Wolfgang Schwarz; Thomas Maxsein; Paul Smith
An example for Gelfand's theory of commutative Banach algebras

Mathematica Slovaca, Vol. 41 (1991), No. 3, 299--310

Persistent URL: http://dml.cz/dmlcz/133212

Math. Slovaca 41, 1991, No 3., 299–310

AN EXAMPLE FOR GELFAND'S THEORY OF COMMUTATIVE BANACH ALGEBRAS

WOLFGANG SCHWARZ — THOMAS MAXSEIN — PAUL SMITH

ABSTRACT. Beginning with the $\mathbf{C}$-vector-spaces $\mathcal{B}$, resp. $\mathcal{D}$, spanned by the Ramanujan sums c_r, resp. the exponential functions $n \mapsto \exp(2\pi i \frac{a}{r} n)$, and using the supremum-norm $\|f\|_u = \sup|f(n)|$, the $\|\cdot\|_u$-closures $\mathcal{B}^u$ and $\mathcal{D}^u$ can be defined. According to Gelfand's theory of commutative Banach algebras, these spaces are isomorphic with the algebra $\mathcal{C}(\Delta)$ of continuous functions on the "maximal ideal space" Δ.

The maximal ideal spaces $\Delta_{\mathcal{B}}$ and $\Delta_{\mathcal{D}}$ are constructed, and the knowledge of these allows to deduce some properties of the function spaces $\mathcal{B}^u$ and $\mathcal{D}^u$.

1. Introduction

Denote by $\mathcal{B}$ (resp. $\mathcal{D}$) the complex vector space of linear combinations of *Ramanujan sums*

$$c_r\colon n \mapsto \sum_{d|\gcd(r,n)} d\mu\left(\tfrac{r}{d}\right) = \sum_{\substack{1\le a\le r,\\ \gcd(a,r)=1}} \exp(2\pi i \tfrac{a}{r} n).$$

(resp. of *exponential functions* $e_{a/r}: n \mapsto \exp(2\pi i \frac{a}{r} n)$). The Ramanujan sum c_r is even mod r, that means, the values $c_r(n)$ only depend on the greatest common divisor of n and r,

$$c_r(n) = c_r(\gcd(n,r)).$$

The closure of $\mathcal{B}$ (resp. $\mathcal{D}$) with respect to the *supremum-norm*

$$\|f\|_u = \sup_{n\in\mathbf{N}} |f(n)| \tag{1.1}$$

is denoted by $\mathcal{B}^u$ (resp. $\mathcal{D}^u$). These vector-spaces are semi-simple commutative Banach algebras with identity 1 (the constant function). Therefore, by Gelfand's

AMS Subject Classification (1985): Primary 11A25, 11L03. Secondary 11R29, 46J10, 46J20.

Key words: Ramanujan sums, Banach algebras, Maximal ideal space, Gelfand transform

theory (see, for example, R u d i n [7], Chapter 18, or R u d i n [8], Chapter 10,11), the space $\mathcal{B}^u$ of *uniformly-almost-even* arithmetical functions is algebraically and topologically isomorph to the algebra $\mathcal{C}(\Delta_{\mathcal{B}})$ of continuous functions on some space $\Delta_{\mathcal{B}}$, the "maximal ideal space"; the same is true for $\mathcal{D}^u$, the space of *uniformly-limit-periodic* arithmetical functions; its maximal ideal space is denoted by $\Delta_{\mathcal{D}}$.

In fact, the determination of $\Delta_{\mathcal{B}}$ was achieved in S c h w a r z-S p i l k e r [10] by an explicit construction, using the Weierstraß approximation theorem, but not using or mentioning Gelfand's theory

It is the aim of this note to give another explicit determination of $\Delta_{\mathcal{B}}$ and $\Delta_{\mathcal{D}}$, now using some simple facts from Gelfand's theory. Of course, these maximal ideal spaces are known (see, for example, [3], [4], [5], [6])

$\Delta_{\mathcal{B}}$ may be de cribed algebraically as the set of algebra-homomorphisms

$$h\colon \mathcal{B}^u \to \mathbf{C}$$

For any f in $\mathcal{B}^u$ we denote by $\operatorname{spec}(f)$ the set of complex λ, for which the function $f - \lambda\cdot 1$ i not invertible in $\mathcal{B}^u$ (similarly for $\mathcal{D}^u$). From R u d i n [7], 18.17, we quote the following simple properties·

(1) $h(f) \in \operatorname{spec}(f)$ for any $f \in \mathcal{B}^u$ and any $h \in \Delta_{\mathcal{B}}$,
(2) $|h(f)| \leq \; f\|_u$
(3) h is continuou on $\mathcal{B}$ and the operator-no m is $\|h| \leq 1$

2. The maximal ideal space of $\mathcal{B}^u$

a) Construction of some homomorphism . Clearly, for any integer $n \in \mathbf{N}$, the *evaluation* $\;_n\cdot f \mapsto f(n)$ are el m nts o $\Delta_{\mathcal{B}}$. Next, for any prime p, and for $f \in \mathcal{B}^u$, th limit

$$f(p^\infty) - \lim_{k\to\infty} f \; p^k)$$

exists[1], and so th functi n

$$h \;_\infty .\; f \mapsto f(p^\infty$$

are e eme t of Δ

Th argument n b u m e ten ively Given e p n nt k_p for p prın $0 \leq k_p \leq \infty$ a (ompl) v l $f\;\mathcal{K})$ can b d fin d f r th ecto

$$\mathcal{K} \quad (k_p \quad \text{pri} \tag{2 1}$$

[1] Given ε , l o F $\mathcal{B}$ i f i $\|f \quad F|_u <$ The functi F s even a $F\; p^k) = \beta$ i con ta t for $k \_ k_0(p\; \varepsilon$, and the efore $|f(p\;) - \beta| < \varepsilon$ fo these k, th ref r the se i nc $\mapsto f(p\;)$ i a auchy seq nce

in the following manner[2] : consider the monotonely increasing sequence n_r of positive integers

$$n_r = \prod_{1 \le \rho \le r} p_\rho^{\min(r, k_{p_\rho})}, \qquad r = 1, 2, \ldots,$$

with the property $n_r | n_{r+1}$ for any r. Then

$$f(\mathcal{K}) = \lim_{r \to \infty} f(n_r) \tag{2.2}$$

exists[3] , and

$$h_{\mathcal{K}}: f \mapsto f(\mathcal{K}) \tag{2.3}$$

is an element of $\Delta_{\mathcal{B}}$. All these functions $h_{\mathcal{K}}$ are different, as can be seen by evaluating $h_{\mathcal{K}}$ on suitable Ramanujan sums c_{q^ℓ}.

Our goal is to show that we got all the elements of $\Delta_{\mathcal{B}}$. Before doing this, we calculate the values of $h_{\mathcal{K}}$ at Ramanujan sums c_{q^ℓ} for prime powers q^ℓ. Obviously (giving the greatest common divisor on the right-hand-side a natural interpretation)

$$h_{\mathcal{K}}(c_{q^\ell}) = c_{q^\ell}\big(\gcd(\prod_p p^{k_p}, q^\ell)\big),$$

and this equals

$$\begin{cases} = c_{q^\ell}(q^\ell) = \varphi(q^\ell), & \text{if} \quad k_q \ge \ell, \\ = c_{q^\ell}(q^{\ell-1}) = -q^{\ell-1}, & \text{if} \quad k_q = \ell - 1, \\ = 0, & \text{if} \quad k_q < \ell - 1. \end{cases} \tag{2.4}$$

b) Determination of $\Delta_{\mathcal{B}}$. We are going to prove

Theorem 2.1. *The maximal ideal space $\Delta_{\mathcal{B}}$ consists exactly of the functions $h_{\mathcal{K}}$, defined in (2.3), where $\mathcal{K}$ runs through the set of vectors $(k_p)_{p \text{ prime}}$, with $0 \le k_p \le \infty$.*

Assume $h \in \Delta_{\mathcal{B}}$; h being continuous it is sufficient to know the values of h on the subalgebra $\mathcal{B}$ of $\mathcal{B}^u$. The Ramanujan sums c_r, considered as functions of the index r, are multiplicative. Therefore it is sufficient to know the values

$$h(c_{q^\ell}) \qquad \text{for prime-powers} \quad q^\ell.$$

[2] We think of the sequence of primes being ordered according to their size. An integer n may be described as a special vector $\mathcal{K}$, where at most finitely k_ρ are non-zero and none is infinity.
[3] Given $\varepsilon > 0$, choose $F \in \mathcal{B}$ satisfying $\|f - F\|_u < \varepsilon$. The function F is even, and so $F(n_r) = \beta$ is constant for $r \ge r_0(\varepsilon)$, and so the sequence $r \mapsto f(n_r)$ is a Cauchy sequence.

Since $h(f) \in \operatorname{spec}(f)$, and $\operatorname{spec}(c_{q^\ell})$ is $\{\varphi(q^\ell),\ -q^{\ell-1},\ 0\}$, if $\ell > 1$, and $\{\varphi(q),\ -1\}$, if $\ell = 1$, and $\{1\}$ if $\ell = 0$, there are only a few (at most three) possibilities for choosing the value $h(c_{q^\ell})$.

However, not every choice is admissible. The relations

$$c_{p^m} \cdot c_{p^\ell} = \varphi(p^\ell) \cdot c_{p^m}, \quad \text{if} \quad m > \ell, \tag{2.5}$$

and[4]

$$c_{p^\ell} \cdot c_{p^\ell} = \varphi(p^\ell) \cdot (c_1 + c_p + \cdots + c_{p^{\ell-1}}) + (p^\ell - 2p^{\ell-1}) \cdot c_{p^\ell} \tag{2.6}$$

imply (using the fact that h is an algebra-homomorphism; q denotes a prime)

(a) $h(c_{q^m}) = 0$, if $h(c_{q^\ell}) = 0$ and $m > \ell$,
(b) $h(c_{q^m}) = \varphi(q^m)$, if $h(c_{q^\ell}) \neq 0$ and $0 \leq m < \ell$,
(c) $h(c_{q^\ell}) < 0$ is possible for at most one ℓ (q fixed),
(d) if $h(c_{q^{\ell+1}}) = 0$, but $h(c_{q^\ell}) \neq 0$, then $h(c_{q^\ell}) = -p^{\ell-1} < 0$.

Therefore either $h(c_{q^m}) = \varphi(q^m)$ for any $m \geq 0$ (define $k_q = \infty$ in that case), or there exists an exponent k_q such that

$$h(c_{q^\ell}) = \begin{cases} \varphi(q^\ell), & \text{if} \quad \ell \leq k_q\,, \\ -p^{\ell-1}, & \text{if} \quad \ell = k_q + 1\,, \\ 0\,, & \text{if} \quad \ell > k_q + 1\,. \end{cases}$$

Then, for the vector $\mathcal{K} = (k_q)_{q \text{ prime}}$, we obtain $h = h_{\mathcal{K}}$, and so $\Delta_{\mathcal{B}}$ is completely determined.

c) Topology. The Gelfand topology of $\Delta_{\mathcal{B}}$ is the weakest topology the makes every Gelfand transform

$$\hat{f}\colon \Delta_{\mathcal{B}} \to \mathbf{C}, \quad \hat{f}(h) = h(f),$$

continuous.

So, for any prime power q^ℓ, the sets

$$\hat{c}_{q^\ell}{}^{-1}(\mathcal{O}) = \{h \in \Delta;\ h(c_{q^\ell}) \in \mathcal{O}\}$$

are open for any open $\mathcal{O}$ in $\mathbf{C}$. Therefore, using (2.4), the sets

$$\{h_{\mathcal{K}};\ k_p \quad \text{arbitrary for} \quad p \neq q,\ k_q \geq \ell\}$$

and

$$\{h_{\mathcal{K}};\ k_p \quad \text{arbitrary for} \quad p \neq q,\ k_q = \ell - 1\}$$

[4] By the way, the second relation implies $h(c_{p^\ell}) \in \{0,\ -p^{\ell-1},\ \varphi(p^\ell)\}$.

are open. Choosing these sets as a subbasis for the topology, we see that every $\hat{f}$ is continuous. For:
Given $\varepsilon > 0$ and f, choose $g = \sum_{1\leq r\leq R} \gamma_r \cdot c_r$ satisfying $\|f-g\|_u < \frac{1}{2}\varepsilon$. Assume that $h \in \Delta_B$, $h = h_{\mathcal{K}}$, $\mathcal{K} = (k_p(h))$, is given. An open neighbourhood $U(h)$ of h is defined by the condition

$$h^* \in U(h) \quad \text{iff} \quad h^* = h_{\mathcal{K}^*}, \quad \text{and} \quad k_p(h^*) = k_p(h) \quad \text{for any} \quad p \leq R.$$

Then $h(g) = h^*(g)$ for any h^* in $U(h)$, and so

$$|\hat{f}(h) - \hat{f}(h^*)| = |h(f) - h(f^*)| \leq$$
$$|h(f) - h(g)| + |h^*(f) - h^*(g)| \leq \|f-g\|_u + \|f-g\|_u < \varepsilon$$

(to get from the first to the second line, property (2) from §1 was used).

Therefore $\hat{f}$ is continuous, and so the topology of Δ_B is completely determined. It coincides with the product topology on the space

$$\prod_p \{1, p, p^2, \ldots, p^\infty\},$$

where each factor is the Alexandroff-one-point-compactification of the discrete (and locally compact) space $\{1, p, p^2, \ldots\}$.

d) Main result. For functions f in $\mathcal{B}^u$ obviously $\|f^2\|_u = \|f\|_u^2$, and so we obtain from 11.12 in [8]

Theorem 2.2. *The Banach-algebra $\mathcal{B}^u$ is semi-simple, and the Gelfand transform $f \mapsto \hat{f}$ is an isometric algebra-isomorphism from $\mathcal{B}^u$ onto $C(\Delta_B)$.*

By the way, semi-simplicity immediately also follows from the fact that the evaluation homomorphisms $h_n\colon f \mapsto f(n)$ are in Δ_B, and so the assumption $f \in \text{radical}(\mathcal{B}^u) = \bigcap_{h\in\Delta_B} \text{kernel}(h)$ implies $f = 0$.
Next, [8] 11.20 implies[5].

Corollary 2.1. *If $f \in \mathcal{B}^u$ is real-valued, and if $\inf_{n\in\mathbb{N}} f(n) > 0$, then there exists a (real-valued) square-root g of f in $\mathcal{B}^u$.*

e) Applications. The following result is well known and may be derived from the Weierstraß approximation theorem also; we deduce it from our knowledge of Δ_B.

[5] The result can be deduced from the Weierstraß approximation theorem also.

Corollary 2.2. *Assume $f \in \mathcal{B}^u$. Then $1/f \in \mathcal{B}^u$ if and only if there exists some positive constant δ, for which $\|f\|_u \geq \delta$.*

Proof. If $1/f \in \mathcal{B}^u$ then this function is bounded and so $|f|$ is bounded from below.

On the other hand, according to Gelfand's theory (see Rudin [7], 18.17) $1/f \in \mathcal{B}^u$, if for any $h \in \Delta_{\mathcal{B}}$ the value $h(f)$ is not zero. The values $h(f)$ are given as certain limits in section 2, and the condition $|f| \geq \delta$ obviously implies that all these limits are non-zero, and the corollary is proved.

These corollaries may be considerably extended, using known results on Banach algebras.

Theorem 2.3. *Let $f \in \mathcal{B}^u$ be given. If the function F is holomorphic in some region of $\mathbf{C}$ including the range $\hat{f}(\Delta_{\mathcal{B}})$ of $\hat{f}$, then the composed function $F \circ \hat{f}$ is in $C(\Delta_{\mathcal{B}})$ and thus equal to some $\hat{g}$, $g \in \mathcal{B}^u$. Therefore $F \circ f$ is in $\mathcal{B}^u$ again.*

Except for the last sentence, this is a specialization of L. H. Loomis, *Abstract Harmonic Analysis*, Princeton 1953, **24 D**. Next, $\hat{g} = F \circ \hat{f}$ implies $h(g) = F\big(h(f)\big)$ for any h in $\Delta_{\mathcal{B}}$, and so the assertion is true if F is a polynomial (then $F\big(h(f)\big) = h\big(F(f)\big)$). The general case follows from this.

Theorem 2.4. *Let $f \in \mathcal{B}^u$ be given. If $\delta > 0$ and f is multiplicative, then $f(p^k) = 0$ is possible for at most finitely primes p.*

The same argument gives the following stronger version.

Theorem 2.5. *Let $f \in \mathcal{B}^u$ be given. If $\delta > 0$ and f is multiplicative, then there are at most finitely many primes with the property $|f(p^k) - 1| > \delta$ for some k.*

Proof. $\hat{f}(h_{\mathcal{K}-0}) = 1$, where $\mathcal{K}_0 = (k_p)$, $k_p = 0$ for any p. Given $\varepsilon = \frac{1}{2}\delta$, then there is some neighbourhood $\mathcal{U}_0$ of $h_{\mathcal{K}_0}$ with the property $|\hat{f}(h) - 1| < \varepsilon$ for any h in $\mathcal{U}_0$. But this neighbourhood contains all $h_{\mathcal{K}}$ with k_p arbitrary except for finitely many primes; for these exceptional primes $k_p = 0$ may be taken. Next, f being multiplicative,

$$\hat{f}(h) = \lim_{L \to \infty} \prod_{p \leq L} f\big(p^{\min(k_p, L)}\big),$$

and this implies, by a suitable choice of the k_p, noticing $|\hat{f}(h) - 1| < \varepsilon$, that $|f(p^k) - 1| > \varepsilon$ is impossible for any "non-exceptional" prime and any k.

3. The maximal ideal space $\Delta_{\mathcal{D}}$ of $\mathcal{D}^u$

a) Embedding of $\Delta_{\mathcal{D}}$ in $\prod_{r\in\mathbf{N}} \mathbb{Z}/r\mathbb{Z}$.

Define, with the abbreviation $\omega_r = \exp(2\pi i/r)$, an element $f_r \in \mathcal{D}$ by $f_r(n) = \omega_r^n$. The set of functions

$$\{f_r^\ell,\ 1 \le \ell \le r,\ \gcd(\ell, r) = 1,\ r = 1, 2, \dots\}$$

is a basis of $\mathcal{D}$. A function f in $\mathcal{D}$ is r-periodic for a suitable r, and so $1/f$ is again r-periodic and so in $\mathcal{D} \subset \mathcal{D}^u$ if f does not assume the value zero. Therefore

$$\operatorname{spec}(f_r) = \{\omega_r^j,\ 1 \le j \le r\}.$$

If $h \in \Delta_{\mathcal{D}}$, then, by (1), §1

$$h(f_r) = \omega_r^{j(r,h)}, \tag{3.1}$$

where $j(r,h)$ is some uniquely determined integer modulo r depending on h. Thus we obtain a map

$$\varphi\colon \Delta_{\mathcal{D}} \to \prod_{r\in\mathbf{N}} \mathbb{Z}/r\mathbb{Z},$$

defined by $\varphi(h) = (j(r,h))_{r=1,2,\dots}$, where h and j are related by (3.1). Obviously φ is *injective.*

b) The Prüfer Ring $\hat{\mathbb{Z}}$.

For any $n \in \mathbf{N}$ consider the residue class ring $\mathbb{Z}/n\mathbb{Z}$ with the discrete topology. If $m|n$, then there is a continuous projection

$$\pi_{m,n}\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}, \qquad (a \mod n) \mapsto (a \mod m).$$

If $X = \prod_{r\in\mathbf{N}} \mathbb{Z}/r\mathbb{Z}$ with the product topology, then X is a compact Hausdorff space, and the set

$$\hat{\mathbb{Z}} = \{(\alpha_n) \in X,\ \alpha_n \in \mathbb{Z}/n\mathbb{Z} \text{ and } \pi_{m,n}(\alpha_n) = \alpha_m,\ \text{if } m|n\}$$

is a closed subspace of X and therefore again compact (and Hausdorff). Note that $\mathbf{N}$ is dense in $\hat{\mathbb{Z}}$; the reason is that, given an element $(\alpha_r)_r$ in $\hat{\mathbb{Z}}$ and positive integers $r_1, \dots, r_N$, there exists an integer $m \in \mathbf{N}$ satisfying $m \equiv \alpha_{r_i} \mod r_i$ for $1 \le i \le N$.

Since $f_{r\cdot s}^s = f_r$ it follows that $j(r\cdot s, h) \equiv j(r,h) \mod r$ for any $h \in \Delta_{\mathcal{D}}$. Therefore the image of the map φ is contained in $\hat{\mathbb{Z}}$.

c) Surjectivity of $\varphi\colon \Delta_{\mathcal{D}} \to \hat{\mathbb{Z}}$.

Let some element $(\alpha_r)_r$ in $\hat{\mathbb{Z}}$ be given. Our aim is to construct an algebra-homomorphism $h \in \Delta_{\mathcal{D}}$ satisfying $\varphi(h) = (\alpha_r)_r$. Define a linear map $h\colon \mathcal{D} \to \mathbf{C}$ on the elements of the basis of $\mathcal{D}$ by

$$h(f_r^k) = \omega_r^{k\cdot\alpha_r},\ 1 \le k \le r,\ \gcd(k,r) = 1,\ r = 1,2,\dots,$$

and extend h linearly to $\mathcal{D}$. Then h is multiplicative on $\mathcal{D}$; assume first $\gcd(r,s) = 1$; then the relation

$$s\cdot k\cdot\alpha_r + r\cdot\ell\cdot\alpha_s \equiv (s\cdot k + r\cdot\ell)\cdot\alpha_{r\cdot s} \quad \mathrm{mod}\ r\cdot s$$

implies

$$h(f_r^k\cdot f_s^\ell) = h(f_r^k)\cdot h(f_s^\ell).$$

This is also true if $\gcd(r,s) \ne 1$; without loss of generality, r and s may be assumed to be powers of the same prime, and then the as ertion is easily checked. Furthermore h is continuous on $\mathcal{D}$. Given an element $\psi \in \mathcal{D}$, $\psi = \sum_{1\le\nu\le N} a_\nu \cdot f_{r_\nu}^{k_\nu}$, satisfying $\|\psi\| \le 1$, there exists an $m \in \mathbf{N}$, for which $m \equiv \alpha_{r_\nu}$ mod r_ν, for $1 \le \nu \le N$. Since $h(\psi) = \psi(m)$, we obtain

$$|h(\psi)| \le |\psi(m)| \le \|\psi\|_u \le 1,$$

and so h is continuous on $\mathcal{D}$. This space being dense in $\mathcal{D}$, h may be contin uously extended to an algebra-homomorphism of $\mathcal{D}^u$, and $\varphi(h) = (\alpha_r)_{r=1,2,}$.

d) Continuity of $\varphi\colon \Delta_{\mathcal{D}} \to \hat{\mathbb{Z}}$.

Fix $\alpha_k \in \mathbb{Z}/k\mathbb{Z}$, $1 \le k \le N$ with the property $\alpha_n \equiv \alpha_m$ mod m if m n Then

$$V(\alpha_1,\dots,\alpha_N) = \{(\beta_n) \in \hat{\mathbb{Z}},\ \beta_k = \alpha_k \text{ for } 1 \le k \le N\}$$

is a typical basis element of the (product-) topology of $\hat{\mathbb{Z}}$. Moreover $h \in \varphi^{-1}(V(\alpha_1,\dots,\alpha_N))$ if and only if $h(f_k) = \omega_k^{\alpha_k}$ for any k in $1 \le k \le N$ This is equivalent with $\hat{f}_k(h) = \omega_k^{\alpha_k}$, $1 \le k \le N$, where f_k is the Gelfand transform of f_k, defined by $\hat{f}(H) = H(f)$ for any $H \in \Delta_{\mathcal{D}}$.

If U_k is a neighbourhood of $\omega_k^{\alpha_k}$, not containing any other kth root of unity then it follows that

$$\varphi^{-1}(V(\alpha_1,\dots,\alpha_N)) = \bigcap_{k=1}^{N} \hat{f}_k^{-1}(U_k)$$

is an open set in the Gelfand topology of $\Delta_{\mathcal{D}}$, and so φ is continuous. Since $\Delta_{\mathcal{D}}$ and $\hat{\mathbb{Z}}$ are compact Hausdorff spaces, φ is a homeomorphism. Thus we got

Theorem 3.1. *The maximal space $\Delta_{\mathcal{D}}$ is homeomorphic with $\hat{\mathbb{Z}}$, defined in* **3b**.

4. On the characterization of additive and multiplicative functions in $\mathcal{B}^u$

In [1] N. G. De Bruijn characterized multiplicative almost-periodic arithmetical functions. Additive almost-periodic functions were characterized by E. R. Van Kampen in "*On uniformly almost periodic multiplicative and additive functions*", Amer. J. Math. 62, (1940), 107–114; see also [11] and the paper of J. Knopfmacher quoted there. The results are as follows.

Theorem 4.1. *Assume f to be fibre-constant.*[6] *Then f is in $\mathcal{B}^u$ if and only if $\lim_{k\to\infty} f(p^k)$ exists (for any prime).*

Theorem 4.2. *An additive function is in $\mathcal{B}^u$ if and only if*

$$\lim_{k\to\infty} f(p^k) \qquad \text{exists for any prime} \tag{4.1}$$

and

$$\sum_p \sup_k |f(p^k)| < \infty . \tag{4.2}$$

Theorem 4.3. *A multiplicative function is in $\mathcal{B}^u$ if and only if* (4.1) *holds and if*

$$\sum_p \sup_k |f(p^k) - 1| < \infty \tag{4.3}$$

is true.

We give proofs for these known theorems, using the isomorphy of $\mathcal{B}^u$ with $\mathcal{C}(\Delta)$. However, the ideas used are more or less also well known.

Remark. If f is in $\mathcal{B}^u$, then the Gelfand transform $\hat{f}$ is continuous at $h_{\mathcal{K}}$, where $\mathcal{K} = (k_p)_p$, and $k_q = \infty$, $k_p = 0$, if $p \neq q$. All the functions $h_{\mathcal{K}'}$, where $k'_p = k_p = 0$ for $p \neq q$, and $k'_q = L$, L sufficiently large, are near $h_{\mathcal{K}}$, and so the limes relation (4.1) is true.

The proof of Theorem 4.1. now follows from the preceding remark and the fact, that for fibre-constant functions $\hat{f}(h)$ may be defined in an obvious manner, using the limes relation (4.1) at q. The resulting functions $\hat{f}$ obviously is continuous, and so f is in $\mathcal{B}^u$.

We now use the following notation: Given any arithmetical function, define, with an obvious interpretation of the greatest common divisor,

$$f_{(p)}(n) = f\big(\gcd(n, p^\infty)\big), \qquad \text{if } p \text{ is prime} \tag{4.4}$$

[6] f is called fibre-constant if there is a prime q such that $f(n) = f(\gcd(n, q^\infty))$ for any n. Obviously, $\lim_{k\to\infty} f(p^k)$ exists for any prime $p \neq q$ trivially.

and

$$F_R(n) = f\big(\gcd(n, \prod_{p>R} p^{\infty})\big)\,. \tag{4.5}$$

The functions $f_{(p)}$ are fibre-constant.

Proof of Theorem 4.2.

(a) Assume that (4.1) and (4.2) hold. f being additive,

$$f = \sum_{p\le R} f_{(p)} + F_R\,, \tag{4.6}$$

and the functions $f_{(p)}$ are in $\mathcal{B}^u$ by Theorem 4.1. Next

$$|F_R(n)| = |f(n) - \sum_{p\le R} f_{(p)}(n)| \le \sum_{p>R} \sup_k |f(p^k)| < \varepsilon\,,$$

if R is sufficiently large, and so $f \in \mathcal{B}^u$.

(b) If $\mathcal{K} = (0,0,\dots)$, $\mathcal{K}' = (k_p)_p$, where k_p is arbitrary for $p > R$ and $k_p = 0$ if $p \le R$, then $h_{\mathcal{K}'}$ is near $h_{\mathcal{K}}$. Since f is additive, we obtain $\hat{f}(h_{\mathcal{K}}) = 0$; $\hat{f}$ is continuous, and so $|\hat{f}(h_{\mathcal{K}'})| < \varepsilon$, if R is sufficiently large. Therefore, evaluating $\hat{f}(h_{\mathcal{K}'})$, one gets

$$|\sum_{R<p<R'} f(p^{k_p})| < \varepsilon$$

for any system k_p of exponents ($k_p = \infty$ is admissible, $f(p^{\infty}) = \lim_k f(p^k)$), and so every subseries of

$$\sum_p f(p^{k_p})$$

is convergent, therefore this series is absolutely convergent (see, for example, Pólya—Szegö, Aufgaben und Lehrsätze aus der Analysis, III, 51) for any choice of the exponents. This implies (4.2).

Proof of Theorem 4.3.

(a) Assume that (4.1) and (4.3) hold. Being multiplicative,

$$f = \prod_{p\le R} f_{(p)} \cdot F_R\,,$$

where the fibre-constant functions $f_{(p)}$ are in $\mathcal{B}^u$. Next, using (4.3),

$$|\prod_{p\le R} f_{(p)}(n)| \le \exp\left\{\sum_{p\le R}{}^{*} \left(|f_{(p)}(n)| - 1\right)\right\} \le C\,,$$

uniformly in R, where * means that summation is only over those primes for which $|f_{(p)}(n)| \geq 1$. And

$$|f(n) - \prod_{p \leq R} f_{(p)}(n)| \leq C|F_R(n)| < C \cdot \varepsilon ,$$

uniformly in n, if R is large, again using (4.3).

Therefore f is in $\mathcal{B}^u$.

(b) If f is in $\mathcal{B}^u$ and multiplicative, then the proof is similar to the corresponding proof of Theorem 4.2. The details, a little more complicated than before, are omitted. One needs that absolute convergence of a product $\prod x_i$ is equivalent with the absolute convergence of the series $\sum\{x_i - 1\}$.

5. Another Application

Using our knowledge of $\Delta_{\mathcal{B}}$ and the Tietze extension theorem (see for example Hewitt—Stromberg, Real and abstract analysis) we prove

Theorem 5.1. *Given a sequence $\{n_j\}$ of (pairwise distinct) integers greater than one with the property*

$$\textit{the minimal prime-divisors } p_{\min}(n_j) = p_j \textit{ of } n_j \textit{ tend to } \infty \textit{ as } j \to \infty, \tag{5.1}$$

and given complex numbers a_j converging to $a \in \mathbf{C}$, then there exists a function f in $\mathcal{B}^u$ assuming the values a_j at n_j.

Proof. Condition (5.1) implies that $\lim_{j \to \infty} h_{n_j} = h_1$ in $\Delta_{\mathcal{B}}$. The subset $\mathcal{K}$ of $\Delta_{\mathcal{B}}$, $\mathcal{K} = \{h_1\} \cup \{h_{n_j}\}$ is closed and therefore compact. Define a complex-valued function F on $\mathcal{K}$ by

$$F(h_1) = a, \quad \text{and} \quad F(h_{n_j}) = a_j .$$

It is easy to check that F is continuous on $\mathcal{K}$, and Tietze's extension theorem gives the existence of a continuous function F^* on $\Delta_{\mathcal{B}}$ extending F, which is the image of some f in $\mathcal{B}^u$ under the Gelfand transform, and

$$f(n_j) = \hat{f}(h_{n_j}) = F(h_{n_j}) = a_j .$$

REFERENCES

[1] DE BRUIJN, N. G.: Bijna periodieke multiplicative functies. Nieuw Arch. Wiskd. 32 (1943), 81–95.

[2] GELFAND, I. M.: Normed Rings. Mat. Sb. 9 (1941), 3 24.

[3] HEWITT, E.—ROSS, K. A.: Abstract Harmonic Analysis, I, II. Berlin-Heidelberg-New York 1963, 1970.

[4] KRYŽIUS, Z.: Almost even arithmetical functions on semigroups (Russian). Litov. Mat. Sb. 25 (1985), No. 2, 90-101.

[5] KRIŽIUS, Z.: Limit periodic arithmetical functions (Russian). Litov. Mat. Sb. 25 (1985), No. 3, 93-103.

[6] MAUCLAIRE, J. L.: Intégration et Théorie des Nombres. Paris 1986.

[7] RUDIN, W.: Real and Complex Analysis. New York, St. Louis et al , 1966.

[8] RUDIN, W.: Functional Analysis. New York, St. Louis et al., 1973.

[9] SCHWARZ, W.: Remarks on the theorem of Elliott and Daboussi, and applications. In Proc. 20th sem. Warszawa 1982. Banach Center Publications 17, Warszawa 1985, pp. 463-498.

[10] SCHWARZ, W.—SPILKER, J.: Eine Anwendung des Approximationssatzes von Weierstraß-Stone auf Ramanujan-Summen. Nieuw Arch. Wiskd. (3) 19 (1971), 198–209.

[11] SCHWARZ, W.—SPILKER, J.: Mean values and Ramanujan expansions of almost even arithmetical functions. In Coll. Math. Soc. J. Bolyai 13. Topics in Number Theory Debreczen, 1974, pp. 315–357.

Received September 18, 1989

Department of Mathematics
Johann Wolfgang Goethe University Frankfurt
Robert-Mayer-Straße 10
D 6000 Frankfurt am Main
Federal Republic Germany