Rastislav Jurga

Some combinatorial results on the classification of lines in Desarguesian Hjelmslev planes

Mathematica Slovaca, Vol. 48 (1998), No. 1, 79--85

Persistent URL: http://dml.cz/dmlcz/132955

Math. Slovaca, 48 (1998), No. 1, 79–85

SOME COMBINATORIAL RESULTS ON THE CLASSIFICATION OF LINES IN DESARGUESIAN HJELMSLEV PLANES

RASTISLAV JURGA

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. In this paper, we investigate some combinatorial results on the classification of lines in the Hjelmslev plane arising from given regular conic of this plane. The main result is Theorem 2.3 on the number of tangents, secants, imaginary tangents, nonsecants and zero tangents.

In this paper, we will introduce and prove some combinatorial results on the number of tangents (in three meanings), secants and nonsecants of a given regular conic in the Desarguesian Hjelmslev plane.

1. Introductory notes and definitions

A *special local ring* is a finite commutative local ring R of which the ideal I of divisors of zero is principal. We will call the elements of the ideal I *singular elements* of R. The element $a \in I$ is a *singular nonsquare* if there does not exist $b \in I$ such that $a = b^2$. Let us denote by the symbol R^* the set of regular elements of R, thus $R^* = R - I$. Let g be the generator of the ideal I. The smallest integer $\nu \in \mathbb{N}$ such that $g^\nu = 0$ is called the *index of nilpotency* of the ring R. We will suppose that R is not a field, and that the characteristic of the ring R is odd. The symbol $\overline{R}$ denotes the factor ring R/I, and Φ is the canonical homomorphism of R onto $\overline{R}$. Next let us denote by $H(R)$ the Desarguesian Hjelmslev plane over the special local ring R.

We will call a *conic* Q in $H(R)$ the set of all points $[x_1; x_2; x_3] \in H(R)$ whose coordinates satisfy the relation

$$\sum_{i,j=1}^{3} a_{ij} x_i x_j = 0 . \tag{1}$$

AMS Subject Classification (1991): Primary 51E30.
Key words: tangent, imaginary tangent, zero tangent of a conic.

We will suppose that the conic Q is regular, i.e., $\det[a_{ij}] \notin I$.

We observe that in a suitable coordinate system, the conic given by (1) satisfies the following equation

$$ax^2 + by^2 + cz^2 = 0\,, \qquad a, b, c \in R - I\,. \tag{2}$$

It is known that a conic in the projective plane $\pi(\overline{R})$ over the skewfield $\overline{R}$ has exactly $|\overline{R}|+1$ points. It can be proved that a conic in the Hjelmslev plane has exactly $|R| + |I|$ points.

Let Q be the conic given by (1), and consider the point $P = [x_1; y_1; z_1] \in H(R)$. Then the *polar* of the point P to the conic Q is the line

$$\begin{aligned} x(a_{11}x_1 + a_{12}y_1 + a_{13}z_1) + y(a_{21}x_1 + a_{22}y_1 + a_{23}z_1) \\ + z(a_{31}x_1 + a_{32}y_1 + a_{33}z_1) = 0\,. \end{aligned}$$

We will call a line t a *tangent* of Q if t intersects the conic Q in more that two points. The line t_0 will be called a *zero tangent* of Q if t_0 is the polar of a point on the conic.

In what follows, we will need the result:

THEOREM 1.1. *The line $Ax+By+Cz = 0$ is a (zero) tangent of the conic* (1) *if and only if*

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & A \\ a_{21} & a_{22} & a_{23} & B \\ a_{31} & a_{32} & a_{33} & C \\ A & B & C & 0 \end{bmatrix} = n^2\,, \qquad n \in I\,,\ (=0)\,. \tag{3}$$

P r o o f. See [4]. □

We will call a line s a *secant* of the conic Q if s intersects Q at exactly two points. We will call the line s a *nonsecant* of Q if s does not intersect the conic Q. We will call the line t an *imaginary tangent* of the conic Q if the determinant (3) is a singular nonsquare.

Remark. We introduced the concept of an imaginary tangent for the following reason. The imaginary tangent in the plane $H(R)$ is mapped into a "real tangent" in the projective plane $\pi(\overline{R})$ by the canonical homomorphism.

2. The classification of lines of $H(R)$ The combinatorial point of view

The connection between the classification of lines with respect to a given conic with the determinant (3) is given by the following theorem.

THEOREM 2.1. *Let t be the line with equation $Ax + By + Cz = 0$, and let Q be the conic given by (1). Then*

1. *t is a tangent of Q if and only if the determinant (3) is a singular square,*
2. *t is an imaginary tangent of Q if and only if the determinant (3) is a singular nonsquare,*
3. *s is a secant of Q if and only if the determinant (3) is a regular square,*
4. *s is a nonsecant of Q if and only if the determinant (3) is a nonsquare.*

Proof. Let us investigate the mutual position of the line and the conic in $H(R)$.

Let the line have the equation

$$Ax + By + Cz = 0\,.$$

We will take the equation of the conic to be

$$ax^2 + by^2 + cz^2 = 0\,.$$

Let us seek their intersection. We get the equation

$$x^2(B^2a + A^2b) + 2ACbxz + z^2(bC^2 + cB^2) = 0\,.$$

The mutual position of the conic and the line depends on the discriminant of the last equation

$$D = 4B^2(-abC^2 - acB^2 - bcA^2)\,,$$

where $-abC^2 - acB^2 - bcA^2$ is the determinant of the matrix (3), from which the statement follows immediately. □

The element $a \in I$ is a singular square if there exists $b \in I$ such that $a = b^2$.

LEMMA 2.1. *Let $\overline{u}' \in R$ be a single root of the polynomial $x^2 + \overline{p}x + \overline{q} \in \overline{R}[x]$. Then the polynomial $x^2 + px + q \in R[x]$ has a root u in R for which $\overline{u} = \overline{u}' = \Phi(u')$.*

Proof. See [3]. □

LEMMA 2.2. *Let R be the special local ring, and let ν be the index of nilpotency of R. Then the number W of singular squares in R is given by the relations*

a) *if $\nu = 2k$,*

$$W = 1 + \frac{|\overline{R}|^{\nu-1} - |\overline{R}|}{2(|\overline{R}| + 1)}, \tag{4}$$

b) *if $\nu = 2k+1$,*

$$W = 1 + \frac{|\overline{R}|^{\nu-1} - 1}{2(|\overline{R}| + 1)}. \tag{5}$$

Proof.

a) Let ν be even. Let us consider classes of squares $g^2, g^4, \dots, g^{\nu-2}, g^\nu = 0$ (the classes are regular multiples of g^{2i}). The class generated by the element g^2 includes

$$\frac{|R| - |I|}{2|\overline{R}|^2}$$

squares. Then all squares in the class generated by g^2 have the form x^2g^2, $x \in R - I$.

Let $x_0 \in R - I$. Let us search the number of elements $x \in R - I$ for which $x^2g^2 = x_0^2g^2$. Consequently $(x^2 - x_0^2)g^2 = 0$. Then $x^2 - x_0^2 \in \operatorname{An} g^2$, and so $x^2 - x_0^2 = kg^2$ (we want to find the number of possibilities for k). Let us consider the equation $\overline{x}^2 - \overline{x}_0^2 = 0$. This has two solutions. So the equation $x^2 - x_0^2 = kg^2$ has exactly $2|[g^2]|$ solutions. Consequently, in the class $[g^2]$, there are exactly

$$\frac{|R| - |I|}{2|\overline{R}|^2}$$

squares. Similarly, the class g^{2i} includes exactly

$$\frac{|R| - |I|}{2|\overline{R}|^{2i}}$$

squares. Then the number of all singular squares is determined by the expression

$$W = 1 + \frac{|R| - |I|}{2|\overline{R}|^2} + \frac{|R| - |I|}{2|\overline{R}|^4} + \dots + \frac{|R| - |I|}{2|\overline{R}|^{\nu-2}},$$

and after reordering the above expression, we get

$$W = 1 + \frac{|\overline{R}|^{\nu-1} - |\overline{R}|}{2(|\overline{R}| + 1)}.$$

b) Similarly: we will get the number of singular squares of the special local ring R by an addition of the number of squares in the classes $0, g^2, g^4, \dots, g^{\nu-1}$. This equals the sum

$$1 + \frac{|R| - |I|}{2|\overline{R}|^2} + \frac{|R| - |I|}{2|\overline{R}|^4} + \dots + \frac{|R| - |I|}{2|\overline{R}|^{\nu-1}},$$

from which, after reordering, we get

$$W = 1 + \frac{|\overline{R}|^{\nu-1} - 1}{2(|\overline{R}| + 1)},$$

as required. □

DEFINITION 2.1. Let $n \in I$. By the *quasiconic* $Q_{[a,b,c]}(n)$ we will understand the set of points $[x; y; z]$ for which

$$ax^2 + by^2 + cz^2 = r^2 n, \qquad r \in R - I. \tag{6}$$

Let $n \in I$, then we denote by $[n]$ the set $\{z \in R;\ z = r^2 n,\ r \notin I\}$.

LEMMA 2.3. $Q_{[a,b,c]}(n) = |Q| \cdot |[n]|$.

Proof. Let $P = [x_1; y_1; z_1]$ be the point of the conic $Q(0)$, then $ax_1^2 + by_1^2 + cz_1^2 = 0$. Let, e.g., $x_1 \notin I$. Suppose that $P = [1; y_1; z_1]$. The point P is mapped by the canonical homomorphism into the point $\overline{P} = [1; \overline{y}_1; \overline{z}_1]$. We will prove that for every $r \in R - I$ there is a unique triple $(1; y_1; \tilde{z}_1)$ such that

$$a + by_1^2 + c\tilde{z}_1^2 = r^2 n,$$

i.e., the point $[1; y_1; z_1]$ is on the quasiconic $Q_{[a,b,c]}(n)$, and

$$[1; \overline{y}_1; \overline{z}_1] = [1; \overline{y}_1; \overline{\tilde{z}}_1]. \tag{7}$$

Let us consider the equation

$$a + by_1^2 + cz^2 = r^2 n. \tag{8}$$

Because the equation $\overline{a} + \overline{b}\overline{y}_1^2 + \overline{c}\overline{z}^2 = 0$ has two solutions in $\overline{R}$, equation (8) (according to Lemma 2.1) also has two solutions. It follows from this that the unique triple $(1; y_1; \tilde{z}_1)$ is mapped into (7). Every triple $(1, y_1, z_1)$ determines a point of the quasiconic $Q_{[a,b,c]}(n)$. These triples are distinct if the elements $r^2 n$ are distinct. Hence exactly $|[n]|$ points of the quasiconic $Q_{[a,b,c]}(n)$ correspond to points of the conic $Q(0)$. □

THEOREM 2.2. *There are exactly $|I| \cdot W$ neighbouring tangents to the given tangent of the conic.*

Proof. The line $Ax + By + Cz = 0$ is a tangent of the conic $ax^2 + by^2 + cz^2 = 0$ if and only if $A^2bc + B^2ac + C^2ab = -n^2$. Then the point A, B, C is a point of the quasiconic $Q_{[-bc,-ac,-ab]}(n^2)$. Exactly, $|[n^2]||I|$ points are neighbouring with the point $[A; B; C]$ on the given quasiconic $Q_{[-bc,-ac,-ab]}(n^2)$. Because n runs through all the singular elements, the number of triples considered on all the quasiconics is given by

$$|I| \sum |[n^2]| = |I| W$$

as required. □

The classification of lines in the plane $H(R)$ is introduced by the following theorem.

THEOREM 2.3. *In the Hjelmslev plane $H(R)$,*

1. *there are $(|R|+|I|)W$ tangents,*
2. *the number of secants is exactly*
$$\frac{|\overline{R}|(|\overline{R}|+1)}{2}\cdot|I|^2,$$
3. *the number of imaginary tangents is exactly*
$$(|\overline{R}|+1)|I|(|I|-W),$$
4. *the number of nonsecants is*
$$\frac{|R|(|R|+|I|)}{2}+|I|^2-(|R|-|I|)W,$$
5. *the number of zero tangents is $|R|+|I|$.*

Proof.

1. According to Theorem 1.1, the line $Ax+By+Cz=0$ is a tangent of the conic $ax^2+by^2+cz^2=0$ if and only if
$$A^2bc+B^2ac+C^2ab=-n^2, \qquad n\in I. \tag{9}$$
The triple $(A;B;C)$ satisfies (9) if and only if $[A,B,C]$ is a point of the quasiconic, thus
$$-A^2bc-B^2ac-C^2ab=(R^*)^2n^2. \tag{10}$$
According to Lemma 2.2, for fixed $n\in I$ the quasiconic (10) has exactly $|[n^2]|\cdot(|R|+|I|)$ points. For every $n\in I$ the number of points satisfying (10) is equal to $(|R|+|I|)\sum|[n^2]|$. However, $\sum|[n^2]|=W$, from which the result follows.

2. We will determine the number of secants in $\pi(\overline{R})$. Because there are exactly $|\overline{R}|+1$ points on the conic in the projective plane, every point of the conic is incident to exactly $|\overline{R}|$ secants, consequently, the number of all secants (in $\pi(\overline{R})$) is
$$\frac{|\overline{R}|(|\overline{R}|+1)}{2}.$$
$|I|^2$ lines of the plane $H(R)$ are mapped into every line in $\pi(\overline{R})$, thus in $H(R)$, there are exactly
$$\frac{|\overline{R}|(|\overline{R}|+1)}{2}\cdot|I|^2$$
secants.

3. We have proved that there are exactly $(|R|+|I|)W$ tangents to the conic in the plane $H(R)$, and we know that there are exactly $|\overline{R}|+1$ nonneighbouring points on the conic in $H(R)$. In one class, there are exactly $|I|W$ tangents. We

know that every point in the plane $H(R)$ is incident to $|I|^2$ lines that are mapped into one line in $\pi(\overline{R})$. Consequently, the number $|I|^2 - |I| \cdot W$ is the number of imaginary tangents in one class. Because the number of classes is $|\overline{R}| + 1$, then the number of imaginary tangents is given by the relation $(|\overline{R}|+1) \cdot |I|(|I| - W)$.

4. We will determine the number of nonsecants as the difference of the number of all the lines, the number of tangents and secants. Consequently,

$$|R|^2 + |R||I| + |I|^2 - (|R| + |I|)W - \frac{|\overline{R}|(|\overline{R}| + 1)}{2} \cdot |I|^2,$$

from which, after reordering, we get

$$\frac{|R|(|R| + |I|)}{2} + |I|^2 - (|R| - |I|) \cdot W.$$

5. Let the conic have equation $ax^2 + by^2 + cz^2 = 0$. Then the line $Ax + By + Cz = 0$ is a zero tangent of the conic if and only if (Theorem 1.1)

$$A^2bc + B^2ac + C^2ab = 0. \tag{11}$$

Clearly, (11) is the equation of the conic for the variables A, B, C. Then the number of zero tangents is equal to the number of triples $(A; B; C)$, which is the number of points of the conic (11), which equals $|R| + |I|$ according to [3]. □

REFERENCES

[1] DEMBOWSKI, P.: *Finite Geometries*, Springer-Verlag, New York Inc., 1968.

[2] HUGHES, D. R.—PIPER, F. C.: *Projective Planes*, Springer-Verlag, New York-Heidelberg-Berlin, 1973.

[3] JURGA, R.: *Thesis*, Košice, 1990. (Slovak)

[4] JURGA, R.: *Some combinatorial properties of conics in the Hjelmslev plane*, Math. Slovaca **45** (1995), 219–226.

[5] RAY-CHADHURI, D. K.: *Some results on quadrics in finite projective geometry based on Galois fields*, Canad. J. Math. **14** (1962), 129–138.

[6] SEGRE, B.: *Ovals in a finite projective plane*, Canad. J. of Math. **7** (1955), 414–416.

Received September 6, 1994
Revised November 11, 1995

Katedra hospodárskej informatiky
a matematiky
Podnikovohospodárska fakulta
v Košiciach
Ekonomickej univerzity
v Bratislave
Tajovského 11
SK-041 30 Košice
SLOVAKIA