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Semirings Embedded in a Completely Regular Semiring

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Abstract

Recently, we have shown that a semiring S is completely regular if and only if S is a union of skew-rings. In this paper we show that a semiring S satisfying $a^2 = na$ can be embedded in a completely regular semiring if and only if S is additive separative.

Key words: Completely regular semiring, skew-ring, b-lattice, archimedean semiring, additive separative semiring.

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1 Introduction

Recall that a semiring $(S, +, \cdot)$ is a type (2,2) algebra whose semigroup reducts $(S, +)$ and $(S, \cdot)$ are connected by ring like distributivity, that is,

$$a(b + c) = ab + ac \quad \text{and} \quad (b + c)a = ba + ca$$

for all $a, b, c \in S$. A semiring $(S, +, \cdot)$ is called a Boolean semiring if $a^2 = a$ for all $a \in S$. A semiring S is called additive cancellative if the additive reduct $(S, +)$ is a cancellative semigroup, i.e., for $a, b, c \in S$, $a + b = a + c$ implies $b = c$.

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In this paper, we call an element a of a semiring $(S, +, \cdot)$ completely regular if there exists an element $x \in S$ satisfying the following conditions:

$$(i) \qquad a = a + x + a$$

$$(ii) \qquad a + x = x + a$$

$$(iii) \qquad a(a + x) = a + x$$

Naturally, a semiring $(S, +, \cdot)$ is a completely regular semiring if every element a of S is completely regular. There are plenty of examples of completely regular semirings, for example, every ring is a completely regular semiring and every distributive lattice is also a completely regular semiring. By definition, if $(S, +, \cdot)$ is a completely regular semiring then its additive reduct $(S, +)$ is a completely regular semigroup but the converse may not be true. For example, if we let $(S, +, \cdot)$ be a semiring whose additive reduct $(S, +)$ is an idempotent semigroup and the multiplicative reduct $(S, \cdot)$ is not a band, then we can immediately see that $(S, +)$ is completely regular but the semiring $(S, +, \cdot)$ itself is not completely regular. Throughout this paper, we denote the set of all inverse elements of a in the regular semigroup $(S, +)$ by $V^+(a)$. As usual, we denote the Green's $\mathcal{H}$-relations on $(S, +)$ by $\mathcal{H}^+$

The following useful concept is due to M. P. Grillet [2].

Definition 1.1 A semiring $(S, +, \cdot)$ is called a skew-ring if its additive reduct $(S, +)$ is a group, not necessarily an abelian group.

We have obtained the following result in [4].

Theorem 1.2 *The following statements on a semiring S are equivalent.*
(I) S is completely regular.
(II) Every $\mathcal{H}^+$-class is a skew-ring.
(II) S is union (disjoint) of skew-rings.

Corollary 1.3 *An additive commutative semiring S is completely regular if and only if S is union of rings.*

2 b-lattice decomposition

We consider the additive commutative semiring $(S, +, \cdot)$ such that for each $a \in S$ there exists a positive integer n such that

$$a^2 = na. \qquad (A)$$

Clearly, every Boolean semiring is a semiring which satisfies condition (A). Also the semiring of all natural numbers is a semiring of this kind which is not Boolean.

We now consider the following examples:

Example 2.1 Let $S = \mathbb{N}\times\{1,2,3\}$. On S we define addition and multiplication by

$$(a,i)+(b,j)=(a+b,\max\{i,j\})$$

and

$$(a,i)\cdot(b,j)=(ab,\min\{i,j\}).$$

Then $(S,+,\cdot)$ is a semiring satisfying condition (A).

Example 2.2 Let $S=\{0,a,b\}$ be a semiring with the following Cayley tables:

$+$	0	a	b
0	0	a	b
a	a	0	b
b	b	b	b

$\cdot$	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

Then $(S,+,\cdot)$ is a semiring which satisfies condition (A) but not Boolean.

Definition 2.3 A semiring $(S,+,\cdot)$ is called a b-lattice if $(S,+)$ is a semilattice and $(S,\cdot)$ is a band. Moreover, a congruence ρ on a semiring S is called a b-lattice congruence if S/ρ is a b-lattice. A semiring S is called a b-lattice Y of semirings $S_\alpha\,(\alpha\in Y)$ if S admits a b-lattice congruence ρ on S such that $Y=S/\rho$ and each S_α is a ρ-class.

Definition 2.4 Let $(S,+,\cdot)$ be a semiring. We define a relation η on S by $a\,\eta\,b$ if and only if there exist $x,y\in S^0$ and positive integers m,n such that $a+x=mb$ and $b+y=na$. Also, we define a relation σ on S by $a\,\sigma\,b$ if and only if there exists a positive integer n such that $a+nb=(n+1)b$ and $b+na=(n+1)a$.

It should be noted that if there exist positive integers m,n such that $a+mb=(m+1)b$ and $b+na=(n+1)a$ then $a\,\sigma\,b$. For if, say $m<n$, then we can add $a+mb=(m+1)b$ by $(n-m)b$ and obtain $a+nb=(n+1)b$.

Definition 2.5 A semiring S is called archimedean if $(S,+)$ is an archimedean semigroup i.e., for any $a,b\in S$ there exist $x,y\in S$ and positive integers m,n such that $a+x=mb$ and $b+y=na$.

Lemma 2.6 *Let S be a semiring satisfying (A). Then*

(i) η is a congruence on S and S/η is the maximal b-lattice homomorphic image of S.

(ii) S is uniquely expressible as a b-lattice T of archimedean semirings $S_\alpha(\alpha\in T)$. The b-lattice T is isomorphic with the maximal b-lattice homomorphic image S/η of S and $S_\alpha(\alpha\in T)$ are equivalent classes of η in S.

Proof (i) From Theorem 4.12 in [1], it follows that η is a semilattice congruence on $(S,+)$. Let $a\,\eta\,b$ and $c\in S$. Then there exist $x,y\in S^0$ and positive integers m,n such that $a+x=mb$ and $b+y=na$. This leads to $ac+xc=m(bc)$ and $bc+yc=n(ac)$. Thus $ac\,\eta\,bc$. Similarly, we can show that $ca\,\eta\,cb$. Hence η is a congruence on the semiring S.

Since S satisfies $a^2 = na$ so $a^2 \,\eta\, na$. Again since η is a semilattice congruence on $(S,+)$, it follows that $na \,\eta\, a$. Thus, $a^2 \,\eta\, a$ and hence η is a b-lattice congruence on the semiring S.

S/η is the maximal homomorphic image of S follows from Theorem 4.12 in [1].

(ii) By (i) of this Lemma, η is a b-lattice congruence on S. By Theorem 4.13 in [1], each η-class $S_\alpha (\alpha \in S/\eta)$ is archimedean semigroup under addition. We show that each S_α is a semiring. For this let $b, c \in \eta(a)$, where $\eta(a)$ is the η-class of $a \in S$. Then $b \,\eta\, a$ and $c \,\eta\, a$. This leads to $bc \,\eta\, a^2 \,\eta\, a$. So $bc \in \eta(a)$ and hence $(S_\alpha, +, \cdot)$ is an archimedean semiring. Thus, S is a b-lattice T of archimedean semirings. Unique expression of S as a b-lattice of archimedean semirings follows from Theorem 4.13 in [1].

The last part of the theorem follows from the Theorem 4.13 in [1].

Definition 2.7 A congruence ρ on a semiring S is said to be additive separative (AS-congruence) if S/ρ is an additive separative semiring (AS-semiring) i.e., $(a + b) \,\rho\, (a + a) \,\rho\, (b + b)$ implies $a \,\rho\, b$.

Lemma 2.8 *The relation σ defined in Definition 2.4 is a congruence on a semiring S and S/σ is the maximal additive separative homomorphic image of S.*

Proof By Theorem 4.14 in [1], σ is a congruence on $(S, +)$. Let $a \,\sigma\, b$ and $c \in S$. Then there exist positive integers m, n such that $a + nb = (n + 1)b$ and $b+ma = (m+1)a$. This leads to $ac+n(bc) = (n+1)bc$ and $bc+m(ac) = (m+1)ac$. Hence $ac \,\sigma\, bc$. Similarly, one can show that $ca \,\sigma\, cb$. Thus, σ is a congruence on S.

Last part follows from Theorem 4.14 in [1].

Corollary 2.9 *Let S be an additive separative semiring. If a, b are elements of S such that $a + mb = (m + 1)b$ and $b + na = (n + 1)a$ for some positive integers m and n, then $a = b$.*

Theorem 2.10 *A semiring S satisfying the condition (A) can be embedded in a completely regular semiring if and only if S is additive separative.*

Proof First suppose that S can be embedded in a completely regular semiring. Then the additive reduct $(S, +)$ of the semiring S can be embedded in a completely regular semigroup. Then by Theorem 4.19 in [1], we have the semigroup reduct $(S, +)$ is separative, i.e., S is additive separative semiring.

Conversely, assume that S is additive separative. Since the semiring S satisfies the condition $a^2 = na$ so S can be expressed as a b-lattice of archimedean semirings. Let $S = \bigcup_{\alpha \in T} S_\alpha$ be the expression of S as a b-lattice T of its archimedean components $S_\alpha (\alpha \in T)$. Since S is additive separative, by Theorem 4.16 in [1] we have S_α is additive cancellative. So by Theorem 5.11 in [3] S_α can be embedded in a ring R_α. Since S_α are mutually disjoint, we can assume that R_α are mutually disjoint. Now every element of R_α can be expressed in the form $a_1 - a_2$ with $a_1, a_2 \in S_\alpha$ and that $a_1 - a_2 = c_1 - c_2$ if and only if $a_1 + c_2 = a_2 + c_1$.

Let $S' = \bigcup_{\alpha \in T} R_\alpha$. On S' we define $\oplus$ and $\odot$ as follows:

$$a \oplus b = (a_1 + b_1) - (a_2 + b_2)$$

and

$$a \odot b = (a_1 b_1 + a_2 b_2) - (a_1 b_2 + b_2 a_1),$$

where $a = a_1 - a_2$ and $b = b_1 - b_2$.

We first show that the operations are well defined. For this let $a = a_1 - a_2 = c_1 - c_2$ and $b = b_1 - b_2 = d_1 - d_2$. So $a_1 + c_2 = a_2 + c_1$ and $b_1 + d_2 = b_2 + d_1$. Now,

$$(a_1+b_1)+(c_2+d_2) = (a_1+c_2)+(b_1+d_2) = (a_2+c_1)+(b_2+d_1) = (a_2+b_2)+(c_1+d_1)$$

This leads to,

$$(a_1 + b_1) - (a_2 + b_2) = (c_1 + d_1) - (c_2 + d_2),$$
$$(a_1 - a_2) \oplus (b_1 - b_2) = (c_1 - c_2) \oplus (d_1 - d_2).$$

So $\oplus$ is well defined.

Again,

$$a_1 b_1 + c_2 b_1 + a_2 b_2 + c_1 b_2 = a_2 b_1 + c_1 b_1 + a_1 b_2 + c_2 b_2,$$
$$(a_1 b_1 + a_2 b_2) + (c_2 b_1 + c_1 b_2) = (c_1 b_1 + c_2 b_2) + (a_2 b_1 + a_1 b_2),$$
$$(a_1 b_1 + a_2 b_2) - (a_2 b_1 + a_1 b_2) = (c_1 b_1 + c_2 b_2) - (c_2 b_1 + c_1 b_2),$$
$$(a_1 - a_2) \odot (b_1 - b_2) = (c_1 - c_2) \odot (b_1 - b_2).$$

Similarly, we can show that

$$(c_1 - c_2) \bigodot (b_1 - b_2) = (c_1 - c_2) \bigodot (d_1 - d_2).$$

Thus,

$$(a_1 - a_2) \bigodot (b_1 - b_2) = (c_1 - c_2) \bigodot (d_1 - d_2).$$

Hence $\bigodot$ is well defined.

Clearly, if $a \in R_\alpha$ and $b \in R_\beta$ $(\alpha, \beta \in T)$ then $a \oplus b \in R_{\alpha+\beta}$ and $a \bigodot b \in R_{\alpha\beta}$.

The associativity under $\oplus$ and $\bigodot$ is easily verified. Also, we can show the distributivity. Hence S' is indeed a semiring which contains S. Since S' is union of rings so by Corollary 1.3, S' is a completely regular semiring.

We now show that if a and b are elements of S then $a \oplus b$ and $a \bigodot b$ are respectively the same as the original operation $a + b$ and $a.b$ respectively in S. Let $a \in R_\alpha$ and $b \in R_\beta (\alpha, \beta \in T)$. Then $a = 2a - a$ and $b = 2b - b$ so that $a \oplus b = (2a - a) \oplus (2b - b) = (2a + 2b) - (a + b) = 2(a + b) - (a + b) = a + b$ and $a \bigodot b = (2a - a) \bigodot (2b - b) = \big((2a)(2b) + ab\big) - (2ab + 2ab) = 5ab - 4ab = a \cdot b$, as desired.

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