and $b \geqslant (5a-6)/3$. Then there exists a connected graph G such that* $\operatorname{rc}(G) = a$ *and* $\operatorname{src}(G) = b$.

Proof. Let $n = 3b - 3a + 6$ and let W_n be the wheel consisting of an n-cycle $C_n\colon v_1, v_2, \ldots, v_n, v_1$ and another vertex v joined to every vertex of C_n. Let G be the graph constructed from W_n and the path $P_{a-1}\colon u_1, u_2, \ldots, u_{a-1}$ of order $a-1$ by identifying v and u_{a-1}.

First, we show that $\operatorname{rc}(G) = a$. Since $b \geqslant (5a-6)/3$ and $a \geqslant 4$, it follows that $b > a$ and so $n = 3b - 3a + 6 \geqslant 7$. By Proposition 2.2, we then have $\operatorname{rc}(W_n) = 3$. Define a coloring c of the graph G by

$$c(e) = \begin{cases} i & \text{if } e = u_i u_{i+1} \text{ for } 1 \leqslant i \leqslant a-2, \\ a & \text{if } e = v_i v \text{ and } i \text{ is odd}, \\ a-1 & \text{if } e = v_i v \text{ and } i \text{ is even}, \\ 1 & \text{otherwise}. \end{cases}$$

Since c is a rainbow a-coloring of the edges of G, it follows that $\operatorname{rc}(G) \leqslant a$.

It remains to show that $\operatorname{rc}(G) \geqslant a$. Assume, to the contrary, that $\operatorname{rc}(G) \leqslant a-1$. Let c' be a rainbow $(a-1)$-coloring of G. Since the path $u_1, u_2, \ldots, u_{a-1}$ is the only $u_1 - u_{a-1}$ path in G, the edges of this path must be colored differently by c'. We may assume, without loss of generality, that $c'(u_i u_{i+1}) = i$ for $1 \leqslant i \leqslant a-2$. For each j with $1 \leqslant j \leqslant 3b - 3a + 6$, there is a unique $u_1 - v_j$ path of length $a-1$ in G and so $c'(vv_j) = a-1$ for $1 \leqslant j \leqslant 3b - 3a + 6$. Consider the vertices v_1 and v_{a+1}. Since $b \geqslant (5a-6)/3$, any $v_1 - v_{a+1}$ path of length $a-1$ or less must contain v and thus two edges colored $a-1$, contradicting our assumption that c' is a rainbow $(a-1)$-coloring of G. This implies that $\operatorname{rc}(G) \geqslant a$ and so $\operatorname{rc}(G) = a$.

Next, we show that $\operatorname{src}(G) = b$. Since $n = 3b - 3a + 6 = 3(b - a + 2) \geqslant 7$, it follows by Proposition 2.3 that $\operatorname{src}(W_n) = b - a + 2$. Let c_1 be a strong rainbow $(b-a+2)$-coloring of W_n. Define a coloring c of the graph G by

$$c(e) = \begin{cases} c_1(e) & \text{if } e \in E(W_n), \\ b - a + 2 + i & \text{if } e = u_i u_{i+1} \text{ for } 1 \leqslant i \leqslant a-2. \end{cases}$$

Since c is a strong rainbow b-coloring of G, it follows that $\operatorname{src}(G) \leqslant b$.

It remains to show that $\operatorname{src}(G) \geqslant b$. Assume, to the contrary, that $\operatorname{src}(G) \leqslant b-1$. Let c^* be a strong rainbow $(b-1)$-coloring of G. We may assume, without loss of generality, that $c^*(u_i u_{i+1}) = i$ for $1 \leqslant i \leqslant a-2$. For each j with $1 \leqslant j \leqslant 3b - 3a + 6$, there is a unique $u_1 - v_j$ geodesic in G, implying $c^*(vv_j) \in C = \{a-1, a, \ldots, b-1\}$. Let $S = \{vv_j\colon 1 \leqslant j \leqslant 3b - 3a + 6\}$. Then $|S| = 3b - 3a + 6$ and $|C| = b - a + 1$. Since at most three edges in S can be colored the same, the $b - a + 1$ colors in C can color at most $3(b-a+1) = 3b-3a+3$ edges, producing a contradiction. Therefore, $\mathrm{src}(G) \geqslant b$ and so $\mathrm{src}(G) = b$. □

Combining Propositions 1.1, 2.2, 2.3 and Theorem 3.1, we have the following.

Corollary 3.2. *Let a and b be positive integers. If $a = b$ or $3 \leqslant a < b$ and $b \geqslant (5a-6)/3$, then there exists a connected graph G such that $\mathrm{rc}(G) = a$ and $\mathrm{src}(G) = b$.*

We conclude with two conjectures and a result.

Conjecture 3.3. *Let a and b be positive integers. Then there exists a connected graph G such that $\mathrm{rc}(G) = a$ and $\mathrm{src}(G) = b$ if and only if $a = b \in \{1, 2\}$ or $3 \leqslant a \leqslant b$.*

It is easy to see that if H is a connected spanning subgraph of a nontrivial (connected) graph G, then $\mathrm{rc}(G) \leqslant \mathrm{rc}(H)$. We have already noted that if, in addition, $\mathrm{diam}(H) = 2$, then $\mathrm{src}(G) \leqslant \mathrm{src}(H)$. However, the question arises as to whether this is true when $\mathrm{diam}(H) \geqslant 3$.

Conjecture 3.4. *If H is a connected spanning subgraph of a nontrivial (connected) graph G, then $\mathrm{src}(G) \leqslant \mathrm{src}(H)$.*

If Conjecture 3.4 is true, then for every nontrivial connected graph G of order n,

$$\mathrm{diam}(G) \leqslant \mathrm{rc}(G) \leqslant \mathrm{src}(G) \leqslant n-1.$$

The following can be proved immediately.

Proposition 3.5. *For each triple d, k, n of integers with $2 \leqslant d \leqslant k \leqslant n-1$, there exists a connected graph G of order n with $\mathrm{diam}(G) = d$ such that $\mathrm{rc}(G) = \mathrm{src}(G) = k$.*

References

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Authors' addresses: *Kathy McKeon*, Connecticut College, Department of Mathematics, 270 Mohegan Avenue, Box 5561, New London, CT 06320, USA; *Garry L. Johns*, Saginaw Valley State University, University Center, Department of Mathematical Sciences, MI 48710-0001, USA; *Ping Zhang*, *Gary Chartrand*, Western Michigan University, Department of Mathematics, Kalamazoo, MI 49008, USA, e-mail: `ping.zhang@wmich.edu`.