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COVERING DENSITIES

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ABSTRACT. In this paper a modification of Buck's measure density is investigated. Some properties of this set function are proved.

Introduction

The notion of measure density was introduced in 1946 by R. C. Buck [3]. The purpose of this paper is to introduce a more general concept of density, the covering density, and to describe some properties of this set function. The measure density of Buck will be a special case of the covering density. In the first two parts we recall the well known notion of a strong submeasure and introduce the notion of covering density. In the third part we shall prove a formula for evaluation of the covering density and show some of its applications. The algebra of measurable sets will be the object of investigation in the next part. Especially, we establish the Darboux property of the covering density on the algebra of measurable sets. The fifth part is devoted to the uniform distribution of sequences, in the sense of I. Niven and its connection with covering density. In 1976 T. Estrada and R. Canvall [9] proved that an infinite series with nonnegative elements converges if and only if this series converges on every set of indexes, with asymptotic density 0. A generalization of this result is in [22]: An infinite series with nonnegative elements converges if and only if this series converges on every set of indexes which belongs to the zero system of a compact submeasure (the system of all sets with submeasure 0). The definition of compact submeasure is also in [22]. The upper asymptotic density and also the Buck's measure density are examples of compact submeasure. It is interesting that the zero system of measure density is much smaller as the system of all sets with asymptotic density 0. In the last part we exhibit also some examples of covering densities with zero system smaller than the zero system of measure density.

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1. Strong submeasure

Let X be an arbitrary set. Denote by $P(X)$ the system of all subsets of the set X.

The set function

$$m^*\colon P(X) \to [0,1]$$

will be called a *strong submeasure* if it has the following properties:

(i) $A \subset B \implies m^*(A) \le m^*(B)$,

(ii) $m^*(A \cup B) + m^*(A \cap B) \le m^*(A) + m^*(B)$,

(iii) $m^*(\emptyset) = 0$, $m^*(X) = 1$,

for every $A, B \in P(X)$.

The set $A \in P(X)$ will be called *measurable* if

$$m^*(A) + m^*(X \setminus A) = 1 . \tag{1}$$

Let us denote the system of all the measurable sets by D_m. From the properties (i) – (iii) it follows that D_m is an algebra of sets, and the function

$$m = m^* \mid D_m$$

is a finitely additive probability measure on D_m. In what follows we give a lot of examples of strong submeasures m^* such that D_m is not σ-algebra and m is not σ-additive. But a certain modified form of σ-additivity still remains valid:

THEOREM 1. *Let A_n, $n = 1,2,\dots$ be a disjoint system of sets such that $A_n \in D_m$. Let*

$$\lim_{n\to\infty} m^*\left(\bigcup_{k=n}^{\infty} A_k\right) = 0 . \tag{2}$$

Then the set $A = \bigcup\limits_{k=1}^{\infty} A_k$ belongs to D_m and

$$m(A) = \sum_{n=1}^{\infty} m(A_n) . \tag{3}$$

Proof. Clearly

$$1 \le m^*(A) + m^*(X \setminus A) . \tag{4}$$

For every $n = 1,2,\dots$ it holds that

$$A_1 \cup A_2 \cup \dots \cup A_n \subset A = A_1 \cup A_2 \cup \dots \cup A_n \cup \bigcup_{k=n+1}^{\infty} A_k .$$

Therefore

$$\sum_{k=1}^{n} m(A_k) \leq m^*(A) \leq \sum_{k=1}^{n} m(A_k) + m^*\Big(\bigcup_{k=n+1}^{\infty} A_k\Big).$$

For $n \to \infty$ we obtain

$$\sum_{k=1}^{\infty} m(A_k) = m^*(A). \tag{5}$$

In the following for $n = 1, 2, \dots$ we have

$$X \setminus A \subset X \setminus (A_1 \cup A_2 \cup \dots A_n),$$

thus

$$m^*(X \setminus A) \leq 1 - \sum_{j=1}^{n} m(A_j).$$

For $n \to \infty$ we have, according to (5)

$$m^*(X \setminus A) \leq 1 - m^*(A).$$

From (4) it follows that

$$m^*(A) + m^*(X \setminus A) = 1.$$

The proof is complete.

COROLLARY. *Let A_n, $n = 1, 2, \dots$ be a disjoint system of sets from D_m. Let $B \in D_m$ be a set for which $A_i \subset B$, $i = 1, 2, \dots$. If*

$$\sum_{i=1}^{\infty} m(A_i) = m(B),$$

then the set $A = \bigcup_{i=1}^{\infty} A_i$ belongs to D_m and

$$m(A) = m(B).$$

P r o o f. For $n = 1, 2, \dots$ it holds that

$$\bigcup_{i=n+1}^{\infty} A_i \subset B \setminus (A_1 \cup A_2 \cup \dots \cup A_n),$$

therefore

$$\lim_{n \to \infty} m^*\Big(\bigcup_{i=n+1}^{\infty} A_i\Big) = 0.$$

Thus, according to Theorem 1 we have $A \in D_m$ and $m(A) = \sum_{i=1}^{\infty} m(A_i) = m(B)$.

2. Covering densities

Let $\mathbb{N}$ be the set of all positive integers. Let the symbol $a + \langle d \rangle$, for $d \in \mathbb{N}$ and a a nonnegative integer denote the arithmetic progression $\{a + nd;\ n = 0, 1, 2 \dots\}$. Instead of $0 + \langle d \rangle$ we simply write $\langle d \rangle$. The greatest common divisor and the least common multiple of the numbers $a, b \in \mathbb{N}$ will be denoted by (a, b) and $[a, b]$, respectively.

The set $A \subset \mathbb{N}$ will be called *closed with respect to divisibility* if it satisfies the following conditions:

(iv) For $a \in A$, $b \in \mathbb{N}$, we have $b \,|\, a \implies b \in A$, and

(v) for $a_1, a_2 \in A$ we have $[a_1, a_2] \in A$.

A trivial example of the set closed with respect to divisibility is the set $\mathbb{N}$. For p prime, also the set $\{p^n;\ n = 0, 1, \dots\}$ is closed according to divisibility. A lot of examples of such sets will be constructed in the Section 6.

In what follows we shall assume that A is an infinite set closed with respect to divisibility. Denote by the symbol S_A the system of all the arithmetic progressions $a + \langle d \rangle$, where $d \in A$. For an arithmetic progression $H = a + \langle d \rangle \in S_A$ put

$$\Delta(H) = \frac{1}{d}.$$

Let H denote also the set of elements of H.

Let $S \subset \mathbb{N}$. Then the value

$$\mu_A^*(S) = \inf\left\{ \sum_{i=1}^{k} \Delta(H_i);\ S \subset \bigcup_{i=1}^{k} H_i \wedge H_i \in S_A \right\}$$

will be called the *covering density* of the set S according to A (or, briefly covering density of S).

If we denote by $\mu^*(S)$ the measure density of the set S, as introduced in [3], then clearly for every $S \subset \mathbb{N}$ it holds that $\mu^*(S) = \mu_{\mathbb{N}}^*(S)$. Therefore the notion of the covering density according to A is a generalization of μ^*.

It follows trivially from the definition that for every set S it holds

$$\mu^*(S) \le \mu_A^*(S). \tag{6}$$

For a set function m let $Z(m)$ be the zero system of m, i.e. the system of all sets $S \subset \mathbb{N}$ for which $m(A) = 0$. The relation (6) implies

$$Z(\mu_A^*) \subset Z(\mu^*). \tag{7}$$

In the following we give many examples when the equality in (7) does not hold. These cases are interesting in connection with the Theorem proved in [22] (see Introduction).

3. The limit formula

In this part we shall prove one formula for evaluation of the covering density $\mu_A^*(S)$. This formula is a generalization of Theorem 1 from [23]. Using it we shall establish some properties of μ_A^*.

The sequence $\{B_n\}$ of positive integers will be called *complete in* A if

(vi) $B_n \in A$, for $n = 1, 2, \dots$, and

(vii) for every $d \in A$ there exists an index n_0 such that for $n \geq n_0$ we have $d \mid B_n$.

We have assumed that A is an infinite set. Let $A = \{A_1 < A_2 < \dots\}$. Then according to (v) the sequence

$$\{[A_1, \dots, A_n]\} \tag{8}$$

forms an example of a sequence which is complete in A. Let $\{m_n\}$ be an increasing subsequence, of (8). Then $\{m_n\}$ is also a sequence which is complete in A, moreover $m_i < m_{i+1}$ and $m_i \mid m_{i+1}$, $i = 1, 2, \dots$. From this we immediately obtain the inequality

$$m_i \geq 2^{i-1}, \qquad i = 1, 2, \dots . \tag{9}$$

For $a, b \in \mathbb{N}$ denote by $a \mod b$ the remainder obtained by dividing a by b. For a set $S \subset \mathbb{N}$ and $b \in \mathbb{N}$ put

$$S \mod b = \{s \mod b;\ s \in S\}.$$

This set will be called the system of representatives of the set S *modulo* b. Let $R(S, b)$ be the number of elements of the set $S \mod b$. Clearly $R(S, b) \leq b$.

THEOREM 2. *Let $\{B_n\}$ be a sequence which is complete in A. Then for every $S \subset \mathbb{N}$ we have*

$$\mu_A^*(S) = \lim_{n \to \infty} \frac{R(S, B_n)}{B_n}.$$

P r o o f. Let $\{a_1, \dots, a_{k(n)}\}$ be the system of representatives of the S modulo B_n, $n = 1, 2, \dots$. Then

$$k(n) = R(S, B_n)$$

and

$$S \subset \bigcup_{j=1}^{k(n)} a_j + \langle B_n \rangle.$$

According to (vi) $B_n \in A$, $n = 1,2,\dots$, thus from the definition of μ_A^* it follows that

$$\mu_A^*(S) \le \frac{R(S,B_n)}{B_n}, \qquad n = 1,2,\dots. \tag{10}$$

Let $\varepsilon > 0$. Then according to the definition of $\mu_A^*(S)$ there exists a disjoint system of arithmetic progressions $a_1 + \langle d_1\rangle, \dots, a_k + \langle d_k\rangle \in S_A$ such that

$$S \subset \bigcup_{i=1}^{k} a_i + \langle d_i\rangle \tag{11}$$

and

$$\frac{1}{d_1} + \dots + \frac{1}{d_k} \le \mu_A^*(S) + \varepsilon. \tag{12}$$

By (v), $[d_1,\dots,d_k] \in A$. Therefore according to (vii) there exists n_0 such that for $n \ge n_0$, $d_i \mid B_n$, $i = 1,2,\dots,k$. This divisibility relation implies that the arithmetic progression $a_i + \langle d_i\rangle$, $i = 1,2,\dots,k$ can be represented as a disjoint union of arithmetic progressions of the form

$$a_i + \langle d_i\rangle = \bigcup_{r=0}^{k_i^{(n)}} a_i + rd_i + \langle B_n\rangle,$$

where $k_i^{(n)} = \frac{B_n}{d_i} - 1$, $i = 1,2,\dots,k$, $n \ge n_0$. Consequently

$$\bigcup_{i=1}^{k} a_i + \langle d_i\rangle = \bigcup_{j=1}^{R_n} b_j^n + \langle B_n\rangle, \qquad n \ge n_0,$$

where $b_1^n,\dots,b_{R_n}^n \in \mathbb{N}$, $n \ge n_0$. In addition,

$$\frac{R_n}{B_n} = \frac{1}{d_1} + \dots + \frac{1}{d_k}, \qquad n \ge n_0. \tag{13}$$

The system of representatives of the set S has $R(S,B_n)$ elements. Two integers contained in the same arithmetic progression $b + \langle B_n\rangle$ are congruent modulo B_n. Therefore according to (11) we have

$$R(S,B_n) \le R_n, \qquad n \ge n_0.$$

Thus, by (10), (12) and (13) we have for $n \ge n_0$

$$\mu_A^*(S) \le \frac{R(S,B_n)}{B_n} \le \mu_A^*(S) + \varepsilon.$$

The proof is complete.

Let $\{m_n\}$ be an increasing sequence, selected from the sequence (8). According to Theorem 2 for every $S \subset \mathbb{N}$ we have

$$\mu_A^*(S) = \lim_{n\to\infty} \frac{R(S, m_n)}{m_n} . \tag{14}$$

For $a \in \mathbb{N}$ and $S \subset \mathbb{N}$ put

$$aS = \{as\,;\ s \in S\} .$$

From (14) we have immediately:

COROLLARY. *If $(a, d) = 1$, for every $d \in A$, then for every set $S \subset \mathbb{N}$ it holds that*

$$\mu_A^*(aS) = \mu_A^*(S) .$$

Let $M = \{m_1, m_2, \dots\}$ be the set of elements of the sequence $\{m_n\}$. Then according to (14) and (9) we have

$$\mu_A^*(M) \le \lim_{n\to\infty} \frac{n+1}{2^n} = 0 .$$

Consider the set

$$G_M = \{n + m_n\,;\ n = 1, 2, \dots\} .$$

It is easy to see that for every $j = 1, 2, \dots$ the numbers $k + m_k\,;\ k = j, \dots, j + m_j - 1$ are incongruent modulo m_j. Therefore it follows from (14) that

$$\mu_A^*(G_M) = 1 .$$

But by (9) the asymptotic density of the set G_M is equal to zero.

Choose $a \in \mathbb{N}$ such that $(a, d) = 1$ for every $d \in A$. Consider the set

$$G_M^a = \{na + m_n\,,\ n = 1, 2, \dots\} .$$

In a similar way we can prove

$$\mu_A^*(G_M^a) = 1 .$$

All elements from G_M^a are relatively prime to a. Thus from the definition of Buck's measure density we have

$$\mu^*(G_M^a) \leq \frac{\varphi(a)}{a} < 1\,,$$

where φ is the known Euler function. From (9) it is also easily seen that the asymptotic density of G_M^a is zero.

It is clear that for S_1, $S_2 \subset \mathbb{N}$ and $n = 1, 2, \dots$ it holds that

$$\begin{gathered} R(S_1, m_n) + R(S_2, m_n) \geq R(S_1 \cup S_2, m_n) + R(S_1 \cap S_2, m_n) \\ S_1 \subset S_2 \implies R(S_1, m_n) \leq R(S_2, m_n)\,. \end{gathered}$$

Therefore according to (14) the set function μ_A^* is a strong submeasure in the sense of chapter 1.

4. Measurable sets

The algebra of measurable sets according to μ_A^* will be denoted by D_A, (instead of D_{μ_A}). Only in the case $A = \mathbb{N}$ we shall use the symbol D_μ, according to Buck's notation in [3]. On the algebra D_A we have the finitely-additive probability measure

$$\mu_A = \mu_A^* \mid D_A\,.$$

If $S \in D_A$, then from (6) it follows that

$$1 \leq \mu^*(S) + \mu^*(\mathbb{N} \setminus S) \leq \mu_A^*(S) + \mu_A^*(\mathbb{N} \setminus S) = 1\,. \tag{15}$$

Thus $\mu^*(S) + \mu^*(\mathbb{N} \setminus S) = 1$. Consequently $D_A \subset D_\mu$. According to (15) it follows that for every $S \in D_A$ it holds that

$$\mu(S) = \mu_A(S)\,.$$

Therefore by virtue of Theorem 2 from [23] every set from D_A has asymptotic density equal to the covering density of this set according to A. In what follows we shall show that $D_A = D_\mu$ if and only if $A = \mathbb{N}$.

For an arbitrary $d \in \mathbb{N}$ let

$$h(d) = \sup\{k = 0, 1, \dots\,;\ d^k \in A\}\,.$$

For every positive integer m with representation as a product of primes

$$m = p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k}$$

we can define a positive integer

$$g_A(m) = p_1^{r_1} \cdot \ldots \cdot p_k^{r_k} ,$$

where

$$r_i = \min\{\alpha_i, h(p_i)\} , \qquad i = 1, 2, \ldots, k .$$

Clearly $p_i^{r_i} \in A$, $i = 1, 2, \ldots, k$ and so $g_A(m) \in A$ according to (v), for every $m \in \mathbb{N}$. From this we see that for every $m \in \mathbb{N}$ there exists j_0 such that

$$g_A(m) \mid m_j ; \qquad j \geq j_0 .$$

Therefore for every m_j and $a \in \mathbb{N}$, $S \subset \mathbb{N}$ there holds

$$R(a + S, m_j) = R(S, m_j) .$$

Now, (14) and the above imply:

LEMMA 1. *For $a \in \mathbb{N}$ and $S \subset \mathbb{N}$ it holds*

$$\mu_A^*(a + S) = \mu_A^*(S) .$$

LEMMA 2. *For every positive integer m and $d \in A$ it holds that*

$$g_A(m) \mid d \implies (m, d) = g_A(m) .$$

P r o o f. It is trivial that $g_A(m) \mid m$. Therefore from the condition

$$g_A(m) \mid d$$

we have $g_A(m) \mid (d, m)$.

Let p be a prime, such that p has in the representation of m as a product of primes the exponent α. Let $p^\beta \mid (m, d)$, then $\beta \leq \alpha$ and $\beta \leq h(p)$. Thus

$$p^\beta \mid g_A(m) .$$

Considering all primes from the representation of (m, d) we have $(m, d) \mid g_A(m)$. The proof is complete.

THEOREM 3. *For every $m \in \mathbb{N}$ and a nonnegative integer we have*

$$\mu_A^*(a + \langle m \rangle) = \frac{1}{g_A(m)} .$$

P r o o f . According to Lemma 1 it is sufficient to prove that

$$\mu_A^*(\langle m \rangle) = \frac{1}{g_A(m)} .$$

The value $R(\langle m \rangle, m_k)$, $k = 1, 2, \dots$ denotes the number of such $j \in \{0, \dots, m_k - 1\}$, for which the congruence

$$xm \equiv j \pmod{m_k}$$

has a solution. There are exactly the j's which are divisible by (m, m_k). The number of such j is exactly

$$\frac{m_k}{(m, m_k)} .$$

From (14) it follows that

$$\mu_A^*(\langle m \rangle) = \lim_{k \to \infty} \frac{1}{(m, m_k)} .$$

According to the condition (vii) we see that there exists k_0, such that for $k \geq k_0$ it holds that $g_A(m) \mid m_k$. From Lemma 2 we have for $k \geq k_0$

$$(m, m_k) = g_A(m) .$$

The proof is complete.

COROLLARY 1. *If $H = a + \langle d \rangle \in S_A$, then*

$$\mu_A^*(H) = \Delta(H) .$$

COROLLARY 2. *If m is a positive integer such that $(m, d) = 1$ for every $d \in A$, then for every nonnegative integer a we have*

$$\mu_A^*(a + \langle m \rangle) = 1 .$$

LEMMA 3. *Let S_1, $S_2 \subset \mathbb{N}$. If there exists $a+\langle d\rangle \in S_A$ such that $S_1 \subset a+\langle d\rangle$ and $S_2 \cap a+\langle d\rangle = \emptyset$, then*

$$\mu_A^*(S_1 \cup S_2) = \mu_A^*(S_1) + \mu_A^*(S_2).$$

Proof. There exists k_0 such that for $k \geq k_0$ we have $d \mid m_k$. Therefore for $k \geq k_0$

$$R(S_1 \cup S_2, m_k) = R(S_1, m_k) + R(S_2, m_k).$$

According to Theorem 2 we have the assertion. The proof is complete.

According to Corollary 1 and Lemma 3 for the disjoint system $H_1, H_2, \dots, H_k \in S_A$ it holds that

$$\mu_A^*\big(H_1 \cup \dots \cup H_k\big) = \Delta(H_1) + \dots + \Delta(H_k).$$

Immediately from these facts we have $S_A \subset D_A$.

Assume that $A \neq \mathbb{N}$. Then there exists a prime p such that $h(p) < \infty$. Let $h(p) = k$. Consider the arithmetic progression $\langle p^{k+1}\rangle$. According to Theorem 3 we have that

$$\mu_A^*(\langle p^{k+1}\rangle) = \frac{1}{p^k} > \mu^*(\langle p^{k+1}\rangle).$$

Thus $\langle p^{k+1}\rangle$ does not belong to D_A. It is trivial that $\langle p^{k+1}\rangle \in D_\mu$, therefore $D_A \neq D_\mu$. In this case the algebra D_A does not contain all arithmetic progressions.

COROLLARY 3. *Let $S \subset \mathbb{N}$. Then $\mu_A^*(S) = 1$ if and only if for every $H \in S_A$ we have $S \cap H \neq \emptyset$.*

In Buck's paper [3] it is proved that

$$\big\{\mu(S);\ S \in D_\mu\big\} = \langle 0, 1\rangle.$$

Using an analogous method we prove a more general result that the measure μ_A has the Darboux property on the algebra D_A:

THEOREM 4. *Let $S \in D_A$. Then for every $\alpha \in \big\langle 0, \mu_A(S)\big\rangle$ there exists a set $S_1 \subset S$ such that $S_1 \in D_A$ and $\mu_A(S_1) = \alpha$.*

Proof. If $\mu_A(S) = \alpha$, the assertion is trivial. Let $\alpha < \mu_A(S)$. Then there exists $\varepsilon > 0$ such that

$$\alpha \leq \mu_A(S) - \varepsilon.$$

From the condition $S \in D_A$ it follows that there exists a disjoint system of arithmetic progressions $a_1 + \langle d \rangle, \dots, a_k + \langle d \rangle \in S_A$ such that

$$\bigcup_{i=1}^{k} a_i + \langle d \rangle \subset S$$

and

$$\frac{k}{d} > \mu_A(S) - \varepsilon \geq \alpha . \tag{16}$$

Put $d_0 = d$. Let $\{d_i\}_{i=0}^{\infty}$ be such a sequence of positive integers that $d_0 \dots d_i \in A$, for $i = 1, 2, \dots$. (An example of such sequence is $d_0 = m_0$, $d_1 = \frac{m_1}{m_0}$, $d_2 = \frac{m_2}{m_1}, \dots$.) We can express the number α by Cantor's series

$$\alpha = \sum_{j=0}^{\infty} \frac{c_j}{d_0 \dots d_j}, \qquad 0 \leq c_j < d_j, \quad j = 0, 1, \dots .$$

According to (16) we have $c_0 < k$. Put

$$H_0 = \bigcup_{i=1}^{c_0} a_i + \langle d \rangle .$$

Then $H_0 \cap a_k + \langle d \rangle = \emptyset$ and $\mu_A(H_0) = \frac{c_0}{d_0}$. Let us denote for $n = 1, 2, \dots$

$$H_n = \bigcup_{j=1}^{c_n} a_k + j d_0 \dots d_{n-1} + \langle d_0 \dots d_n \rangle .$$

The union on the right-hand side is disjoint and therefore

$$\mu_A(H_n) = \frac{c_n}{d_0 \dots d_n} .$$

Assume that for $m < n$ it holds

$$H_n \cap H_m \neq \emptyset .$$

Then there exist numbers $j, j_1, h, h_1 \in \mathbb{N}$, such that $0 < j \leq c_m$, $0 < j_1 \leq c_n$ and

$$j_1 d_0 \dots d_{n-1} + h_1 d_0 \dots d_n = j d_0 \dots d_{m-1} + h d_0 \dots d_m .$$

Therefore $d_0 \dots d_m \mid j d_0 \dots d_{m-1}$ – a contradiction. It is obvious that for $n > m$

$$\bigcup_{k=n}^{\infty} H_n \subset a_k + \langle d_0 \dots d_{n-1} \rangle .$$

Therefore

$$\lim_{n \to \infty} \mu_A^* \left(\bigcup_{k=n}^{\infty} H_k \right) = 0 .$$

According to Theorem 1 we now see that the set

$$S_1 = \bigcup_{n=0}^{\infty} H_n$$

belongs to D_A and $\mu_A(S_1) = \alpha$. It is trivial that $S_1 \subset S$. The proof is complete.

COROLLARY 1. *We have*

$$\{\mu_A(S);\ S \in D_A\} = \langle 0, 1 \rangle .$$

COROLLARY 2. *We have*

$$\{\mu_A^*(S);\ S \subset G_M\} = \langle 0, 1 \rangle .$$

Proof. Let $\alpha \in \langle 0, 1 \rangle$. Then there exists $S \in D_A$ such that $\mu_A(S) = \alpha$. Let

$$S = \{a_1 < a_2 < \dots\} .$$

Put

$$H = \{a_n + m_{a_n};\ n = 1, 2, \dots\} .$$

It is trivial that $H \subset G_M$. The sequence $\{m_{a_n}\}$ is complete in A. For every k we have

$$a_n + m_{a_n} \equiv a_n \pmod{m_{a_k}}; \qquad n \geq k .$$

Therefore

$$R(H, m_{a_k}) = R(S, m_{a_k}) + O(k) .$$

This equation according to (9) and (14) implies

$$\mu_A^* = \alpha .$$

The proof is complete.

Denote by S^0 the system of sets $S \subset \mathbb{N}$, having the asymptotic density zero. Then immediately from Corollary 2 we have

$$\{\mu_A^*(S);\ S \in S^0\} = \langle 0, 1 \rangle .$$

5. Uniform Distribution

In 1961 I. Niven in [18], introduced the notion of a sequence *uniformly distributed* modulo m, where $m \in \mathbb{N}$, in the following way: The sequence of positive integers $\{x_n\}$ is uniformly distributed modulo m if and only if for every $j \in \mathbb{N}$ it holds that

$$\frac{1}{N} \sum_{\substack{n \le N \\ x_n \equiv j \pmod{m}}} 1 \to \frac{1}{m} \qquad \text{for} \quad N \to \infty .$$

Let $H \subset \mathbb{N}$. We say that the sequence of positive integers $\{x_n\}$ is *uniformly distributed in* H if it is uniformly distributed modulo m for every $m \in H$.

The following theorem shows a natural connection between the uniform distribution by A and μ_A^*.

THEOREM 5. *Let $S \subset \mathbb{N}$. Then $\mu_A^*(S) = 1$ if and only if S can be arranged in form of a sequence $\{x_n\}$ which is uniformly distributed in A.*

For the proof we will use the following lemma:

LEMMA 4. *Let $\{x_n\}$ be a sequence of positive integers satisfying the condition*

$$x_k \equiv k \pmod{m_k}, \qquad k = 1, 2, \dots .$$

Then $\{x_n\}$ is uniformly distributed in A.

Proof. Let $m \in A$. Then there exists n_0 such that $m \mid m_n$ for $n > n_0$. Thus for $n > n_0$

$$x_n \equiv n \pmod{m}.$$

Then for $N \ge n_0$ and $j \in \mathbb{N}$, $0 \le j < m$ we have

$$\begin{aligned}\frac{1}{N} \sum_{\substack{n \le N \\ x_n \equiv j \pmod{m}}} 1 &= \frac{1}{N} \sum_{\substack{n \le n_0 \\ x_n \equiv j \pmod{m}}} 1 + \frac{1}{N} \sum_{\substack{n_0 < n \le N \\ x_n \equiv j \pmod{m}}} 1 \\ &= \frac{1}{N} \sum_{\substack{n \le N \\ n \equiv j \pmod{m}}} 1 + O\Big(\frac{1}{N}\Big) \to \frac{1}{m} \qquad \text{as} \quad N \to \infty ,\end{aligned}$$

and the lemma follows.

Proof of Theorem 5. If $S = \{x_1, x_2, \dots\}$ is a uniformly distributed sequence in A, then by virtue of Corollary 3 of Theorem 3 we have $\mu_A^*(S) = 1$.

If $\mu_A^*(S) = 1$, then S has a non-empty intersection with every arithmetic sequence from S_A. Therefore for every $n \in \mathbb{N}$ there exists $y_n \in S$ such that

$$y_n \equiv n \pmod{m_n}.$$

Then the lemma implies that $\{y_n\}$ is uniformly distributed in A. We can assume that the sequence $\{y_n\}$ is increasing. If the set $S \setminus \{y_n\,;\ n = 1,2,\dots\}$ is finite, then the proof is complete.

Suppose therefore that the set

$$S \setminus \{y_n\,;\ n = 1,2,\dots\} = \{y'_k\,;\ k = 1,2,\dots\}$$

is infinite. Define

$$x_n = \begin{cases} y_n\,, & \text{for } n \neq k^2, \\ y_{k^2}\,, & \text{for } n = (2k)^2, \\ y'_k\,, & \text{for } n = (2k+1)^2, \end{cases} \qquad \text{for} \quad n = 1,2,\dots.$$

Clearly $\{x_n\,;\ n = 1,2,\dots\} = S$. Let $j \in \mathbb{N}$ and $m \in A$. Then for $N \to \infty$

$$\frac{1}{N} \sum_{\substack{n \le N \\ x_n \equiv j \pmod m}} 1 = \frac{1}{N} \sum_{\substack{n \le N \\ y_n \equiv j \pmod m}} 1 + O(N^{-\frac{1}{2}}) \to \frac{1}{m}.$$

Thus the sequence $\{x_n\}$ is uniformly distributed in A. The proof of Theorem 5 is complete.

COROLLARY. *Let $S \subset \mathbb{N}$. Then $\mu^*(S) = 1$ if and only if S can be arranged in form of a sequence $\{x_n\}$ which is uniformly distributed in $\mathbb{N}$.*

Let a be a positive integer such that $(a,m) = 1$ for every $m \in A$. Consider the set

$$G_M^a = \{an + m_n\,;\ n = 1,2,\dots\}.$$

Then according to the results of the third section G_M^a can be arranged in a sequence uniformly distributed in A, but not in a sequence which is uniformly distributed in $\mathbb{N}$. The arithmetic progression $\langle a \rangle$ has also a similarly property.

We shall finish this part by pointing out one more analogy between the uniform distribution by A and the uniform distribution mod 1.

Let $\{x_n\}$ be a sequence of positive integers. Given $S \in P(\mathbb{N})$ and $k \in \mathbb{N}$ let

$$Q(S,\{x_n\},k) = \sum_{\substack{n \le k \\ x_n \in S}} 1.$$

The concept of the uniform distribution by A gives us a further possibility to characterize the algebra D_A. Using a simple estimation directly from definitions we can prove:

THEOREM 6. *The sequence $\{x_n\}$ of positive integers is uniformly distributed in A if and only if for every set $S \in D_A$*

$$\lim_{k\to\infty} \frac{Q(S,\{x_n\},k)}{k} = \mu_A(S).$$

In the proof of the next theorem the following notion will be used: Let $S \in P(\mathbb{N})$ and $n \in \mathbb{N}$. The set $S' \subset S$ will be called a remainder system of the set S modulo n if

(viii) for every $a \in S$ there exists an $a' \in S'$ such that $a \equiv a' \pmod{n}$,

(ix) for every $a', a'' \in S'$ $a' \equiv a'' \pmod{n} \implies a' = a''$.

It is obvious that two remainder systems of the set S modulo n have the same number of elements and that this number is equal to the number of elements of the system of representatives of the set S modulo n.

THEOREM 7. *Let $S \subset \mathbb{N}$. If for every sequence $\{x_n\}$ uniformly distributed in A we have*

$$\lim_{N\to\infty} \frac{Q(S,\{x_n\},N)}{N} = \mu_A^*(S),$$

then $S \in D_A$.

Proof. Let $S \notin D_A$. Then

$$1 - \mu_A^*(\mathbb{N} \setminus S) < \mu_A^*(S). \tag{17}$$

Suppose that the sequence $\{B_n\}$ is complete in A. Suppose that this sequence also satisfies the condition

$$B_n \mid B_{n+1}, \qquad n = 1,2,\dots .$$

Let S'_n be a remainder system of the set S modulo B_n, for $n = 1,2,\dots$. Put $S_1 = S'_1$ and

$$S_n = S'_{n-1} \cup \{y \in S'_n ;\ \forall x \in S_{n-1},\ x \not\equiv y \pmod{B_n}\} \qquad \text{for} \quad n = 2,3,\dots .$$

In this way an increasing sequence of sets S_n

$$S_1 \subset S_2 \subset \dots \subset S_n \subset \dots$$

of remainder systems of the set S modulo B_n can be constructed.

Similarly, there exists a sequence

$$\overline{S}_1 \subset \overline{S}_2 \subset \dots \subset \overline{S}_n \dots$$

such that $\overline{S}_n$ is a remainder system of the set $\mathbb{N}\backslash S$ modulo B_n for $n = 1, 2, \dots$.

Construct the sequence $\{C(B_n)\}$ of sets as follows: The set $C(B_n)$ is the complete remainder system modulo B_n ($n = 1, 2, \dots$) which consists of the elements of $\overline{S}_n$ and $B_n - R(\mathbb{N} \setminus S, B_n)$ elements of S_n.

Clearly

$$C(B_1) \subset C(B_2) \subset \cdots \subset C(B_n) \subset \dots .$$

Put $D_1 = B_1$. Let us rearrange the set $C(D_1)$ into a (finite) sequence

$$C'(D_1) = \{x_0, \dots, x_{D_1 - 1}\}$$

in such a way that $x_j \equiv j \pmod{D_1}$, for $j = 0, \dots, D_1 - 1$. Let

$$D_2 = \min\{B_n\,;\ x_1 < B_n, \dots, x_{D_1-1} < B_n\}.$$

Rearrange the set $C(D_2)$ into the (finite) sequence

$$C'(D_2) = \{x_0, \dots, x_{D_1-1}, x_{D_1}, \dots, x_{D_2-1}\},$$

where $x \equiv j \pmod{D_2}$, $D_1 \le j < D_2$. In this way we can construct a sequence $\{D_n\}$, which is complete in A, and the system of finite sequences

$$C'(D_n) = \{x_0, \dots, x_{D_{n-1}-1}, x_{D_{n-1}}, \dots, x_{D_n - 1}\}$$

in which $x_j \equiv j \pmod{D_n}$, $D_{n-1} \le j < D_n$.

Consider the sequence

$$\{x_n\} = \bigcup_{n=1}^{\infty} C'(D_n),$$

in which the elements are written in such a way that we begin with elements of the sequence $C'(D_1)$, then there follow the remaining elements of the sequence $C'(D_2)$, etc.. For $d \in A$ there exists n_0 such that $d \mid D_{n_0}$. Therefore for $j > D_{n_0}$ we have

$$x_j \equiv j \pmod{d}.$$

This implies that the sequence $\{x_n\}$ is uniformly distributed in A.

If $n = 1, 2, \dots$, then

$$Q(S, \{x_j\}, D_n) = D_n - R(\mathbb{N} \setminus S, D_n).$$

Because of (17) and Theorem 2 we have

$$\lim_{n \to \infty} \frac{Q(S, \{x_j\}, D_n)}{D_n} < \mu_A^*(S).$$

The proof is complete.

6. Special cases of system A

Put for $k = 1, 2, \ldots$

$$P_k = \{n^k \,;\ n = 1, 2, \ldots\}.$$

In the paper by B u c k [3] it was proved that $\mu^*(P_2) = 0$.

Let p be an odd prime. In [3] it was proved that for every $n = 1, 2, \ldots$

$$R(P_2\,;\,p^n) = \frac{p^{n+1}}{2p+2} + O(1). \tag{20}$$

Consider the set $A = \{1, p, p^2, \ldots\}$ and the sequence $\{m_n\}$ in the form

$$m_n = p^n, \qquad n = 1, 2, \ldots.$$

According to (14) a (20) we have in this case

$$\mu_A^*(P_2) = \frac{p}{2p+2} > 0.$$

Thus $Z(\mu_A^*)$ is a nontrivial subset of $Z(\mu^*)$.

Let $H \subset \mathbb{N}$. Denote by the symbol $A(H)$ the set which is closed according to divisibility generated by H in the following sense: The set A closed according to divisibility is generated by H if and only if

(x) $H \subset A$, and

(xi) if A_1 is closed according to divisibility and $H \subset A_1$ then $A \subset A_1$.

It is easy to see that if $p_1 \neq p_2$ are primes and $H = \{p_1, p_2\}$, then $A(H) = \{1, p_1, p_2, p_1 p_2\}$.

Let us remark that for every set A closed according to divisibility we have $A = A(H)$, where

$$H = \bigcup_{\substack{p \in A \\ h(p) < \infty}} \{p^{h(p)}\} \cup \bigcup_{\substack{p \in A \\ h(p) = \infty}} \left(\bigcup_{n=1}^{\infty} \{p^n\} \right), \qquad p \text{ – prime}.$$

It is also easy to see that if $H = \{p_1, p_2, \ldots\}$, where p_i, $i = 1, 2, \ldots$ are primes, then $A(H)$ is the set of all numbers in the form

$$\prod_{i=1}^{\infty} p_i^{\alpha_i},$$

where $\alpha_i \in \{0, 1\}$ and only for a finite number of i we have $\alpha_i \neq 0$.

As a sequence $\{m_n\}$ we can in this case consider the sequence

$$m_n = p_1 \ldots p_n, \qquad n = 1, 2, \ldots.$$

In what follows we shall use the following lemma:

LEMMA 5. *If* $(a.b) = 1$, *then for* $k = 1, 2, \dots$ *we have*

$$R(P_k, ab) = R(P_k, a) \cdot R(P_k, b).$$

Proof. Clearly if $c \in P_k \mod ab$ then $c \mod a \in P_k \mod a$, $c \mod b \in P_k \mod b$.

Thus we can define a mapping

$$F \colon P_k \mod ab \to (P_k \mod a) \times (P_k \mod b)$$
$$F(c) = (c \mod a, c \mod b).$$

It is sufficient to prove that F is a bijection. From the condition $(a, b) = 1$ it follows that F is an injection.

Let $c_1 \in P_k \mod a$, $c_2 \in P_k \mod b$. From the Chinese remainder theorem we have that there exists such a $c \in \{0, \dots, ab-1\}$ that $c \equiv c_1 \pmod{a}$ and $c \equiv c_2 \pmod{b}$. Therefore there exists d_1, d_2 such that $c \equiv d_1^k \pmod{a}$ a $c \equiv d_2^k \pmod{b}$. Again according to the Chinese Remainder Theorem we obtain that there exists d such that $d \equiv d_1 \pmod{a}$ and $d \equiv d_2 \pmod{b}$. Therefore $c \equiv d^k \pmod{a}$ and $c \equiv d^k \pmod{b}$. Thus $c \equiv d^k \pmod{ab}$, and so $c \in P_k \mod ab$. Clearly, $F(c) = (c_1, c_2)$, therefore F is a bijection. The proof is complete.

Using the Dirichlet theorem on primes in the arithmetic progression (see [10]) we have immediately

LEMMA 7. *If k is an odd positive integer, then there exists an infinite system of primes p satisfying the condition*

$$(k, p-1) = 1. \tag{21}$$

LEMMA 8. *For every $k = 1, 2, \dots$ there exists an infinite system of primes p satisfying the condition*

$$k \mid p - 1. \tag{22}$$

THEOREM 8. *Let k be odd and $H = \{p_1, p_2, \dots\}$, where p_i, $i = 1, 2, \dots$ are primes satisfying* (21). *Let $A = A(H)$. Then*

$$\mu_A^*(P_k) = 1.$$

Proof. Put $m_j = p_1 \cdot \ldots \cdot p_j$, $j = 1, 2, \dots$. We show that for every j there holds

$$R(P_k, m_j) = m_j.$$

According to Lemma 6 it is sufficient to prove

$$R(P_k, p_i) = p_i, \qquad i = 1, 2, \dots . \tag{23}$$

Assume that for $x, y \in \mathbb{N}$

$$x^k \equiv y^k \pmod{p_i}. \tag{24}$$

If $x \equiv 0 \pmod{p_i}$, then $y \equiv 0 \pmod{p_i}$. Let $x \not\equiv 0 \pmod{p_i}$ and $y \not\equiv 0 \pmod{p_i}$. Let g_i be a primitive root modulo p_i. Then there exist r, l such that

$$x \equiv g_i^r \pmod{p_i}, \qquad y \equiv g_i^l \pmod{p_i}.$$

From (24) we have

$$g_i^{rk} \equiv g_i^{lk} \pmod{p_i}.$$

Therefore

$$(l - r)k \equiv 0 \pmod{p_i - 1},$$

thus according to (21) $l \equiv r \pmod{p_i - 1}$, which implies $x \equiv y \pmod{p_i}$. We have proved (23) and so according to Lemma 6 and Theorem 2 the proof is complete.

In contrast to Theorem 8 we prove the following assertion:

THEOREM 9. *Let $k > 1$ and $H = \{p_1, p_2, \dots\}$, where p_i, $i = 1, 2, \dots$ are primes satisfying (22). Put $A = A(H)$. Then*

$$\mu_A^*(P_k) = 0.$$

Proof. Put again $m_j = p_1 \cdot \ldots \cdot p_j$. Let g_i be a primitive root modulo p_i. Denote

$$h_j = g_j^{\frac{p_j - 1}{k}}.$$

Then $h_j \not\equiv 1 \pmod{p_j}$, thus 1 and h_j are two different roots of the congruence $x^k \equiv 1 \pmod{p_j}$. Then for every $a \in \mathbb{N}$, $(a, p_j) = 1$, we have

$$a \not\equiv a h_j \pmod{p_j},$$

but

$$a^k \equiv (a h_j)^k \pmod{p_j}, \qquad j = 1, 2, \dots .$$

From this it follows for $j = 1, 2, \dots$ that

$$R(P_k, p_j) \le \frac{p_j - 1}{2} + 1 = \frac{p_j + 1}{2}.$$

From Lemma 6 we have

$$\frac{R(P_k, m_j)}{m_j} \le \frac{1}{2^j}\left(1 + \frac{1}{p_1}\right)\dots\left(1 + \frac{1}{p_j}\right). \tag{25}$$

But

$$\left(1 + \frac{1}{p_1}\right)\dots\left(1 + \frac{1}{p_j}\right) \le \left(\frac{3}{2}\right)^j.$$

From this according to (14) and (25) we obtain

$$\mu_A^*(P_k) = 0.$$

The proof is complete.

According to (6) we obtain

COROLLARY. *For every $k > 1$ we have*

$$\mu^*(P_k) = 0.$$

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