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Weak and Strong Convergence Theorems of Common Fixed Points for a Pair of Nonexpansive and Asymptotically Nonexpansive Mappings *

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Abstract

The purpose of this paper is to establish some weak and strong convergence theorems of modified three-step iteration methods with errors with respect to a pair of nonexpansive and asymptotically nonexpansive mappings in uniformly convex Banach spaces. The results presented in this paper generalize, improve and unify a few results due to Chang [1], Liu and Kang [5], Osilike and Aniagbosor [7], Rhoades [8] and Schu [9], [10] and others. An example is included to demonstrate that our results are sharp.

Key words: Nonexpansive mappings, asymptotically nonexpansive mappings, common fixed points, modified three-step iteration methods with errors with respect to a pair of mappings.

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1 Introduction

In 1972, Goebel and Kirk [3] introduced the concept of asymptotically nonexpansive mappings and proved that if K is a nonempty closed bounded subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping of K has a fixed point. After that, some authors studied a few iterative approximation methods of fixed points for asymptotically nonexpansive mappings. In 1991, Schu [9], [10] introduced the modified Ishikawa iteration methods and modified Mann iteration methods and proved that the modified Mann iteration sequence converges strongly to some fixed points of asymptotically nonexpansive mappings in Hilbert spaces. Rhoades [8] extended the results in [9] to uniformly convex Banach spaces and to modified Ishikawa iteration methods. Chang [1], Liu and Kang [5] and Osilike and Aniagbosor [7] also established some strong and weak convergence theorems of modified Ishikawa iteration methods with errors and three-step iteration methods with errors for asymptotically nonexpansive mappings.

Inspired and motivated by the work in [1], [5] and [7]–[10], in this paper we introduce a new iterative method, called modified three-step iteration method with errors with respect to a pair of mappings, and establish some strong and weak convergence theorems of the modified three-step iteration method with errors with respect to nonexpansive and asymptotically nonexpansive mappings in nonempty closed convex subsets of uniformly convex Banach spaces. The results presented in this paper generalize, improve and unify a few results due to Chang [1], Liu and Kang [5], Osilike and Aniagbosor [7], Rhoades [8] and Schu [9], [10] and others. An example is included to demonstrate that our results are sharp.

2 Preliminaries

Let E be a uniformly convex Banach space, K be a nonempty subset of E and $S, T : K \to K$ be two mappings. I stands for the identity mapping, $F(T)$ and $F(S,T)$ denote the sets of fixed points of T and common fixed points of S and T, respectively. Let $J : E \to 2^E$ be the normalized duality mapping defined by

$$J(x) = \{f \in E^*, \langle x, f\rangle = \|x\| \cdot \|f\|, \|f\| = \|x\|\}, \quad \forall x \in E.$$

Let us recall the following concepts and results.

Definition 2.1 [2] A mapping $T : K \to K$ is said to be

(1) *asymptotically nonexpansive* if there exists a sequence $\{k_n\}_{n\geq 1} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$, $\forall x, y \in K$, $n \geq 1$;

(2) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in K$;

(3) *uniformly L-Lipschitzian* if there exists a constant $L \geq 1$ satisfying $\|T^n x - T^n y\| \leq L\|x - y\|$, $\forall x, y \in K$, $n \geq 1$;

(4) *semi-compact* if K is closed and for any bounded sequence $\{x_n\}_{n\geq 1}$ in K with $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}_{i\geq 1} \subset \{x_n\}_{n\geq 1}$ and $x \in K$ such that $\lim_{i\to\infty} x_{n_i} = x$.

It is easy to see that if T is an asymptotically nonexpansive mapping with a sequence $\{k_n\}_{n\geq 1} \subset [1,\infty)$ such that $\lim_{n\to\infty} k_n = 1$, then it must be uniformly L-Lipschitzian with $L = \sup\{k_n : n \geq 1\}$.

Definition 2.2 A mapping T with domain $D(T)$ and range $R(T)$ in E is called *demiclosed* at a point $p \in D(T)$ if whenever $\{x_n\}_{n\geq 1}$ is a sequence in E which converges weakly to a point $x \in E$ and $\{Tx_n\}_{n\geq 1}$ converges strongly to p, then $Tx = p$.

Definition 2.3 [6] A Banach space E is called to satisfy *Opial's condition* if for each sequence $\{x_n\}_{n\geq 1}$ in E which converges weakly to a point $x \in E$

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|, \quad \forall y \in E - \{x\}.$$

Definition 2.4 Let K be a nonempty convex subset of a normed linear space E and $S, T : K \to K$ be two mappings. For an arbitrary $x_1 \in K$, the *modified three-step iteration sequence* with errors $\{x_n\}_{n\geq 1}$ with respect to S and T is defined by

$$\begin{aligned} z_n &= a_n'' S x_n + b_n'' T^n x_n + c_n'' w_n, \\ y_n &= a_n' S x_n + b_n' T^n z_n + c_n' v_n, \\ x_{n+1} &= a_n S x_n + b_n T^n y_n + c_n u_n, \quad \forall n \geq 1, \end{aligned} \tag{2.1}$$

where $\{u_n\}_{n\geq 1}$, $\{v_n\}_{n\geq 1}$ and $\{w_n\}_{n\geq 1}$ are bounded sequences in K, $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$, $\{c_n\}_{n\geq 1}$, $\{a_n'\}_{n\geq 1}$, $\{b_n'\}_{n\geq 1}$, $\{c_n'\}_{n\geq 1}$, $\{a_n''\}_{n\geq 1}$, $\{b_n''\}_{n\geq 1}$ and $\{c_n''\}_{n\geq 1}$ are sequences in $[0,1]$ satisfying

$$a_n + b_n + c_n = a_n' + b_n' + c_n' = a_n'' + b_n'' + c_n'' = 1, \quad \forall n \geq 1. \tag{2.2}$$

Remark 2.1 In case $S = I$ and $b_n'' = c_n'' = 0$ for $n \geq 1$, then the sequence $\{x_n\}_{n\geq 1}$ generated in (2.1) reduces to the usual modified Ishikawa sequence with errors.

Lemma 2.1 [4] *Let E be a Banach space satisfying Opial's condition and K be a nonempty closed convex subset of E. If $T : K \to K$ is an asymptotically nonexpansive mapping, then $I - T$ is demiclosed at zero.*

Lemma 2.2 [10] *Let E be a uniformly convex Banach space, $\{t_n\}_{n\geq 1} \subseteq [b,c] \subset (0,1)$, $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be sequences in E. If $\limsup_{n\to\infty}\|x_n\| \leq a$, $\limsup_{n\to\infty}\|y_n\| \leq a$ and $\lim_{n\to\infty}\|t_n x_n + (1-t_n)y_n\| = a$ for some constant $a \geq 0$, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Lemma 2.3 [2] *Let E be a normed linear space. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall x, y \in E,\ j(x+y) \in J(x+y).$$

Lemma 2.4 [11] *Let $p > 1$ and $r > 0$ be two constants. Then a Banach space E is uniformly convex if and only if there exists a continuous and strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|ax + (1-a)y\|^p \le a\|x\|^p + (1-a)\|y\|^p - w_p(a)g(\|x-y\|)$$

for each $x, y \in B(\theta, r) = \{x : \|x\| \le r \text{ and } x \in E\}$, $a \in [0,1]$ and

$$w_p(a) = a^p(1-a) + a(1-a)^p$$

Lemma 2.5 [7] *Let $\{a_n\}_{n\ge1}$, $\{b_n\}_{n\ge1}$ and $\{c_n\}_{n\ge1}$ be sequences of nonnegative numbers satisfying the inequality*

$$a_{n+1} \le (1+c_n)a_n + b_n, \quad \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}_{n\ge1}$ has a subsequence which converges to zero, then $\lim_{n\to\infty} a_n = 0$.

3 Main Results

Lemma 3.1 *Let K be a nonempty convex subset of a normed linear space E. Let $S : K \to K$ be a mapping and $T : K \to K$ be uniformly L-Lipschitzian. Then*

$$\begin{aligned}\|x_{n+1} - Tx_{n+1}\| \le \|x_{n+1} - T^{n+1}x_{n+1}\| + L^2(L^2+2L+2)\|x_n - T^n x_n\| \\ + L(L+1)[(L^2+L+1)\|Sx_n - x_n\| + c_n\|u_n - x_n\| \\ + b_n c'_n L\|v_n - x_n\| + b_n b'_n c''_n L^2\|w_n - x_n\|]\end{aligned}$$

for $n \ge 1$, where $\{x_n\}_{n\ge1}$ is defined by (2.1).

Proof Set $A_{n+1} = \|x_{n+1} - T^{n+1}x_n\|$, $B_{n+1} = \|Sx_{n+1} - x_{n+1}\|$ for $n \ge 1$. It follows that

$$\|z_n - x_n\| \le a''_n\|Sx_n - x_n\| + b''_n\|T^n x_n - x_n\| + c''_n\|w_n - x_n\|, \tag{3.1}$$

$$\begin{aligned}\|y_n - x_n\| &\le a'_n\|Sx_n - x_n\| + b'_n(L\|z_n - x_n\| + \|T^n x_n - x_n\|) + c'_n\|v_n - x_n\| \\ &\le a'_n B_n + b'_n L\|z_n - x_n\| + b'_n A_n + c'_n\|v_n - x_n\| \end{aligned}\tag{3.2}$$

and

$$\begin{aligned}\|x_{n+1} - Tx_{n+1}\| &\le \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - Tx_{n+1}\| \\ &\le A_{n+1} + L\|T^n x_{n+1} - x_{n+1}\| \\ &\le A_{n+1} + L(\|T^n x_{n+1} - T^n x_n\| + \|T^n x_n - x_{n+1}\|) \\ &\le A_{n+1} + L^2\|x_{n+1} - x_n\| + L\|T^n x_n - x_{n+1}\| \\ &\le A_{n+1} + L^2 a_n B_n + L^2 b_n(\|T^n y_n - T^n x_n\| \\ &\quad + \|T^n x_n - x_n\|) + L^2 c_n\|u_n - x_n\| + La_n B_n + La_n A_n \\ &\quad + b_n L^2\|y_n - x_n\| + Lc_n A_n + Lc_n\|u_n - x_n\| \\ &\le A_{n+1} + L(L+1)a_n B_n + L(Lb_n + a_n + c_n)A_n \\ &\quad + L^2 b_n(L+1)\|y_n - x_n\| + Lc_n(L+1)\|u_n - x_n\| \end{aligned}\tag{3.3}$$

for $n \geq 1$. Substituting (3.1) and (3.2) into (3.3), we obtain that

$$\|x_{n+1} - Tx_{n+1}\| \leq A_{n+1} + L^2(L^2 + 2L + 2)A_n + L(L+1)[(L^2 + L + 1)B_n + c_n\|u_n - x_n\| + b_n c'_n L\|v_n - x_n\| + b_n b'_n c''_n L^2\|w_n - x_n\|]$$

for $n \geq 1$. This completes the proof of Lemma 2.1. □

Remark 3.1 Lemma 1.2 in [7], Lemma 3.1 in [5], Lemma 1.4 in [8] and Lemma 1.4 in [10] are special cases of Lemma 3.1.

Lemma 3.2 *Let K be a nonempty convex subset of a normed linear space E. Let $S : K \to K$ be a nonexpansive mapping and $T : K \to K$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subseteq [1, \infty)$ satisfying* $\lim_{n\to\infty} k_n = 1$ *and $F(S,T) \neq \emptyset$. If the following conditions*

$$\sum_{n=1}^{\infty}(k_n - 1) < \infty \tag{3.4}$$

and

$$\sum_{n=1}^{\infty} b_n b'_n c''_n < \infty, \quad \sum_{n=1}^{\infty} b_n c'_n < \infty, \quad \sum_{n=1}^{\infty} c_n < \infty \tag{3.5}$$

hold, then $\lim_{n\to\infty} \|x_n - q\|$ *exists for any $q \in F(S,T)$, where $\{x_n\}_{n\geq 1}$ is defined by (2.1).*

Proof Let $q \in F(S,T)$ and $L = \sup\{k_n : n \geq 1\}$. Note that $\{u_n - q\}_{n\geq 1}$, $\{v_n - q\}_{n\geq 1}$ and $\{w_n - q\}_{n\geq 1}$ are bounded. It follows that $M = \sup\{\|u_n - q\|, \|v_n - q\|, \|w_n - q\| : n \geq 1\} < \infty$. Since S is nonexpansive and T is asymptotically nonexpansive, by (2.1) we know that

$$\begin{aligned}
\|x_{n+1} - q\| &= \|a_n S x_n + b_n T^n y_n + c_n u_n - q\| \\
&\leq a_n\|x_n - q\| + b_n k_n\|y_n - q\| + c_n\|u_n - q\| \\
&\leq a_n\|x_n - q\| + b_n k_n(a'_n\|x_n - q\| + b'_n k_n\|z_n - q\| + c'_n\|v_n - q\|) + c_n M \\
&\leq (a_n + b_n k_n a'_n)\|x_n - q\| + b_n b'_n k_n^2(a''_n\|x_n - q\| + b''_n k_n\|x_n - q\| \\
&\quad + c''_n\|w_n - q\|) + b_n c'_n k_n M + c_n M \\
&\leq [a_n + b_n k_n a'_n + b_n b'_n k_n^2(a''_n + b''_n k_n)]\|x_n - q\| \\
&\quad + (b_n b'_n c''_n k_n + b_n c'_n k_n M + c_n)M \\
&\leq [1 - b_n + b_n k_n(1 - b'_n) + b_n b'_n k_n^2(1 - b''_n + b''_n k_n)]\|x_n - q\| \\
&\quad + (L b_n b'_n c''_n + LM b_n c'_n + c_n)M \\
&\leq [1 + b_n(k_n - 1)(1 + L + L^2)]\|x_n - q\| \\
&\quad + (L b_n b'_n c''_n + LM b_n c'_n + c_n)M
\end{aligned} \tag{3.6}$$

for $n \geq 1$. It follows from Lemma 2.5, (3.4) and (3.5) that $\lim_{n\to\infty} \|x_n - q\|$ exists. This completes the proof. □

Remark 3.2 Lemma 3.2 generalizes Lemma 3.2 in [5], Lemma 3 in [7] and Lemma 1.2 in [10].

Lemma 3.3 *Let K be a nonempty convex subset of a uniformly convex Banach space E. Let $S : K \to K$ be a nonexpansive mapping and $T : K \to K$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subseteq [1, \infty)$ satisfying (3.4),* $\lim_{n\to\infty} k_n = 1$, *$F(S,T) \neq \emptyset$ and*

$$\|x - Ty\| \leq \|Sx - Ty\|, \quad \forall x, y \in K. \tag{3.7}$$

Suppose that

$$\sum_{n=1}^{\infty} c'_n < \infty, \quad \sum_{n=1}^{\infty} b'_n c''_n < \infty, \quad \sum_{n=1}^{\infty} c_n < \infty, \tag{3.8}$$

$$(1 + \limsup_{n\to\infty} b''_n) \cdot \limsup_{n\to\infty} b'_n < 1, \tag{3.9}$$

$$0 < a \leq b_n \leq b < 1, \quad \forall n \geq 1, \tag{3.10}$$

where a and b are constants. Then $\lim_{n\to\infty} \|x_n - Sx_n\| = \lim_{n\to\infty} \|x_n - Tx_n\| = 0$, *where $\{x_n\}_{n\geq 1}$ is defined by (2.1).*

Proof Let $q \in F(S,T)$. Lemma 3.2 ensures that $\lim_{n\to\infty} \|x_n - q\|$ exists. Set $\lim_{n\to\infty} \|x_n - q\| = d$. Since $\{u_n\}_{n\geq 1}$, $\{v_n\}_{n\geq 1}$ and $\{w_n\}_{n\geq 1}$ are bounded sequences, it follows that

$$M = \sup\{\|u_n - q\|, \|v_n - q\|, \|x_n - v_n\|, \|x_n - w_n\|, \|x_n - u_n\| : n \geq 1\} < \infty.$$

Observe that

$$\begin{aligned} \lim_{n\to\infty} \|x_{n+1} - q\| &= \lim_{n\to\infty} \|(1 - b_n - c_n)Sx_n + b_n T^n y_n + c_n u_n - q\| \\ &= \lim_{n\to\infty} \|(1 - b_n)[Sx_n - q - c_n(Sx_n - u_n)] \\ &\quad + b_n[T^n y_n - q - c_n(Sx_n - u_n)]\|. \end{aligned} \tag{3.11}$$

From the nonexpansivity of S and (3.8), we deduce that

$$\begin{aligned} &\limsup_{n\to\infty} \|Sx_n - q - c_n(Sx_n - u_n)\| \\ &\leq \limsup_{n\to\infty} (\|x_n - q\| + c_n\|x_n - q\| + c_n\|u_n - q\|) \\ &\leq \limsup_{n\to\infty} [(1 + c_n)\|x_n - q\| + c_n M] \leq d. \end{aligned} \tag{3.12}$$

Since S is nonexpansive and T is asymptotically nonexpansive, by (2.1) we derive that

$$\begin{aligned} &\|T^n y_n - q - c_n(Sx_n - u_n)\| \\ &\leq k_n\|y_n - q\| + c_n\|Sx_n - q\| + c_n\|u_n - q\| \\ &\leq k_n[a'_n\|x_n - q\| + b'_n k_n\|z_n - q\|] + (c'_n k_n + c_n)M + c_n\|x_n - q\| \\ &\leq (a'_n k_n + c_n)\|x_n - q\| + b'_n k_n^2\|z_n - q\| + (c'_n k_n + c_n)M \\ &\leq [a'_n k_n + c_n + b'_n k_n^2(a''_n + b''_n k_n)]\|x_n - q\| + (c'_n k_n + c_n + b'_n c''_n k_n^2)M \\ &\leq [k_n + b'_n k_n(k_n - 1) + b'_n k_n^2 b''_n(k_n - 1) + c_n]\|x_n - q\| \\ &\quad + (c'_n k_n + c_n + b'_n c''_n k_n^2)M \end{aligned} \tag{3.13}$$

for $n \geq 1$. In view of (3.4), (3.8) and (3.13), we conclude that

$$\limsup_{n\to\infty} \|T^n y_n - q - c_n(Sx_n - u_n)\| \leq d. \tag{3.14}$$

On account of (3.10)–(3.12), (3.14) and Lemma 2.2, we see that

$$\begin{aligned} &\lim_{n\to\infty} \|Sx_n - T^n y_n\| = \\ &= \lim_{n\to\infty} \|[Sx_n - q - c_n(Sx_n - u_n)] - [T^n y_n - q - c_n(Sx_n - u_n)]\| \\ &= 0, \end{aligned} \tag{3.15}$$

which implies that

$$\lim_{n\to\infty} \|x_n - T^n y_n\| = 0 \tag{3.16}$$

by (3.7). Notice that

$$\|Sx_n - x_n\| \leq \|Sx_n - T^n y_n\| + \|x_n - T^n y_n\|, \quad \forall n \geq 1.$$

Thus (3.15) and (3.16) mean that

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0. \tag{3.17}$$

It is easy to verify that

$$\begin{aligned} \|x_n - T^n x_n\| &\leq \|x_n - T^n y_n\| + k_n\|y_n - x_n\| \\ &\leq \|x_n - T^n y_n\| + k_n[a'_n\|Sx_n - x_n\| + b'_n k_n\|z_n - x_n\| \\ &\quad + b'_n\|T^n x_n - x_n\| + c'_n M] \\ &\leq \|x_n - T^n y_n\| + k_n(a'_n + k_n b'_n a''_n)\|Sx_n - x_n\| \\ &\quad + k_n b'_n(1 + k_n b''_n)\|T^n x_n - x_n\| + k_n(c'_n + k_n b'_n c''_n)M \end{aligned} \tag{3.18}$$

for $n \geq 1$. Note that (3.9) implies that there exists a positive integer N satisfying $k_n b'_n(1 + k_n b''_n) < 1$ for $n \geq N$. It follows from (3.18) that

$$\begin{aligned} \|x_n - T^n x_n\| \leq \frac{1}{1 - k_n b'_n(1 + k_n b''_n)} [\|x_n - T^n y_n\| \\ + k_n(a'_n + k_n b'_n a''_n)\|Sx_n - x_n\| + k_n(c'_n + k_n b'_n c''_n)M] \end{aligned} \tag{3.19}$$

for $n \geq N$. According to (3.8), (3.9), (3.16), (3.17) and (3.19), we conclude that

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \tag{3.20}$$

In terms of (3.8), (3.17), (3.20) and Lemma 3.1, we get that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof. □

Remark 3.3 Lemma 3.3 extends Lemma 3.3 in [6], Lemma 4 in [8] and Theorem 1 in [9].

Theorem 3.1 *Let E be a uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed convex subset of E. Let $S : K \to K$ be a nonexpansive mapping and $T : K \to K$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subseteq [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ and $F(S,T) \neq \emptyset$. If* (3.4) *and* (3.7)–(3.10) *hold, then the modified three-step iteration sequences with errors $\{x_n\}_{n\geq1}$ with respect to S and T defined by* (2.1) *converges weakly to a common fixed point of S and T.*

Proof It follows from Lemma 3.2 that $\{x_n\}_{n\geq1}$ is bounded. Hence $\{x_n\}_{n\geq1}$ has a subsequence $\{x_{n_j}\}_{j\geq1}$, which converges weakly to p. Since $\{x_{n_j}\}_{j\geq1} \subseteq K$ and K is weakly closed, it follows that $p \in K$. From Lemmas 3.3 and 2.1 we deduce that $I - T$ and $I - S$ are demiclosed at zero. Hence $(I - T)p = (I - S)p = 0$. That is, $p \in F(S,T)$. Suppose that $\{x_n\}_{n\geq1}$ does not converge weakly to p . Then there exists another subsequence $\{x_{m_k}\}_{k\geq1}$ of $\{x_n\}_{n\geq1}$ which converges weakly to some $q \neq p$. It is clear that $q \in F(S,T)$, $\lim_{n\to\infty} \|x_n - p\|$ and $\lim_{n\to\infty} \|x_n - q\|$ exist. Let $a = \lim_{n\to\infty} \|x_n - p\|$, $b = \lim_{n\to\infty} \|x_n - q\|$. Because E satisfies Opial's condition, we obtain that

$$\begin{aligned} a &= \liminf_{j\to\infty} \|x_{n_j} - p\| < \liminf_{j\to\infty} \|x_{n_j} - q\| \\ &= b = \liminf_{k\to\infty} \|x_{m_k} - q\| < \liminf_{k\to\infty} \|x_{m_k} - p\| = a, \end{aligned}$$

which is a contradiction. Hence $p = q$ and $\{x_n\}_{n\geq1}$ converges weakly to $p \in F(S,T)$. This completes the proof. □

Lemma 3.4 *Let K be a nonempty bounded closed convex subset of a normed linear space E. Let $S : K \to K$ be a mapping and $T : K \to K$ be uniformly L-Lipschitzian. Then*

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| \leq{} & \|x_{n+1} - T^{n+1}x_{n+1}\| + L\|x_n - T^n x_n\| \\ & + L(L+1)[(1 + L + L^2)(\|x_n - T^n x_n\| + \|Sx_n - x_n\|) \\ & + c_n\|u_n - Sx_n\| + Lb_n c'_n\|v_n - Sx_n\| \\ & + b_n b'_n c''_n L^2\|w_n - Sx_n\|] \end{aligned} \tag{3.21}$$

for $n \geq 1$,where $\{x_n\}_{n\geq1}$ is defined by (2.1).

Proof Put

$$A_n = c_n(u_n - Sx_n),\ B_n = c'_n(v_n - Sx_n),\ C_n = c''_n(w_n - Sx_n),\ \forall n \geq 1. \tag{3.22}$$

Then the sequence $\{x_n\}_{n\geq1}$ defined by (2.1) can be rewritten as

$$\begin{aligned} z_n &= (1 - b''_n)Sx_n + b''_n T^n x_n + C_n, \\ y_n &= (1 - b'_n)Sx_n + b'_n T^n z_n + B_n, \\ x_{n+1} &= (1 - b_n)Sx_n + b_n T^n y_n + A_n, \quad \forall n \geq 1. \end{aligned} \tag{3.23}$$

The rest of the proof is exactly the same as that of Lemma 3.1, and is omitted. This completes the proof. □

Remark 3.4 Lemma 3.4 is an improvement of Lemma 3 in [1] and Lemma 1.2 in [9].

Lemma 3.5 *Let K be a nonempty bounded closed convex subset of a real Banach space E. Let $S : K \to K$ be a nonexpansive mapping and $T : K \to K$ be a uniformly L-Lipschitzian and asymptotically nonexpansive mapping with a sequence $\{k_n\} \subseteq [1, \infty)$ satisfying $\lim_{n\to\infty} k_n = 1$, $F(S, T) \neq \emptyset$, (3.4) and (3.7). Suppose that (3.8), (3.10) and*

$$(1 + L \limsup_{n\to\infty} b''_n) \cdot \limsup_{n\to\infty} b'_n < L^{-1} \tag{3.24}$$

hold. Then $\lim_{n\to\infty} \|Sx_n - x_n\| = \lim_{n\to\infty} \|x_n - T^n x_n\| = 0$, where $\{x_n\}_{n\geq1}$ is defined by (2.1).

Proof Let $\{A_n\}_{n\geq1}$, $\{B_n\}_{n\geq1}$, $\{C_n\}_{n\geq1}$ be defined by (3.22) and $q \in F(S, T)$. Note that K is a nonempty bounded closed convex subset and $\{x_n\}_{n\geq1}$, $\{y_n\}_{n\geq1}$, $\{z_n\}_{n\geq1}$, $\{T^n x_n\}_{n\geq1}$, $\{T^n y_n\}_{n\geq1}$, $\{T^n z_n\}_{n\geq1}$, $\{Sx_n\}_{n\geq1}$ are in K. Then there exists $r > 0$ such that

$$\begin{aligned} K \cup \{&x_n - q, y_n - q, z_n - q, Sx_n - q, Sx_n - u_n, Sx_n - v_n, Sx_n - w_n, \\ &Sx_n - q + A_n, Sx_n - q + B_n, Sx_n - q + C_n, T^n x_n - q + A_n, \\ &T^n y_n - q + B_n, T^n z_n - q + C_n, T^n y_n - q + A_n, T^n y_n - q + C_n\} \\ &\subset B(\theta, r) \end{aligned}$$

for any $n \geq 1$. From Lemma 2.3 we get that

$$\begin{aligned} \|Sx_n - q + A_n\|^2 &\leq \|Sx_n - q\|^2 + 2\langle A_n, j(Sx_n - q + A_n)\rangle \\ &\leq \|x_n - q\|^2 + 2\|A_n\| \cdot \|Sx_n - q + A_n\| \\ &\leq \|x_n - q\|^2 + 2r\|A_n\| \end{aligned} \tag{3.25}$$

for $j(Sx_n - q + A_n) \in J(Sx_n - q + A_n)$ and $n \geq 1$. Similarly we have

$$\|T^n y_n - q + A_n\|^2 \leq \|T^n y_n - q\|^2 + 2r\|A_n\| \leq k_n^2\|y_n - q\|^2 + 2r\|A_n\| \tag{3.26}$$

for $n \geq 1$. It follows from (3.25), (3.26) and Lemma 2.4 that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|(1-b_n)(Sx_n-q+A_n)+b_n(T^ny_n-q+A_n)\|^2 \\
&\leq (1-b_n)\|Sx_n-q+A_n\|^2+b_n(T^ny_n-q+A_n)\|^2 \\
&\quad - w_2(b_n)g(\|Sx_n-T^ny_n\|) \\
&\leq (1-b_n)(\|x_n-q\|^2+2r\|A_n\|)+b_n(\|T^ny_n-q\|^2+2r\|A_n\|) \\
&\quad - b_n(1-b_n)g(\|Sx_n-T^ny_n\|) \\
&= \|x_n-q\|^2+b_n(\|T^ny_n-q\|^2-\|y_n-q\|^2) \\
&\quad + b_n(\|y_n-q\|^2-\|x_n-q\|^2)+2r\|A_n\| \\
&\quad - b_n(1-b_n)g(\|Sx_n-T^ny_n\|) \\
&\leq \|x_n-q\|^2+b_n(k_n^2-1)\|y_n-q\|^2+b_n(\|y_n-q\|^2-\|x_n-q\|^2) \\
&\quad + 2r\|A_n\|-b_n(1-b_n)g(\|Sx_n-T^ny_n\|) \qquad (3.27)
\end{aligned}$$

for $n \geq 1$. Obviously we have

$$\begin{aligned}
&\|z_n-q\|^2-\|x_n-q\|^2 \\
&\leq (1-b''_n)\|x_n-q\|^2+b''_n\|T^nx_n-q\|^2-\|x_n-q\|^2+2r\|C_n\| \\
&\leq b''_n(k_n^2-1)\|x_n-q\|^2+2r\|C_n\| \qquad (3.28)
\end{aligned}$$

and

$$\begin{aligned}
&\|y_n-q\|^2-\|x_n-q\|^2 \\
&\leq (1-b'_n)\|x_n-q\|^2+b'_n\|T^nz_n-q\|^2-\|x_n-q\|^2+2r\|B_n\| \\
&\leq b'_n(k_n^2-1)\|z_n-q\|^2+b'_n(\|z_n-q\|^2-\|x_n-q\|^2)+2r\|B_n\| \quad (3.29)
\end{aligned}$$

for $n \geq 1$. Using (3.27)–(3.29) we obtain that

$$\begin{aligned}
&\|x_{n+1}-q\|^2 \\
&\leq \|x_n-q\|^2+b_n(k_n^2-1)\|y_n-q\|^2+b_nb'_n(k_n^2-1)\|z_n-q\|^2 \\
&\quad + b_nb'_nb''_n(k_n^2-1)\|x_n-q\|^2+2rb_nb'_n\|C_n\|+2rb_n\|B_n\|+2r\|A_n\| \\
&\quad - b_n(1-b_n)g(\|Sx_n-T^ny_n\|) \\
&\leq \|x_n-q\|^2+b_n(k_n^2-1)[\|y_n-q\|^2+b'_n\|z_n-q\|^2 \\
&\quad + b'_nb''_n\|x_n-q\|^2]+2r(b_nb'_n\|C_n\|+b_n\|B_n\|+\|A_n\|) \\
&\quad - b_n(1-b_n)g(\|Sx_n-T^ny_n\|) \qquad (3.30)
\end{aligned}$$

for $n \geq 1$. Since $\{x_n-q\}_{n\geq 1}$, $\{y_n-q\}_{n\geq 1}$ and $\{z_n-q\}_{n\geq 1}$ belong to $B(\theta, r)$, (3.10) and (3.30) ensure that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\leq \|x_n-q\|^2+3r^2(1+\sup\{k_n : n\geq 1\})(k_n-1) \\
&\quad + 2r^2(b_nb'_nc''_n+b_nc'_n+c_n)-a(1-b)g(\|Sx_n-T^ny_n\|) \qquad (3.31)
\end{aligned}$$

for $n \geq 1$. Therefore

$$\begin{aligned}
a(1-b)g(\|Sx_n-T^ny_n\|) &\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2 \\
&+ 3r^2(1+\sup\{k_n : n\geq 1\})(k_n-1)+2r^2(b_nb'_nc''_n+b_nc'_n+c_n)
\end{aligned}$$

for $n \geq 1$. This yields that

$$a(1-b)\sum_{n=1}^{m} g(\|Sx_n - T^n y_n\|) \leq \|x_1 - q\|^2$$
$$+ 3r^2(1+\sup\{k_n : n \geq 1\})\sum_{n=1}^{m}(k_n - 1) + 2r^2\sum_{n=1}^{m}(b_n b'_n c''_n + b_n c'_n + c_n)$$

for $m \geq 1$. Letting $m \to \infty$ in the above inequality, we derive that

$$\sum_{n=1}^{\infty} g(\|Sx_n - T^n y_n\|) < \infty,$$

which implies that

$$\lim_{n\to\infty} g(\|Sx_n - T^n y_n\|) = 0. \tag{3.32}$$

Note that $g : [0,\infty) \to [0,\infty)$ is continuous and strictly increasing with $g(0) = 0$. It follows from (3.32) that

$$\lim_{n\to\infty} \|Sx_n - T^n y_n\| = 0. \tag{3.33}$$

On account of (3.7) and (3.33), we know that

$$\lim_{n\to\infty} \|x_n - T^n y_n\| = 0. \tag{3.34}$$

It follows from (3.33) and (3.34)that

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0. \tag{3.35}$$

By virtue of (3.23) we have

$$\begin{aligned}
&\|x_n - y_n\| \\
&\leq \|x_n - T^n y_n\| + \|T^n y_n - y_n\| \\
&\leq \|x_n - T^n y_n\| + (1-b'_n)\|Sx_n - T^n y_n\| + b'_n L\|z_n - y_n\| + c'_n r \\
&\leq \|x_n - T^n y_n\| + (1-b'_n)\|Sx_n - T^n y_n\| \\
&\quad + b'_n L(\|z_n - x_n\| + \|x_n - y_n\|) + c'_n r \\
&\leq \|x_n - T^n y_n\| + (1-b'_n)\|Sx_n - T^n y_n\| + b'_n L[(1-b''_n)\|Sx_n - x_n\| \\
&\quad + b''_n\|T^n x_n - x_n\| + c''_n r + \|x_n - y_n\|] + c'_n r
\end{aligned} \tag{3.36}$$

for $n \geq 1$. Notice that (3.24) ensures that there exists a positive integer M satisfying

$$b'_n < L^{-1} \quad \text{and} \quad (1+Lb''_n)b'_n < L^{-1} \quad \text{for } n \geq M. \tag{3.37}$$

From (3.36) and (3.37), we conclude that

$$\|x_n - y_n\| \leq \frac{1}{1-b'_n L}[\|x_n - T^n y_n\| + \|Sx_n - T^n y_n\| + b'_n L(\|Sx_n - x_n\|$$
$$+ b''_n\|T^n x_n - x_n\| + rc''_n) + c'_n r]$$

and

$$\begin{aligned}\|T^n x_n - x_n\| &\leq \|T^n x_n - T^n y_n\| + \|x_n - T^n y_n\| \leq L\|x_n - y_n\| + \|x_n - T^n y_n\| \\ &\leq L \cdot \frac{1}{1-b'_n L}\{\|x_n - T^n y_n\| + \|Sx_n - T^n y_n\| \\ &\quad + b'_n L[\|Sx_n - x_n\| + b''_n\|T^n x_n - x_n\| + rc''_n] + c'_n r\} + \|x_n - T^n y_n\|\end{aligned}$$

for $n \geq M$. Simplifying we get that

$$\begin{aligned}\|T^n x_n - x_n\| \leq\ & \frac{1}{1-b'_n L - b'_n b''_n L^2}[(L+1)\|x_n - T^n y_n\| + L\|Sx_n - T^n y_n\| \\ & + b'_n L^2\|Sx_n - x_n\| + L(b'_n L c''_n + c'_n)r] \qquad (3.38)\end{aligned}$$

for $n \geq M$. It follows from (3.8), (3.10), (3.33)–(3.35) and (3.38) that

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0.$$

This completes the proof. □

Remark 3.5 Lemma 3.5 improves Lemma 4 in [1], Theorem 1 in [9] and Lemma 1.4 in [10].

Theorem 3.2 *Let E be a real uniformly convex Banach space, K be a nonempty bounded closed convex subset of E. Let $S : K \to K$ be a nonexpansive mapping and $T : K \to K$ be a uniformly L-Lipschitzian and asymptotically nonexpansive mapping with a sequence $\{k_n\} \subseteq [1,\infty)$ satisfying $\lim_{n\to\infty} k_n = 1$, $F(S,T) \neq \emptyset$, (3.4) and (3.7). Suppose that (3.8), (3.10) and (3.24) hold. If T is semi-compact, then the modified three-step iteration sequences with errors $\{x_n\}_{n\geq 1}$ with respect to S and T defined by (2.1) converges strongly to a common fixed point of S and T.*

Proof It follows from Lemmas 3.4 and 3.5 and (3.8) that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Since T is semi-compact, there exists a subsequence $\{x_{n_i}\}_{i\geq 1} \subset \{x_n\}_{n\geq 1}$ such that $x_{n_i} \to q \in K$ as $i \to \infty$. By the continuity of S and T, (3.8) and Lemma 3.5, we conclude that

$$\lim_{i\to\infty} \|Tx_{n_i} - x_{n_i}\| = \|q - Tq\| = 0, \qquad \lim_{i\to\infty} \|Sx_{n_i} - x_{n_i}\| = \|q - Sq\| = 0.$$

That is, q is a common fixed point of S and T in K. From (3.31) we know that

$$\begin{aligned}\|x_{n+1} - q\|^2 \leq\ & \|x_n - q\|^2 + 3r^2(1 + \sup\{k_n : n \geq 1\})(k_n - 1) \\ & + 2r^2(b_n b'_n c''_n + b_n c'_n + c_n) \qquad (3.39)\end{aligned}$$

for $n \geq 1$. Then (3.4), (3.8), (3.39) and Lemma 2.5 guarantee that $\lim_{n\to\infty}\|x_n - q\|^2 = 0$. That is $\lim_{n\to\infty} x_n = q$. This completes the proof. □

Remark 3.6 Theorems 3.1 and 3.2 extend, improve and unify Theorems 1.1 and 1.2 in [1], Theorem 3.1 in [5], Theorems 1 and 2 in [7], Theorems 2 and 3 in [8], Theorem 1.5 in [9] and Theorems 2.1 and 2.2 in [10] and in the following ways:

(1) the identity mapping in [1], [5], [7]–[10] is replaced by the more general nonexpansive mapping.

(2) the usual modified Mann iteration methods in [10], the usual modified Ishikawa iteration methods in [8] and [9], the usual modified Ishikawa iterations methods with errors in [1] and [7] and the usual modified three-step iteration methods with errors in [5] are extended to the modified three-step iteration methods with errors with respect to a pair of mappings.

(3) the conditions (3.8) and (3.10) are weaker than the conditions $\lim_{n\to\infty} b_n = 0$ and $0 < \epsilon \leq a_n \leq 1-\epsilon$, for all $n \geq 1$, imposed on Theorems 1.1 and 1.2 in [1], Theorem 1 in [7], Theorems 2 and 3 in [8] and Theorem 1.5 in [9].

Remark 3.7 We would like to point out that $\sum_{n=1}^{\infty}(k_n^p - 1) < \infty$ in [9] and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$ in [1] and [10] are equivalent to condition (3.4).

The following example shows that Theorems 3.1 and 3.2 extend substantially the corresponding results in [1], [5] and [7]–[10].

Example 3.1 Let E be the real line with the usual norm $|\cdot|$ and let $K = [0,1]$. Define S and $T : K \to K$ by

$$Tx = \begin{cases} -\sin x, & x \in [0,1], \\ \sin x, & x \in [-1,0) \end{cases} \quad \text{and} \quad Sx = \begin{cases} x, & x \in [0,1], \\ -x, & x \in [-1,0) \end{cases}$$

for $x \in K$. Obviously $F(S,T) = \{0\}$ and T is semi-compact. Now we check that T is nonexpansive. In fact, if x and $y \in [0,1]$ or x and $y \in [-1,0)$, then $|Tx - Ty| = |\sin x - \sin y| \leq |x-y|$; if $x \in [0,1]$ and $y \in [-1,0)$ or $x \in [-1,0)$ and $y \in [0,1]$, then

$$|Tx - Ty| = |\sin x + \sin y| = 2\left|\sin\frac{x+y}{2}\cos\frac{x-y}{2}\right| \leq |x+y| \leq |x-y|.$$

That is, T is nonexpansive. Similarly we can verify that S is nonexpansive. Thus S is uniformly 1-Lispchitzian and asymptotically nonexpansive. In order to show that S and T satisfy (3.7), we have to consider the following cases:

Case 1. Suppose that x and $y \in [0,1]$. It follows that

$$|x - Ty| = |x + \sin y| = |Sx - Ty|;$$

Case 2. Suppose that x and $y \in [-1,0)$ Then we easily see that

$$|x - Ty| = |x - \sin y| \leq |-x - \sin y| = |Sx - Ty|;$$

Case 3. Suppose that $x \in [-1,0)$ and $y \in [0,1]$. It is easy to verify that

$$|x - Ty| = |x + \sin y| \leq |-x + \sin y| = |Sx - Ty|;$$

Case 4. Suppose that $x \in [0,1]$ and $y \in [-1,0)$. It follows that

$$|x - Ty| = |x - \sin y| = |Sx - Ty|.$$

Hence (3.7) is satisfied. Suppose that $\{u_n\}_{n\geq 1}$, $\{v_n\}_{n\geq 1}$ and $\{w_n\}_{n\geq 1}$ are arbitrary sequences in K,

$$\begin{aligned}
&a = \frac{3}{5}, \quad b = \frac{6}{7}, \quad a_n = \frac{2}{5} - \frac{1}{3n+2} - \frac{1}{6n^2},\\
&a'_{2n} = 1 - \frac{1}{3n} - \frac{1}{2n^2+3,} \quad a'_{2n-1} = \frac{1}{2} - \frac{1}{3n+2} - \frac{1}{2n^2+3},\\
&b_n = \frac{3}{5} + \frac{1}{3n+2,} \quad b'_{2n} = \frac{1}{3n}, \quad b'_{2n-1} = \frac{1}{3n+2} + \frac{1}{2},\\
&c_n = \frac{1}{6n^2}, \quad c'_{2n} = c'_{2n-1} = \frac{1}{2n^2+3},\\
&a''_n = \frac{3}{7} + \frac{1}{12n}, \quad b''_n = \frac{4}{7} - \frac{1}{12n} - \frac{1}{4n^2}, \quad c''_n = \frac{1}{4n^2}
\end{aligned}$$

for $n \geq 1$. Thus the conditions of Theorems 3.1 and 3.2 are fulfilled. Hence Theorems 3.1 and 3.2 guarantee that the modified Ishikawa iteration sequences with errors $\{x_n\}_{n\geq 1}$ with respect to S and T defined by (2.1) converges both strongly and weakly to 0, respectively. But the results in [1], [5] and [7]–[10] are not applicable.

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