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# Additive Closure Operators on Abelian Unital *l*-groups

Filip ŠVRČEK

*Department of Algebra and Geometry, Faculty of Science, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic*
*e-mail: filipsvrcek@inf.upol.cz*

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**Abstract**

In the paper an additive closure operator on an abelian unital $l$-group $(G, u)$ is introduced and one studies the mutual relation of such operators and of additive closure ones on the $MV$-algebra $\Gamma(G, u)$.

**Key words:** $MV$-algebra; $l$-group.

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## 1 Introduction

In [6] additive closure (and multiplicative interior) operators on $MV$-algebras were introduced as a natural generalization of topological closure (and interior) operators on Boolean algebras. Closure and interior $MV$-algebras ($MV$-algebras endowed with additive closure or multiplicative interior operators) generalize topological boolean algebras in a natural way.

Let us recall the notions of an $MV$-algebra and of an additive closure operator on an $MV$-algebra.

**Definition 1.1** An algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ of the signature $\langle 2, 1, 0\rangle$ is called *an $MV$-algebra* iff for each $x, y, z \in A$:

(MV1) $\quad x \oplus (y \oplus z) = (x \oplus y) \oplus z;$

(MV2) $\quad x \oplus y = y \oplus x;$

(MV3) $x \oplus 0 = x$;

(MV4) $\neg\neg x = x$;

(MV5) $x \oplus \neg 0 = \neg 0$;

(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$.

**Definition 1.2** Let us consider an $MV$-algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ and a mapping $Cl : A \to A$. Then $Cl$ is called *an additive closure operator* on $\mathcal{A}$ iff for each $a, b \in A$

1. $Cl(a \oplus b) = Cl(a) \oplus Cl(b)$,
2. $a \leq Cl(a)$,
3. $Cl(Cl(a)) = Cl(a)$,
4. $Cl(0) = 0$.

$MV$-algebras, which are an algebraic counterpart of the Łukasiewicz infinite valued logic, are by [3], Chapters 2, 7 in a very close connection with abelian unital $l$-groups.

**Definition 1.3** An algebra $G = (G, +, 0, \vee, \wedge)$ of the signature $\langle 2, 0, 2, 2 \rangle$ is called *an l-group* iff

1. $(G, +, 0)$ is a group,
2. $(G, \vee, \wedge)$ is a lattice,
3. $x + (y \vee z) + w = (x + y + w) \vee (x + z + w) \qquad \forall x, y, z, w \in G$,
   $x + (y \wedge z) + w = (x + y + w) \wedge (x + z + w) \qquad \forall x, y, z, w \in G$.

An element $u \in G$ $(u > 0)$ is called *a strong unit* of the $l$-group $G$ iff

$$(\forall a \in G)(\exists n \in \mathbb{N})\ (a \leq nu),$$

where

$$nu \overset{\text{def}}{=} \underbrace{u + u + \cdots + u}_{n}.$$

If an $l$-group $G$ contains a strong unit $u$, then $(G, u)$ is called *a unital l-group*. Moreover, if the operation "+" of the $l$-group $G$ is commutative, then $G$ is called *an abelian l-group*.

In the following remark we will describe the mutual relation of abelian unital $l$-groups and $MV$-algebras.

**Remark 1.4**

a) Let $(G, +, 0, \vee, \wedge)$ be an abelian $l$-group and let $u \in G, u \geq 0$. If

$$x \oplus y := (x + y) \wedge u, \qquad \neg x := u - x,$$

then $\Gamma(G, u) = ([0, u], \oplus, \neg, 0, u)$ is an $MV$-algebra.

b) On the other hand, Daniele Mundici [5] proved that for every $MV$-algebra $\mathcal{A}$ there exists such an abelian unital $l$-group $(G, u)$ that $\mathcal{A} \cong \Gamma(G, u)$.

The aim of this paper is to introduce an additive closure operator on an abelian unital $l$-group $(G, u)$. That means, we will investigate in introducing of such an operator on abelian unital $l$-groups that it will preferably form a natural counterpart of additive closure operators on $MV$-algebras.

## 2 Relation between additive closure operators on $MV$-algebras and on abelian unital $l$-groups

**Definition 2.1** Let $(G, u)$ be an abelian unital $l$-group. A mapping $\psi^+ : G^+ \to G^+$ such that for each $x, y \in G^+$ it holds

1. $\psi^+(x + y) = \psi^+(x) + \psi^+(y)$,
2. $\psi^+(x \wedge u) = \psi^+(x) \wedge u$,
3. $x \leq \psi^+(x)$,
4. $\psi^+(\psi^+(x)) = \psi^+(x)$,

will be called *an additive closure operator on* $G^+$, where $G^+ = \{x \in G;\, x \geq 0\}$.

**Lemma 2.2** *Let $(G, u)$ be an abelian unital $l$-group and let $\psi^+$ be an additive closure operator on $G^+$. Then we have for each $k \in \mathbb{N}$, $k > 1$ and for each $x, y \in G^+$*

*(i)* $\psi^+(u) = u$,

*(ii)* $\psi^+(ku) = ku$,

*(iii)* $x \leq y \Rightarrow \psi^+(x) \leq \psi^+(y)$.

**Proof**

(i) From the axiom 3 of Definition 2.1 it follows that $u \leq \psi^+(u)$. Moreover, from the second axiom of the same definition we get

$$\psi^+(u) = \psi^+(u \wedge u) = \psi^+(u) \wedge u$$

and further $\psi^+(u) \leq u$. Together we have $u = \psi^+(u)$.

(ii) It follows from the first axiom of Definition 2.1 and from (i).

(iii) Let $x, y \in G^+, x \leq y$. Since $-x + (x \vee y) \in G^+$, it must also be

$$\psi^+(y) = \psi^+(x \vee y) = \psi^+(x + (-x + (x \vee y))) = \psi^+(x) + \psi^+(-x + (x \vee y)),$$

But since

$$\psi^+(-x + (x \vee y)) \in G^+,$$

we finally get

$$\psi^+(x) \leq \psi^+(y).$$

□

**Definition 2.3** Let $(G,u)$ be an abelian unital $l$-group. A mapping $\psi : G \to G$ is called *an additive closure operator on* $G$ iff there exists such an additive closure operator $\psi^+$ on $G^+$, that it holds for each element $a \in G$

1. $\psi\,|_{G^+} = \psi^+$,
2. $\psi(a) = \psi^+(a^+) - \psi^+(a^-)$, where $a^+ = a \vee 0$, $a^- = -a \vee 0$.

**Remark 2.4** It is known that in each $l$-group $G$ we have $a = a^+ - a^-$ for each element $a \in G$. So $G = G^+ - G^+$ holds in each $l$-group $G$. Let us show now that in each $l$-group $G$ all representations of $\psi(a)$ in the form of the difference of $\psi^+(x)$ and $\psi^+(y)$, where $x, y \in G^+$ such that $a = x - y$, are the same as the representation of $\psi(a)$ in the form of the difference of $\psi^+(a^+)$ and $\psi^+(a^-)$.

**Lemma 2.5** *Let $(G,u)$ be an abelian unital $l$-group and let $\psi$ be an additive closure operator on $G$. Then it holds for each element $a \in G$ and for each elements $x, y \in G^+$*

$$[a = x - y] \Longrightarrow [\psi(a) = \psi^+(a^+) - \psi^+(a^-) = \psi^+(x) - \psi^+(y)].$$

**Proof** If $a = x - y$, then $x - y = a^+ - a^-$. From that we have $x + a^- = a^+ + y$ and so $\psi^+(x) + \psi^+(a^-) = \psi^+(a^+) + \psi^+(y)$, and finally $\psi^+(x) - \psi^+(y) = \psi^+(a^+) - \psi^+(a^-) = \psi(a)$. □

In the sequel we will study the mutual relation of additive closure operators on abelian unital $l$-groups and on $MV$-algebras. The properties of additive closure operators on $MV$-algebras were studied in [6].

**Theorem 2.6** *Let us consider an abelian unital $l$-group $(G,u)$ and further an additive closure operator $\psi^+$ on $G^+$. Then $\varphi = \psi^+\,|_{[0,u]}$ is an additive closure operator on the $MV$-algebra $\mathcal{A} = \Gamma(G,u)$.*

**Proof** Since $\psi^+$ is isotone and $\psi^+(u) = u$, it is obvious that $\varphi$ is a mapping from $[0,u]$ into $[0,u]$. We will check now validity of 1.–4. from Definition 1.2. Therefore, let us choose two arbitrary elements $a, b \in [0,u]$ and we have

1. $\varphi(a \oplus b) = \varphi((a+b) \wedge u) = \psi^+((a+b) \wedge u) = \psi^+(a+b) \wedge u = (\psi^+(a) + \psi^+(b)) \wedge u = (\varphi(a) + \varphi(b)) \wedge u = \varphi(a) \oplus \varphi(b)$,
2. $a \leq \psi^+(a) = \varphi(a)$,
3. $\varphi(\varphi(a)) = \psi^+(\varphi(a)) = \psi^+(\psi^+(a)) = \psi^+(a) = \varphi(a)$,
4. $\varphi(0) = \psi^+(0) = 0$, because of $\psi^+(0) = \psi^+(0+0) = \psi^+(0) + \psi^+(0)$. □

Let $\mathcal{A} = \Gamma(G,u)$ be the $MV$-algebra constructed on an abelian unital $l$-group $(G,u)$. Then by [3], Lemma 7.1.3 each element $a \in G^+$ can be uniquely represented in the form

$$a = a_1 + a_2 + \cdots + a_n,$$

where the $n$-tuple $(a_1, a_2, \ldots, a_n) \in [0,u]^n$ is determined by relations

$$a_1 = a \wedge u,\ a_2 = (a - a_1) \wedge u, \ldots,\ a_n = (a - a_1 - \cdots - a_{n-1}) \wedge u.$$

**Remark 2.7** The introduced $n$-tuple $(a_1, a_2, \dots, a_n)$ is *a good sequence* of elements of $MV$-algebra $\Gamma(G,u)$—see [3, Lemma 7.1.3]. Let us recall that *a good sequence* of elements of an $MV$-algebra $\mathcal{A}$ is such a sequence $(a_1, a_2, \dots, a_n, \dots)$ of elements of this algebra that for each $i = 1, 2, \dots$ the identity

$$a_i \oplus a_{i+1} = a_i$$

holds and at the same time there exists such $n \in \mathbb{N}$ that $a_r = 0$ for all $r > n$.

Now, let $\varphi$ be an additive closure operator on the $MV$-algebra $\mathcal{A} = \Gamma(G,u)$ and let us define a mapping $\overline{\varphi} : G^+ \to G^+$, where

$$\overline{\varphi}(a) \stackrel{\text{def}}{=} \varphi(a_1) + \varphi(a_2) + \cdots + \varphi(a_n) \qquad \forall a \in G^+.$$

**Remark 2.8** Let us notice the prescription for the introduced mapping $\overline{\varphi}$. By Remark 2.5 we know that $(a_1, a_2, \dots, a_n)$ is a good sequence of elements of $\Gamma(G,u)$ and for each $i = 1, 2, \dots, n-1$ we have therefore $a_i \oplus a_{i+1} = a_i$. But then also for each $i = 1, 2, \dots, n-1$

$$\varphi(a_i) \oplus \varphi(a_{i+1}) = \varphi(a_i \oplus a_{i+1}) = \varphi(a_i).$$

That means, $(\varphi(a_1), \varphi(a_2), \dots, \varphi(a_n))$ is a good sequence of elements of $\Gamma(G,u)$ again.

**Lemma 2.9** *Let us consider an MV-algebra $\mathcal{A} = \Gamma(G,u)$ constructed on an abelian unital l-group $(G,u)$ and an additive closure operator $\varphi$ on $\mathcal{A}$. Then the mapping $\overline{\varphi}$ is isotone.*

**Proof** Let us choose arbitrary elements $a, b \in G^+$, $a \le b$. It holds ([3, Lemma 7.1.3])

$$a = a_1 + a_2 + \cdots + a_m, \qquad b = b_1 + b_2 + \cdots + b_n,$$

where $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n \in [0,u]$ and $m, n$ are some integers, not necessarily the same. If for example $m > n$, then we put $b_{m-n+1} = \cdots = b_m = 0$. So we can consider $m = n$. Now, if $a \le b$, then for each integer $k$

$$((a - ku) \vee 0) \wedge u \le ((b - ku) \vee 0) \wedge u.$$

Further by [3, Lemma 7.1.3] we have from the last inequality

$$(a - a_1 - a_2 - \cdots - a_k) \wedge u \le (b - b_1 - b_2 - \cdots - b_k) \wedge u,$$

that means $a_{k+1} \le b_{k+1}$ for each integer $k$. From that it follows that $\varphi(a_{k+1}) \le \varphi(b_{k+1})$ for each integer $k$ and finally

$$\overline{\varphi}(a) = \varphi(a_1) + \varphi(a_2) + \cdots + \varphi(a_n) \le \varphi(b_1) + \varphi(b_2) + \cdots + \varphi(b_n) = \overline{\varphi}(b).$$

**Theorem 2.10** *Let $\mathcal{A} = \Gamma(G,u)$ be the MV-algebra constructed on an abelian unital l-group $(G,u)$ and let $\varphi$ be an additive closure operator on $\mathcal{A}$. Then for the mapping $\overline{\varphi}$ and an arbitrary element $a \in G^+$*

- $\overline{\varphi}(a \wedge u) = \overline{\varphi}(a) \wedge u$,
- $a \leq \overline{\varphi}(a)$,
- $\overline{\varphi}(\overline{\varphi}(a)) = \overline{\varphi}(a)$.

**Proof** Let $a \in G^+$ is chosen arbitrarily. Then there exists an $n$-tuple $(a_1, a_2, \dots, a_n)$ of elements from $[0, u]$, where $a = a_1 + a_2 + \dots + a_n$, $a_1 = a \wedge u$, $a_2 = (a - a_1) \wedge u$, ..., $a_n = (a - a_1 - \dots - a_{n-1}) \wedge u$. We have:

- $\overline{\varphi}(a) \wedge u = (\varphi(a_1) + \varphi(a_2) + \dots + \varphi(a_n)) \wedge u = \varphi(a_1) \oplus \varphi(a_2) \oplus \dots \oplus \varphi(a_n) = = \varphi(a_1 \oplus a_2 \oplus \dots \oplus a_n) = \varphi((a_1 + a_2 + \dots + a_n) \wedge u) = \varphi(a \wedge u) = \overline{\varphi}(a \wedge u)$;
- $a = a_1 + a_2 + \dots + a_n \leq \varphi(a_1) + \varphi(a_2) + \dots + \varphi(a_n) = \overline{\varphi}(a)$;
- $\overline{\varphi}(\overline{\varphi}(a)) = \overline{\varphi}(\varphi(a_1) + \varphi(a_2) + \dots + \varphi(a_n)) = \varphi(\varphi(a_1)) + \varphi(\varphi(a_2)) + \dots + \varphi(\varphi(a_n)) = \varphi(a_1) + \varphi(a_2) + \dots + \varphi(a_n) = \overline{\varphi}(a)$, because of $c = \varphi(a_1) + \varphi(a_2) + \dots + \varphi(a_n)$ is just the unique decomposition of the element $c \in G^+$ onto a sum of elements from $[0, u]$, which form a good sequence of $\Gamma(G, u)$.

□

**Remark 2.11** (open problem) In Theorem 2.10, we have proven in fact that the operator $\overline{\varphi}$ fulfils conditions 2, 3 and 4 from Definition 2.1. Not answered stays now the problem, in which condition does $\overline{\varphi}$ fulfil moreover the axiom 1 from Definition 2.1, that means in which condition does $\overline{\varphi}$ become an additive closure operator on $G^+$.

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