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AN EXISTENCE AND MULTIPLICITY RESULT FOR A PERIODIC BOUNDARY VALUE PROBLEM

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Abstract. A periodic boundary value problem for nonlinear differential equation of the second order is studied. Nagumo condition is not assumed on a part of nonlinearity. Existence and multiplicity results are proved using the method of lower and upper solutions. Results are applied to the generalized Liénard oscillator.

Keywords: periodic boundary value problem, multiplicity result, method of lower and upper solutions, Liénard oscillator

MSC 2000: 34C25

The paper deals with the method of lower and upper solutions to prove the existence of at least one solution of the periodic boundary value problem

$$
\begin{gathered}
x'' + kx' = F(t, x, x'), \\
x(a) = x(b), \quad x'(a) = x'(b),
\end{gathered}
$$

where the nonlinearity $F(t,x,y) = f(t,x,y) + g(t,x,y)$ is divided into two parts, $f, g\colon I = [a,b] \times \mathbb{R}^2 \to \mathbb{R}$, which are continuous functions.

The method of lower and upper solutions as a method of proving the existence of a solution of various types of boundary value problems has a relatively long history. Reader can find such methods in papers by Scorza and Dragoni [18], Gaines and Mawhin [3], Rachůnková [13], [14], Rudolf [17], Šeda [19], Thompson [20] and others.

The idea of the existence of a solution in the case when a lower solution is greater then an upper one can be find in papers of Gossez and Omari [4], Habets and Omari [5], Omari [11], Nieto [10], Rachůnková [14], Rudolf and Kubáček [15].

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When the nonlinearity has a suitable form there exist more lower and upper solutions and this fact leads to typical multiplicity results. See Fabry, Mawhin and Nkashama [2], Mawhin and Willem [9], Rachůnková [12], Gaines and Mawhin [3], Rudolf [16], Šeda [19].

Lower and upper solutions give natural bounds of the set on which the theory of the topological degree [3], [7], [8] is used. To this end, an a priori bound of the norm of a possible solution is needed. Therefore a certain form of Nagumo, or Nagumo-Bernstein condition is assumed which controls the growth of the nonlinearity in the y variable.

In the paper of Korman [6], a method how to obtain an existence result for a Dirichlet boundary value problem is given provided that a Nagumo type condition fails on part g of the nonlinearity, instead of it a certain kind of the sign property is assumed.

The goal of this paper is to adapt the idea of Korman to suit periodic boundary value problems and to prove existence and multiplicity results for such problems and apply them to a generalized Liénard oscillator with a damping term with possibly rapid growth.

Preliminaries

We deal with the periodic boundary value problem

$$x'' + kx' = f(t, x, x') + g(t, x, x'), \tag{1}$$

$$x(a) = x(b), \quad x'(a) = x'(b), \tag{2}$$

where $f, g\colon I = [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, f satisfies a Nagumo-Bernstein condition while g in general doesn't.

A solution of (1), (2) is a classical solution $x(t) \in C^2(I)$.

We define lower and upper solutions for the boundary value problem (1), (2) in a standard manner.

Definition. A function $\alpha \in C^2(I)$ is called a lower solution of the problem (1), (2) if

$$\begin{gathered} \alpha''(t) + k\alpha'(t) \geqslant f(t, \alpha(t), \alpha'(t)) + g(t, \alpha(t), \alpha'(t)), \\ \alpha(a) = \alpha(b), \quad \alpha'(a) = \alpha'(b). \end{gathered} \tag{3}$$

Similarly, a function $\beta \in C^2(I)$ is called an upper solution of the problem (1), (2) if

$$\begin{gathered} \beta''(t) + k\beta'(t) \leqslant f(t, \beta(t), \beta'(t)) + g(t, \beta(t), \beta'(t)), \\ \beta(a) = \beta(b), \quad \beta'(a) = \beta'(b). \end{gathered} \tag{4}$$

If strict inequalities hold then α, β are called strict lower and upper solutions.

The main idea of the method of lower and upper solutions in connection with existence and multiplicity proofs based on the topological degree is to give bounds for an open bounded set Ω such that both the original and a suitably perturbed boundary value problem have no solution on the boundary of Ω.

Natural bounds for a solution are given by strict lower and upper solutions. The following lemma means there is no touch between a strict lower or upper solution and a solution of (1), (2).

Lemma 1 [17]. *Let α, β be strict lower and upper solutions and let $x \in C^2(I)$ be a solution of the problem* (1), (2).

Then $\alpha \leqslant x$ implies $\alpha < x$ and $x \leqslant \beta$ implies $x < \beta$.

Proof. Let $x \leqslant \beta$ and $0 = x(t_0) - \beta(t_0)$ at $t_0 \in (a,b)$. Then

$$\begin{aligned}0 \geqslant x''(t_0) - \beta''(t_0) &= f(t_0, x(t_0), x'(t_0)) - \beta''(t_0)\\ &= f(t_0, \beta(t_0), \beta'(t_0)) - \beta''(t_0) > 0,\end{aligned}$$

a contradiction.

The cases $t_0 = a$, $t_0 = b$ lead to the same contradiction due to periodic boundary conditions. $\square$

We use the following operator formulation of the problem (1), (2).

Let $X = C^1(I)$, $\operatorname{dom} L = \{x(t) \in C^2(I),\ x \text{ satisfies } (2)\}$, $Y = C(I)$. We denote

$$\begin{aligned}&L_k\colon \operatorname{dom} L \subset X \to Y, \quad L_k x = x'' + kx',\\ &N\colon X \to Y, \quad Nx(t) = f(t, x(t), x'(t)) + g(t, x(t), x'(t)).\end{aligned}$$

Then the problem (1), (2) is equivalent to the operator equation

$$L_k x = Nx,$$

where the operator N is L_k-compact [7], [8].

Existence results

Our first existence result concerns the case when the lower and upper solutions are constant functions.

Let $r > 0$, $\varrho > 0$ be constants. We set

$$\Omega_{r,\varrho} = \{x(t) \in C^1(I);\ \|x\| < r,\ \|x'\| < \varrho\}.$$

Lemma 2. *Let the following conditions be satisfied:*

(i) *there is a constant $r > 0$ such that $f(t, r, 0) > 0$ and $f(t, -r, 0) < 0$,*

(ii) *there is a constant $c_r > 0$ such that $|f(t, x, y)| \leqslant c_r(1 + y^2)$ for each $t \in I$, $x \in [-r, r]$, $y \in \mathbb{R}$,*

(iii) *$xg(t, x, y) \geqslant 0$ for each $t \in I$, $x \in [-r, r]$, $y \in \mathbb{R}$.*

Then there is a constant $\varrho_0 > 0$ such that the topological degree

$$D(L_k, N, \Omega_{r,\varrho}) = 1 \pmod 2$$

for each $\varrho > \varrho_0$, i.e. there is a solution $x(t)$ of (1), (2) such that $|x(t)| < r$, $|x'(t)| < \varrho$.

P r o o f. We consider a homotopy

$$L_k x = \widetilde{N}(x, \lambda), \qquad \lambda \in [0, 1]$$

defined by a parametric system of equations

$$x'' + kx' = \lambda f(t, x, x') + \lambda g(t, x, x') + (1 - \lambda)x, \tag{5}$$

with boundary conditions (2). Now $-r$, r are respectively a strict lower and a strict upper solution of the problem (5), (2).

Obviously there is a positive constant c_r^* such that $|\lambda f(t, x, y) + (1 - \lambda)x - ky| \leqslant \lambda c_r(1 + y^2) + (1 - \lambda)r + |k||y| \leqslant c_r^*(1 + y^2)$.

Let $x(t)$ be a solution of (5), (2) such that $|x(t)| \leqslant r$. We derive an a priori estimate of $|x'(t)|$ for such a solution.

Let $[t_0, t_1] \subset I$ be an interval such that $x(t) > 0$ for $t \in (t_0, t_1)$, $\max\limits_{t \in [t_0, t_1]} x(t) = x(t_0) = x_0$, $x'(t_0) = 0$ and $x'(t) < 0$ for $t \in (t_0, t_1)$. Then $x(t)$ is invertible on (t_0, t_1). Let $t(x)$ be the inverse function. We denote by $p(x)$ the function $p(x) = x'(t) = x'(t(x))$. Then $x''(t) = p'(x)p(x)$ for each (t, x), $t \in (t_0, t_1)$, $x = x(t)$.

We rewrite the equation (5) to

$$x'' = f^*(t, x, x') + g^*(t, x, x'),$$

where $f^* = \lambda f + (1 - \lambda)x - kx'$, $g^* = \lambda g$.

Now

$$p'(x)p(x) = f^*(t, x, p(x)) + g^*(t, x, p(x)) \geqslant -c_r^*(1 + p^2(x)).$$

We substitute $q = p^2$:

$$\frac{1}{2}q'(x) \geqslant -c_r^*(1 + q(x)).$$

By changing the variable $v = x_0 - x$ and setting $\tilde{q}(v) = q(x_0 - v)$ we obtain

$$\frac{1}{2}\tilde{q}'(v) \leqslant c_r^*(1 + \tilde{q}(v)), \qquad \tilde{q}(0) = 0.$$

From the last inequality we obtain the estimate

$$\tilde{q}(v) \leqslant \mathrm{e}^{2c_r^* v} \leqslant \mathrm{e}^{2c_r^* r} = C(r).$$

The above estimate of $\tilde{q}$ and continuity of x' imply an estimate of $|p|$ and $|x'(t)|$ on $[t_0, t_1]$. Similarly we derive an estimate of $|x'(t)|$ for $x'(t) > 0$ on $[t_2, t_0]$ and also for the case $x(t) < 0$.

On the interval $[a, t_1]$ or $[t_2, b]$ we use periodic continuation of the solution $x(t)$ to estimate $|x'(t)|$.

Dividing the interval I into intervals of monotonicity with constant sign of $x(t)$ and repeating the above estimation of $|x'(t)|$ on each of them we estimate $|x'(t)| < \varrho_0$ on I.

This a priori estimate and Lemma 1 imply that for each $\varrho > \varrho_0$ there is no solution of (5), (2) on $\partial\Omega_{r,\varrho}$.

By the generalized Borsuk theorem [8] we have

$$D(L_k, \widetilde{N}(\cdot, 1), \Omega_{r,\varrho}) = D(L_k, \widetilde{N}(\cdot, 0), \Omega_{r,\varrho}) = 1 \pmod 2.$$

□

Instead of the assumption (iii) it is possible to assume that the function g satisfies a sign property with respect to the variable y, which allows the application to oscillators with a nonlinear damping term with higher than quadratic growth. For such type of assumption the linear damping term kx' is considered as part of the nonlinearity g and we deal with the problem

$$x'' = f(t, x, x') + g(t, x, x'), \tag{6}$$

$$x(a) = x(b), \quad x'(a) = x'(b). \tag{2}$$

Lemma 3. *Let the following conditions be satisfied:*

(i) *there is a constant $r > 0$ such that $f(t, r, 0) > 0$ and $f(t, -r, 0) < 0$,*

(ii) *there is a constant $c_r > 0$ such that $|f(t, x, y)| \leqslant c_r(1 + y^2)$ for each $t \in I$, $x \in [-r, r]$, $y \in \mathbb{R}$,*

(iii) *$yg(t, x, y) \leqslant 0$ for each $t \in I$, $x \in [-r, r]$, $y \in \mathbb{R}$.*

Then there is a constant $\varrho_0 > 0$ such that the topological degree

$$D(L_0, N, \Omega_{r,\varrho}) = 1 \pmod 2$$

for each $\varrho > \varrho_0$, i.e. there is a solution $x(t)$ of (6), (2) *such that $|x(t)| < r$, $|x'(t)| < \varrho$.*

Proof. The proof is similar to the one of Lemma 2. The only difference is that we don't distinguish the cases $x(t) > 0$ and $x(t) < 0$; instead we derive the inequality

$$\frac{1}{2} q'(x) \geqslant -c_r^*(1 + q(x))$$

for the case $p(x) < 0$, and the inequality

$$\frac{1}{2} q'(x) \leqslant c_r^*(1 + q(x))$$

with the change of variable $v = x + x_1$, $x_1 = x(t_1)$ being the minimum of $x(t)$ on the interval $[t_1, t_2]$ of increase of $x(t)$, i.e. for the case $p(x) > 0$. □

Example 1. We consider a periodic boundary value problem for the generalized Liénard oscillator with nonlinear damping

$$\begin{gathered} x'' + k(t, x) \operatorname{sgn}(x')|x'|^m = f(t, x, x'), \\ x(a) = x(b), \quad x'(a) = x'(b), \end{gathered}$$

where $k(t, x) \geqslant 0$ is a continuous function, $m > 0$. Suppose that assumptions (i), (ii) of Lemma 3 on the function f are satisfied. Then Lemma 3 implies that there is a periodic solution of the given problem. This result generalizes the results of [16] where $m \leqslant 2$ is assumed and [2] where $m = 1$.

Remark. The result similar to Lemma 3 can be derived assuming negative damping and

(iii) $yg(t, x, y) \geqslant 0$ for each $t \in I$, $x \in [-r, r]$, $y \in \mathbb{R}$.

Results for constant lower and upper solutions are generalized in a standard manner [3], [14], [17] to the case of nonconstant ones.

For given functions $\alpha, \beta \colon I \to \mathbb{R}$, $\alpha < \beta$ and constant ϱ we set

$$\Omega = \{x \in C^1(I);\ \alpha < x < \beta,\ \|x'\| < \varrho\}.$$

Theorem 1. *Let*

(i) $\alpha < \beta$ *be strict lower and upper solutions of the problem* (1), (2),

(ii) *there exist a positive function* $c(r)\colon \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $|f(t,x,y)| \leqslant c(r)(1+y^2)$ *for each* $t \in I$, $-r < x < r$, $y \in \mathbb{R}$,

(iii) $xg(t,x,y) \geqslant 0$ *for each* $t \in I$, x, $y \in \mathbb{R}$.

Then there is a constant ϱ_0 *such that for each* $\varrho > \varrho_0$ *we have*

$$D(L_k, N, \Omega) = 1 \pmod 2,$$

i.e. there is a solution $x \in \Omega$ *of* (1), (2).

Proof. Set $r = \max\{\|\alpha\|, \|\beta\|\} + 1$, $M > \max|f(t,x,0)|$ for $t \in I$, $|x| \leqslant r$.

We define a perturbation

$$f^*(t,x,y) = \begin{cases} f(t,\beta(t),y) + M(r-\beta(t)), & x > r,\\ f(t,\beta(t),y) + M(x-\beta(t)), & \beta(t) < x \leqslant r,\\ f(t,x,y), & \alpha(t) \leqslant x \leqslant \beta(t),\\ f(t,\alpha(t),y) - M(\alpha(t)-x), & -r \leqslant x < \alpha(t),\\ f(t,\alpha(t),y) - M(\alpha(t)+r), & x < -r \end{cases}$$

and

$$g^*(t,x,y) = \begin{cases} g(t,\beta(t),y), & 0 \leqslant \beta(t) < x,\\ g(t,\beta(t),y)x/\beta(t), & \beta(t) < x < 0,\\ 0, & \beta(t) < 0 \leqslant x,\\ g(t,x,y) & \alpha(t) \leqslant x \leqslant \beta(t),\\ 0, & x \leqslant 0 < \alpha(t),\\ g(t,\alpha(t),y)x/\alpha(t), & 0 < x < \alpha(t),\\ g(t,\alpha(t),y), & x < \alpha(t) \leqslant 0. \end{cases}$$

Now f^*, g^* satisfy conditions (i), (ii), (iii) of Lemma 2. That means there is ϱ_0 such that $D(L_k, N^*, \Omega_{r,\varrho}) = 1$ for each $\varrho > \varrho_0$, where $N^* = f^* + g^*$.

Let x^* be a solution of

$$x'' + kx' = f^*(t,x,x') + g^*(t,x,x'), \tag{7}$$

$$x(a) = x(b), \quad x'(a) = x'(b) \tag{2}$$

and suppose $x^*(t_1) \geqslant \beta(t_1)$. Then $v(t) = x^*(t) - \beta(t)$ attains its nonnegative maximum v_M and $\beta + v_M \geqslant x^*$. It is easy to prove that $\beta(t) + v_M$ is a strict upper solution of (7), (2), but this contradicts Lemma 1.

That means $x^* < \beta$. Simultaneously, $x^* > \alpha$.

Then

$$D(L_k, N^*, \Omega_{r,\varrho}) = D(L_k, N^*, \Omega) = D(L_k, N, \Omega) = 1 \pmod 2$$

and x^* is a solution of (1), (2). $\square$

Again it is possible to replace the sign condition (iii) by a sign condition with respect to variable the y.

Theorem 2. *Let*

(i) $\alpha < \beta$ *be strict lower and upper solutions of the problem* (1), (2),

(ii) *there exist a positive function* $c(r)\colon \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $|f(t,x,y)| \leqslant c(r)(1+y^2)$ *for each* $t \in I$, $-r < x < r$, $y \in \mathbb{R}$,

(iii) $yg(t,x,y) \leqslant 0$ *for each* $t \in I$, $x, y \in \mathbb{R}$.

Then there is a constant ϱ_0 *such that for each* $\varrho > \varrho_0$ *we have*

$$D(L_0, N, \Omega) = 1 \pmod 2$$

i.e., there is a solution $x \in \Omega$ *of* (6), (2).

The proof is similar to the one of Theorem 1, the only difference is that Lemma 3 is used instead of Lemma 2 to prove $D(L_0, N^*, \Omega_{r,\varrho}) = 1$.

R e m a r k. The assumption (iii) of Theorem 2 can be replaced by (iii) $yg(t,x,y) \geqslant 0$ for each $t \in I$, x, $y \in \mathbb{R}$.

The idea of a proof of existence of a solution in the case when a lower solution is greater than an upper one is due to Omari [11], further results have been given by Nieto [10], Rachůnková [14] and others.

In our next theorem we consider a little more general assumption, namely that a strict lower solution isn't less than an upper one.

We remark that the assumption $\alpha \leqslant \beta$, $\alpha(t_0) = \beta(t_0)$, α, β are strict lower and upper solutions, leads to a contradiction.

For given functions $\alpha, \beta\colon I \to \mathbb{R}$ and positive constants r, ϱ we set

$$\widetilde{\Omega} = \{x(t) \in \Omega_{r,\varrho};\ \exists t_a \in I,\ \beta(t_a) < x(t_a),\ \exists t_b \in I,\ x(t_b) < \alpha(t_b)\}.$$

Theorem 3. *Let*

(i) $\alpha \not\leqslant \beta$ *be strict lower and upper solutions of the problem* (1), (2),

(ii) *there exist constants* $A, B \in (0,1)$, $c > 0$ *such that* $|f(t,x,y)| \leqslant c(1 + |x|^A + |y|^B)$ *for each* $t \in I$, $x, y \in \mathbb{R}$,

(iii) $xg(t,x,y) \geqslant 0$ *for each* $t \in I$, x, $y \in \mathbb{R}$.

Then there are constants $r, \varrho_0 > 0$ *such that for each* $\varrho > \varrho_0$ *we have*

$$D(L_k, N, \widetilde{\Omega}) = 1 \pmod 2,$$

i.e., there is a solution $x(t) \in \widetilde{\Omega}$ *of the problem* (1), (2).

P r o o f. We set $r_0 = \max(\|\alpha\|, \|\beta\|)$, $r > r_0$, to be specified later, $M_0 = c(1+r^A)$ and define a perturbation f^* by

$$f^*(t,x,y) = \begin{cases} f(t,r,y) + M_0, & x > r+1, \\ f(t,r,y) + M_0(x-r), & r < x \leqslant r+1, \\ f(t,x,y), & -r \leqslant x \leqslant r, \\ f(t,r,y) + M_0(x+r), & -r-1 \leqslant x < -r, \\ f(t,r,y) - M_0, & x < -r-1. \end{cases}$$

As $f^*(t, r+1, 0) = f(t,r,0) + M_0 > -c(1+r^A) + M_0 = 0$ hence $r+1$ is a strict upper solution of the equation

$$x'' + kx' = f^*(t,x,x') + g(t,x,x') \tag{8}$$

with boundary conditions (2). Similarly $-r-1$ is a strict lower solution of (8), (2).

Moreover, $|f^*| < c^*(r)(1+y^2)$ for each $t \in [a,b]$, $x \in [(-r-1), r+1]$, $y \in \mathbb{R}$. Setting $N^* = f^* + g$ Lemma 2 implies that there is a constant ϱ_0 such that for $\varrho > \varrho_0$ we have

$$D(L_k, N^*, \Omega_{r+1,\varrho}) = 1 \pmod 2.$$

Let now

$$\Omega_l = \{x(t) \in \Omega_{r+1,\varrho};\ x < \beta\},$$
$$\Omega_u = \{x(t) \in \Omega_{r+1,\varrho};\ \alpha < x\}.$$

Then Theorem 1 implies

$$D(L_k, N^*, \Omega_l) = D(L_k, N^*, \Omega_u) = 1 \pmod 2.$$

Set $\Omega_m = \Omega_{r+1,\varrho} \setminus \left(\overline{\Omega_l \cup \Omega_u}\right)$.

As $-r-1$, α are strict lower and $r+1$, β are strict upper solutions, Lemma 2 implies there is no solution $x \in \partial\Omega_m$.

The additivity of the degree yields

$$D(L_k, N^*, \Omega_m) = 1 \pmod 2.$$

Let $x(t) \in \Omega_m$ be a solution of (8), (2). We will prove that $|x(t)| < r$ using the idea of [6].

First, suppose that $k > 0$ and $\max|x(t)| = x(t_0) > r_0$. Then there is an interval $[a_1, b_1]$ such that $x(a_1) = x(b_1) = r_0$, $x(t) > r_0$ on (a_1, b_1), $\max\limits_{t\in[a_1,b_1]} x(t) - r_0 = x(t_0) - r_0 = M$.

We use the estimate

$$|f^*| < c(1 + |x|^A + |y|^B) + c(1 + r^A) \leqslant c(2 + 2r^A + |y|^B),$$

denote $p(x) = x'(t)$ on the interval $[t_0, t_1]$ of monotonicity of $x(t)$ and arguing as in the proof of Lemma 2 we obtain

$$p'(x)p(x) + kp(x) = f^*(t, x, p) + g(t, x, p).$$

Then using (iii), $kp \leqslant 0$ and setting $q = p^2$ we arrive at

$$\frac{1}{2}q'(x) \geqslant -c(2 + 2r^A + |q|^{B/2}).$$

The substitution $v = M + r_0 - x$ leads to the inequality

$$\tilde{q}'(v) \leqslant 2c(2 + 2r^A + |\tilde{q}|^{B/2}), \qquad \tilde{q}(0) = 0,$$

where $\tilde{q}(v) = q(M + r_0 - v)$. Then

$$\tilde{q}(v) \leqslant 4cM(1 + r^A) + 2c\int_0^v |\tilde{q}|^{B/2}\,\mathrm{d}v.$$

The Bihari lemma [1] implies

$$\tilde{q}(v) \leqslant \left[(4cM(1 + r^A))^{1-B/2} + \left(1 - \frac{B}{2}\right)\int_0^v 2c\,\mathrm{d}v\right]^{1/(1-B/2)}.$$

Further

$$p^2(v) \leqslant \left[(4cM(1 + r^A))^{1-B/2} + (2 - B)cM\right]^{1/(1-B/2)}$$

and

$$p(v) \leqslant 2d\left(cM(1+r^A)\right)^{1/2} + d\left((2-B)cM\right)^{1/(2-B)}, \qquad d = 2^{B/(2-B)}.$$

As $M \leqslant (b-a) \max\limits_{v\in[0,M]} |p(v)|$, we have

$$M \leqslant 2(b-a)d\left(cM(1+r^A)\right)^{1/2} + (b-a)d\left((2-B)cM\right)^{1/(2-B)}.$$

Then

$$M^{1/2} \leqslant c_1 r^{A/2} + c_2 M^{\gamma/2}$$

where c_i are positive constants independent of r and $\gamma = B/(2-B) < 1$.

Suppose now that

$$r_0 + M > r.$$

Then

$$M^{1/2} \leqslant c_1(r_0+M)^{A/2} + c_2 M^{\gamma/2}.$$

The last inequality implies that

$$M < M_1$$

where M_1 is independent of r.

We choose $r > M_1 + r_0$ to obtain an a priori estimate of M:

$$M < \max(M_1, r - r_0).$$

We proceed similarly also in the case $\max|x(t)| = -x(t_0) > r_0$ and in the case $k < 0$.

That means $|x(t)| < r$ and

$$D(L_k, N^*, \Omega_m) = D(L_k, N, \widetilde{\Omega}) = 1(\bmod 2).$$

$\square$

A similar existence result holds for the problem (6) (2) if (iii) is replaced by a sign condition on the function g with respect to the variable y.

Theorem 4. *Let*

(i) $\alpha \nleqslant \beta$ *be strict lower and upper solutions of the problem* (6), (2),

(ii) *there exist constants* $A, B \in (0,1)$ *such that* $|f(t,x,y)| \leqslant c(1+|x|^A+|y|^B)$ *for each* $t \in I$, $x, y \in \mathbb{R}$,

(iii) $yg(t,x,y) \geqslant 0$ *for each* $t \in I$, $x, y \in \mathbb{R}$,

Then there are constants $r, \varrho_0 > 0$ *such that for each* $\varrho > \varrho_0$ *we have*

$$D(L_0, N, \widetilde{\Omega}) = 1 \pmod 2,$$

i.e. there is a solution $x(t) \in \widetilde{\Omega}$ *of the problem* (6), (2).

Proof. We follow the proof of Theorem 3 using the same perturbation f^*. The existence of a solution of the perturbed problem (8), (2) follows from Lemma 3. The estimate of the norm of the solution $x(t) \in \Omega_m$ of the perturbed problem is derived similarly to the case of Theorem 3. $\square$

Multiplicity results

First we present a simple perturbation lemma based on Lemma 1. We denote $F(t,x,y) = f(t,x,y) + g(t,x,y)$ and consider the boundary value problem

$$x'' + kx' = F(t,x,x'), \tag{9}$$

$$x(a) = x(b), \quad x'(a) = x'(b). \tag{2}$$

Lemma 4. *Let* α *be a strict lower solution of the problem* (9), (2).
Set

$$\overline{F}(t,x,y) = \begin{cases} F(t,x,y), & x(t) > \alpha(t), \\ F(t,\alpha(t),y), & x(t) \leqslant \alpha(t). \end{cases}$$

Then each solution $x(t)$ *of*

$$x'' + kx' = \overline{F}(t,x,x'), \tag{10}$$

$$x(a) = x(b), \quad x'(a) = x'(b), \tag{2}$$

is a solution of (9), (2).

Proof. Let $x(t)$ be a solution of (10), (2). Suppose $m = \max(\alpha(t) - x(t)) \geqslant 0$. Then $\alpha(t) - m \leqslant x(t)$ and there is t_0 such that $\alpha(t_0) - m = x(t_0)$. As $\alpha(t) - m$ is a strict lower solution of (9), (2), we obtain a contradiction with Lemma 1. $\square$

We prove that a periodic boundary value problem (6), (2) admits at least two different solutions provided there is a lower and an upper solution either in reversed order or possibly crossing each other and both are greater then another lower solution.

Set

$$\begin{aligned}\Omega_{\alpha,r,\varrho} &= \{x(t) \in C^1(I);\ \alpha < x < r, \|x'\| < \varrho\},\\ \Omega_1 &= \{x(t) \in \Omega_{\alpha,r,\varrho};\ x < \beta\},\\ \Omega_2 &= \{x(t) \in \Omega_{\alpha,r,\varrho};\ \exists t_a \in I, \beta(t_a) < x(t_a), \exists t_b \in I, x(t_b) < \alpha_1(t_b)\}.\end{aligned}$$

Theorem 5. *Let*

(i) $\alpha < \beta$, $\alpha < \alpha_1$, $\alpha_1 \nleqslant \beta$, *where* α, α_1 *are strict lower solutions and* β *is a strict upper solution of the problem* (6), (2),

(ii) *there exist a positive function* $c(\tilde r)\colon \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $|f(t,x,y)| \leqslant c(\tilde r)(1 + |y|^2)$ *for each* $t \in I$, $|x| < \tilde r$, $y \in \mathbb{R}$,

(iii) *there exist constants* $A, B \in (0,1)$, $c_1 > 0$ *such that* $f(t,x,y) > -c_1(1 + |x|^A + |y|^B)$ *for each* $t \in I$, $x > \alpha(t)$, $y \in \mathbb{R}$,

(iv) $yg(t,x,y) \leqslant 0$ *for each* $t \in I$, $x, y \in \mathbb{R}$.

Then there are constants $r > \|\alpha\|$, $\varrho_0 > 0$ *such that for each* $\varrho > \varrho_0$ *we have*

$$\begin{aligned}D(L_0, N, \Omega_1) &= 1 \pmod 2,\\ D(L_0, N, \Omega_2) &= 1 \pmod 2,\end{aligned}$$

i.e., there are at least two solutions $x_1(t) \in \Omega_1$, $x_2(t) \in \Omega_2$ *of the problem* (6), (2).

P r o o f. The existence of a solution $x_1 \in \Omega_1$ follows from Theorem 2.

To prove the existence of another solution x_2 we consider the perturbed problem

$$x'' = \overline{f}(t,x,x') + \bar g(t,x,x'), \tag{11}$$

$$x(a) = x(b), \quad x'(a) = x'(b), \tag{2}$$

where

$$\overline{f}(t,x,y) = \begin{cases} f(t,x,y), & x(t) > \alpha(t),\\ f(t,\alpha(t),y), & x(t) \leqslant \alpha(t)\end{cases}$$

and

$$\bar g(t,x,y) = \begin{cases} g(t,x,y), & x(t) > \alpha(t),\\ g(t,\alpha(t),y), & x(t) \leqslant \alpha(t).\end{cases}$$

Now we modify the problem (11), (2) similarly to the case of Theorem 3 setting $r_0 = \max(\|\alpha\|, \|\beta\|)$, $r > r_0$, $M_0 = c_1(1 + r^A)$ and defining a perturbation f^* by

$$f^*(t,x,y) = \begin{cases} \overline{f}(t,r,y) + M_0, & x > r+1,\\ \overline{f}(t,r,y) + M_0(x - r), & r < x \leqslant r+1,\\ \overline{f}(t,x,y), & x \leqslant r.\end{cases}$$

Clearly $r+1$, β are strict upper and α, α_1 are strict lower solutions of

$$x'' = f^*(t,x,x') + \bar{g}(t,x,x'), \tag{12}$$

$$x(a) = x(b), \quad x'(a) = x'(b). \tag{2}$$

Similarly to the proof of Theorem 3 we prove that

$$D(L_0, N^*, \Omega_m) = 1 \pmod 2$$

where $N^* = f^* + \bar{g}$ and $\Omega_m = \{x(t) \in \Omega_{\alpha,r+1,\varrho};\ \exists t_a \in I,\ \beta(t_a) < x(t_a),\ \exists t_b \in I,\ x(t_b) < \alpha_1(t_b)\}$.

For a solution $x(t) \in \Omega_m$ of (12), (2) we prove that $x(t) < r$ proceeding as in the proof of Theorem 3 and using only the one sided estimate from below of the function f from assumption (iii).

Then $x(t) \in \Omega_2$,

$$D(L_0, N^*, \Omega_m) = D(L_0, N^*, \Omega_2) = D(L_0, \overline{N}, \Omega_2) = 1 \pmod 2$$

where $\overline{N} = \bar{f} + \bar{g}$. That means $x(t)$ is a solution of (11), (2). Lemma 4 and the excision property of the degree imply that

$$D(L_0, \overline{N}, \Omega_2) = D(L_0, N, \Omega_2) = 1 \pmod 2$$

and that $x(t)$ is a solution of (6), (2). $\square$

By interchanging the role of lower and upper solutions we obtain the following results.

Lemma 5. *Let β be a strict upper solution of the problem* (9), (2).
Set

$$\overline{F}(t,x,y) = \begin{cases} F(t,x,y), & x(t) < \beta(t) \\ F(t,\beta(t),y), & x(t) \geqslant \beta(t). \end{cases}$$

Then each solution $x(t)$ of

$$x'' + kx' = \overline{F}(t,x,x'),$$
$$x(a) = x(b), \quad x'(a) = x'(b),$$

is a solution of (9), (2).

Set

$$\Omega_{-r,\beta,\varrho} = \{x(t) \in C^1(I);\ -r < x < \beta, \|x'\| < \varrho\},$$
$$\Omega_3 = \{x(t) \in \Omega_{-r,\beta,\varrho};\ \alpha < x\},$$
$$\Omega_4 = \{x(t) \in \Omega_{-r,\beta,\varrho};\ \exists t_a \in I, \beta_1(t_a) < x(t_a), \exists t_b \in I, x(t_b) < \alpha(t_b)\}.$$

Theorem 6. *Let*

(i) $\alpha < \beta$, $\beta_1 < \beta$, $\alpha \nleqslant \beta_1$, *where* α *is a strict lower solution and* β, β_1 *are strict upper solutions of the problem* (6), (2),

(ii) *there exist a positive function* $c(\tilde{r})\colon \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $|f(t,x,y)| \leqslant c(\tilde{r})(1 + |y|^2)$ *for each* $t \in I$, $|x| < \tilde{r}$, $y \in \mathbb{R}$,

(iii) *there exist constants* $A, B \in (0,1)$, $c_1 > 0$ *such that* $f(t,x,y) < c_1(1+|x|^A+|y|^B)$ *for each* $t \in I$, $x < \beta(t)$, $y \in \mathbb{R}$,

(iv) $yg(t,x,y) \leqslant 0$ *for each* $t \in I$, $x, y \in \mathbb{R}$.

Then there are constants $r > \|\beta\|$, $\varrho_0 > 0$ *such that for each* $\varrho > \varrho_0$ *we have*

$$D(L_0, N, \Omega_3) = 1 \pmod 2,$$
$$D(L_0, N, \Omega_4) = 1 \pmod 2,$$

i.e., there are at least two solutions $x_1(t) \in \Omega_3$, $x_2(t) \in \Omega_4$ *of the problem* (6), (2).

We apply the previous theorems to the generalized Liénard oscillator with a nonlinear damping term with possibly rapid growth and a cubic nonlinearity in the variable x.

Example 2. We consider a periodic boundary value problem

$$x'' + k(t,x)\operatorname{sgn}(x')|x'|^m + \varphi(t,x) = h(t), \tag{13}$$
$$x(a) = x(b), \quad x'(a) = x'(b), \tag{2}$$

with a nonlinearity $\varphi(t,x)$ of cubic type.

We assume that $\varphi(t,x)$ is a continuous function such that

$$\lim_{x\to-\infty} \varphi(t,x) = \infty,$$
$$\lim_{x\to\infty} \varphi(t,x) = -\infty$$

uniformly for $t \in I$, and there are constants x_1, x_2, $x_1 < x_2$, such that $\varphi(t,x_1) < \varphi(t,x_2)$ for each $t \in I$.

Then for each $h(t)$ there is $r > \max\{|x_1|, |x_2|\}$ sufficiently large and such that $-r$ is a strict lower and r a strict upper solution of (13), (2). Moreover, for each $h(t)$ such that $\varphi(t,x_1) < h(t) < \varphi(t,x_2)$, x_1 is a strict upper and x_2 a strict lower solution of (13), (2).

Then Theorems 5 and 6 imply that for each $h(t)$, $\varphi(t,x_1) < h(t) < \varphi(t,x_2)$, there are at least three periodic solutions of the problem (13), (2).

For each $h(t)$, $\varphi(t,x_1) \leqslant h(t) \leqslant \varphi(t,x_2)$, there is a sequence $h_n(t)$ such that $\varphi(t,x_1) < h_n(t) < \varphi(t,x_2)$ and $h_n \to h$ in $C(I)$. There are at least two different limits of subsequences of solutions of periodic problems with righthand sides $h_n(t)$ (see e.g. [2], [16]), so there are at least two periodic solutions of the problem (13), (2).

Finally Lemma 3 implies that there is a periodic solution of (13), (2) for each $h(t)$.

It is not possible to replace the nonlinear term $\varphi(t,x)$ in Example 3 by $-\varphi(t,x)$ and use Theorems 5 and 6 because in such a case assumption (iii) is not satisfied. To prove a multiplicity result also in this case we need multiplicity theorems with reverse inequalities in assumption (iii).

Theorem 7. *Let*

(i) $\alpha < \beta$, $\alpha < \alpha_1$, $\alpha_1 \nleqslant \beta$, *where* α, α_1 *are strict lower solutions and* β *is a strict upper solution of the problem* (6), (2),

(ii) *there exist a positive function* $c(r)\colon \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $|f(t,x,y)| \leqslant c(r)(1+|y|^2)$ *for each* $t \in I$, $|x| < r$, $y \in \mathbb{R}$,

(iii) *there exist constants* $A, B \in (0,1)$, $c_1 > 0$ *such that* $f(t,x,y) \leqslant c_1(1+|x|^A+|y|^B)$ *for each* $t \in I$, $x > \alpha(t)$, $y \in \mathbb{R}$,

(iv) $yg(t,x,y) \leqslant 0$ *for each* $t \in I$, x, $y \in \mathbb{R}$.

Then there are constants $r > \|\alpha\|$, $\varrho_0 > 0$ *such that for each* $\varrho > \varrho_0$ *we have*

$$D(L_0, N, \Omega_1) = 1 \pmod 2,$$
$$D(L_0, N, \Omega_2) = 1 \pmod 2,$$

i.e., there are at least two solutions $x_1(t) \in \Omega_1$, $x_2(t) \in \Omega_2$ *of the problem* (6), (2).

P r o o f. The proof is similar to the one of Theorem 5. We use perturbed problems (11), (2) and (12), (2) and prove that

$$D(L_0, N^*, \Omega_m) = 1 \pmod 2$$

on the set

$$\Omega_m = \{x(t) \in \Omega_{\alpha,r+1,\varrho};\ \exists t_a \in I,\ \beta(t_a) < x(t_a),\ \exists t_b \in I,\ x(t_b) < \alpha_1(t_b)\}.$$

To prove the estimate $x(t) < r$ of a solution $x(t) \in \Omega_m$ of (12), (2), we use assumption (iii).

Let r, r_0, M have the same meaning as in the proof of Theorem 3. Similarly to the proof of Theorem 3 we suppose that $\max|x(t)| = x(t_0) > r_0$. Then there is an interval $[a_1,b_1]$ such that $x(a_1) = x(b_1) = r_0$, $x(t) > r_0$ on (a_1,b_1), $\max\limits_{t\in[a_1,b_1]} x(t) - r_0 = x(t_0) - r_0 = M$.

Let $[t_i, t_{i+1}] \subset [a_1, t_0]$ be an interval of increase of the solution $x(t)$.

We use the estimate

$$|f^*| < c_1(1 + |x|^A + |y|^B) + c_1(1 + r^A) = c_1(2 + 2r^A + |y|^B).$$

We denote $p(x) = x'(t)$ on the interval $[t_i, \tau_i]$ and arguing as in the proof of Lemma 2 we obtain

$$\frac{1}{2}q'(x) \leqslant c_1(2 + 2r^A + |q|^{B/2}).$$

Substituting $v = x - x_i$ and setting $\tilde{q}(v) = q(v - x_i)$ with $x_i = x(t_i)$ we obtain inequalities

$$\tilde{q}'(v) \leqslant 2c_1(2 + 2r^A + |\tilde{q}|^{B/2}), \quad \tilde{q}(0) = 0,$$
$$\tilde{q}(v) \leqslant 4cM(1 + r^A) + 2c\int_0^v |\tilde{q}|^{B/2}\,\mathrm{d}v.$$

Let $[t_i, \tau_i]$ for $i = 1, \ldots, k$ be a finite system of disjoint intervals such that $x(t)$ is increasing on each of them, $x(\tau_i) = x(t_{i+1}) = x_{i+1}$, $x(t_1) = r_0$, $x(\tau_k) = r_0 + M$.

The Bihari lemma [1] implies

$$\tilde{q}(v) \leqslant \left[(4cM(1 + r^A))^{1-B/2} + \left(1 - \frac{B}{2}\right)\int_0^v 2c\,\mathrm{d}v\right]^{1/(1-B/2)}$$

on each interval $v \in V_i = [0, x_{i+1} - x_i]$. Then

$$p(v) \leqslant 2d\left(c_1M(1 + r^A)\right)^{1/2} + d\left((2 - B)c_1M\right)^{1/2-B} \text{ with } d = 2^{B/(2-B)}.$$

As $M \leqslant (b - a)\max\limits_{v \in \bigcup V_i} |p(v)|$, we have

$$M \leqslant 2(b - a)d\left(cM(1 + r^A)\right)^{1/2} + (b - a)d\left((2 - B)cM\right)^{1/(2-B)}.$$

Then

$$M^{1/2} \leqslant c_1r^{A/2} + c_2M^{\gamma/2}$$

where c_i are positive constants independent of r and $\gamma = B/(2 - B) < 1$.

Suppose now that

$$r_0 + M > r.$$

Then

$$M^{1/2} \leqslant c_1(r_0 + M)^{A/2} + c_2M^{\gamma/2}.$$

The last inequality implies that

$$M < M_1$$

where M_1 is independent of r.

If we choose initially $r > M_1 + r_0$ we obtain an a priori estimate of M:

$$M < \max(M_1, r - r_0).$$

That means $x(t) < r$. Then

$$\begin{aligned} D(L_0, N^*, \Omega_m) &= D(L_0, N^*, \Omega_2) \\ &= D(L_0, \overline{N}, \Omega_2) = D(L_0, N, \Omega_2) = 1 \pmod 2, \end{aligned}$$

and $x(t)$ is a solution of the original problem (6), (2). □

Theorem 8. *Let*

(i) $\alpha < \beta$, $\beta_1 < \beta$, $\alpha \nleqslant \beta_1$, *where α is a strict lower solution and β, β_1 are strict upper solutions of the problem* (6), (2),

(ii) *there exist a positive function $c(r)\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t,x,y)| \leqslant c(r)(1 + |y|^2)$ for each $t \in I$, $|x| < r$, $y \in \mathbb{R}$,*

(iii) *there exist constants $A, B \in (0,1)$, $c_1 > 0$ such that $f(t,x,y) \geqslant -c_1(1 + |x|^A + |y|^B)$ for each $t \in I$, $x < \beta(t)$, $y \in \mathbb{R}$,*

(iv) *$yg(t,x,y) \leqslant 0$ for each $t \in I$, $x, y \in \mathbb{R}$.*

Then there are constants $r > \|\beta\|$, $\varrho_0 > 0$ such that for each $\varrho > \varrho_0$ we have

$$D(L_0, N, \Omega_3) = 1 \pmod 2,$$
$$D(L_0, N, \Omega_4) = 1 \pmod 2,$$

i.e., there are at least two different solutions $x_1(t) \in \Omega_3$, $x_2(t) \in \Omega_4$ of the problem (6), (2).

Example 3. We consider the problem

$$x'' + k(t,x)\operatorname{sgn}(x')|x'|^m - \varphi(t,x) = h(t), \tag{14}$$

$$x(a) = x(b), \quad x'(a) = x'(b) \tag{2}$$

with another type of cubic nonlinearity.

Again we assume that $k(t,x) \geqslant 0$, $h(t)$ are continuous functions, $m > 0$, and $\varphi(t,x)$ is a continuous function such that

$$\lim_{x \to -\infty} \varphi(t,x) = \infty,$$
$$\lim_{x \to \infty} \varphi(t,x) = -\infty$$

uniformly for $t \in I$, and there are constants x_1, x_2, $x_1 < x_2$ such that $\varphi(t,x_1) < \varphi(t,x_2)$ for each $t \in I$.

Then for each $h(t)$ there is $r > \max\{|x_1|,|x_2|\}$ such that r is a strict lower and $-r$ a strict upper solution of (14), (2), and for each $h(t)$ such that $\varphi(t,x_1) < h(t) < \varphi(t,x_2)$, x_1 is a strict lower and x_2 a strict upper solution of (14), (2).

Then Theorems 7 and 8 imply that for each $h(t)$, $\varphi(t,x_1) < h(t) < \varphi(t,x_2)$ there are at least three periodic solutions of the problem (14), (2).

For each $h(t)$, $\varphi(t,x_1) \leqslant h(t) \leqslant \varphi(t,x_2)$, there is a periodic solution of (14), (2). This solution we obtain as a limit of convergent subsequence of solutions of periodic problems with righthand sides $h_n(t)$, $\varphi(t,x_1) < h_n(t) < \varphi(t,x_2)$, $h_n \to h$ in $C(I)$.

We apply Theorems 7 and 8 also to the case of nonlinearities of quadratic type.

Example 4. We consider the problem

$$x'' + k(t,x)\operatorname{sgn}(x')|x'|^m + x^2 = h(t), \tag{15}$$
$$x(a) = x(b), \quad x'(a) = x'(b) \tag{2}$$

with $k(t,x) \geqslant 0$, $h(t)$ continuous functions, $m > 0$. Suppose $h(t) > 0$ on I. Set $\max h(t) = h_{\max}$. Then 0 is a strict upper solution and for each $r > 0$, $r^2 > h_{\max}$, $-r$, r are strict lower solutions. Theorem 7 implies that for each $h(t) > 0$ there are at least two different periodic solutions of problem (15), (2).

Passing to limits of solutions of problems with positive $h_n(t) \to h(t)$ we obtain that for each $h(t) \geqslant 0$ there is a periodic solution of problem (15), (2).

Let $h(t) < 0$ and let $x(t)$ be a solution of (15). Integrating equation (13) on the interval $[t_0,t_1]$ of increase of $x(t)$ with t_0 a point of local minimum we obtain a contradiction. So for each $h(t) < 0$ there is no solution of (15), (2).

We generalize the problem (15), (2) to

$$x'' + k(t,x)\operatorname{sgn}(x')|x'|^m + \varphi(t,x) = h(t), \tag{13}$$
$$x(a) = x(b), \quad x'(a) = x'(b), \tag{2}$$

where we assume that the nonlinearity $\varphi(t,x)$ is a continuous function of quadratic type, i.e.

$$\lim_{x\to-\infty}\varphi(t,x)=\lim_{x\to\infty}\varphi(t,x)=\infty$$

uniformly for $t\in I$.

Then there is a set H with nonempty interior $\operatorname{int} H$ and complement H^c such that for each $h(t)\in\operatorname{int} H$ there are at least two solutions, for each $h(t)\in\partial H$ there is a solution, and for each $h(t)\in H^c$ there is no solution of (14), (2).

If $h_0\in\partial H$ then each $h>h_0$ satisfies $h\in\operatorname{int} H$ and each $h<h_0$ satisfies $h\in H^c$.

Remark. A similar result we obtain replacing $\varphi(t,x)$ by $-\varphi(t,x)$.

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