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TWO REMARKS ON DUALLY RESIDUATED LATTICE ORDERED SEMIGROUPS

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ABSTRACT. It is proved that in a system of axioms of a dually residuated lattice ordered semigroup, the identity $x - x \geq 0$ is implied by the remaining axioms. Further, S w a m y 's problem of whether certain autometrics in a dually residuated lattice ordered semigroup are identical is solved.

Dually residuated lattice ordered semigroups are certain ordered algebraic structures that generalize simultaneously abelian lattice ordered groups and Brouwerian algebras. They were introduced in the mid-60's by K. L. N. S w a m y [2].

An algebra $A = (A; 0; +; -; \wedge; \vee)$ of type $\langle 0; 2; 2; 2; 2\rangle$ is a Dually Residuated Lattice Ordered Semigroup (abbreviated as a DRℓ-semigroup) if the following holds:

(i) $(A; 0; +; \wedge; \vee)$ is a commutative lattice ordered monoid, that is:

- (a) $(A; 0; +)$ is a commutative monoid,
- (b) $(A; \wedge; \vee)$ is a lattice (the induced order is denoted by $\leq$),
- (c) $(x \wedge y) + z = (x + z) \wedge (y + z)$ for all $x, y, z \in A$,
- (d) $(x \vee y) + z = (x + z) \vee (y + z)$ for all $x, y, z \in A$,

(ii) $(x - y) + y \geq x$, and if $z + y \geq x$, then $z \geq x - y$ for all $x, y, z \in A$,

(iii) $(x - y) \vee 0 + y \leq x \vee y$ for all $x, y \in A$,

(iv) $x - x \geq 0$ for each $x \in A$.

1. LEMMA. (cf. also [2; Lemma 2]) *Let* $A = (A; 0; +; -; \wedge; \vee)$ *be an algebra of type* $\langle 0; 2; 2; 2; 2\rangle$ *satisfying the conditions* (i), (ii) *and* (iii). *If* $x \in A$ *and* $x \leq 0$, *then* $(0 - x) \vee 0 + x = 0$.

P r o o f . From (iii) it follows that $(0 - x) \vee 0 + x \leq x \vee 0 = 0$ and (i) yields $(0 - x) \vee 0 + x = \big[(0 - x) + x\big] \vee x \geq 0 \vee x = 0$. □

A M S S u b j e c t C l a s s i f i c a t i o n (1991): Primary 06F05.
K e y w o r d s: autometric, Brouwerian algebra, lattice ordered group, lattice ordered monoid.

2. THEOREM. *In an algebra* $A = (A; 0; +; -; \wedge; \vee)$ *of type* $\langle 0; 2; 2; 2; 2 \rangle$ *satisfying the conditions* (i), (ii) *and* (iii), *the following identity holds:*

$$x - x = 0.$$

P r o o f . From $0 + x \geq x$ it follows $0 \geq x - x$ and $\big[(x-x)+(x-x)\big] + x = (x-x) + \big[(x-x)+x\big] \geq (x-x)+x \geq x$ yields $(x-x)+(x-x) \geq x-x$. By Lemma 1 we conclude $x - x = \big\{[0-(x-x)] \vee 0 + (x-x)\big\} + (x-x) = \big[0-(x-x)\big] \vee 0 + \big[(x-x)+(x-x)\big] \geq \big[0-(x-x)\big] \vee 0 + (x-x) = 0$. Hence $x - x = 0$. □

3. COROLLARY. *The axiom* (iv) *is not independent.*

In a DRℓ-semigroup, the following autometrics were introduced by S w a m y , [2] and [3]:

$$x \star y = (x-y) \vee (y-x) \qquad \text{and} \qquad x \star y = (x-y) \vee 0 + (y-x) \vee 0.$$

The following theorem offers a solution of S w a m y 's problem ([3]), whether these autometrics are identical.

4. THEOREM. *In any DRℓ-semigroup, the following identity holds:*

$$(x-y) \vee 0 + (y-x) \vee 0 = (x-y) \vee (y-x).$$

P r o o f . [2; Lemmas 1, 4, 5, 11 and 15] yield $(x-y)\vee 0+(y-x)\vee 0 = (x-y)\vee(y-y)+(y-x)\vee(y-y) = (x\vee y-y)+(y-x\wedge y) = (x\vee y-x\wedge y) = (x-y)\vee(y-x)$. □

5. COROLLARY. *Swamy's autometrics are identical.*

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