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SEMIGROUPS AND THEIR NATURAL ORDER

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ABSTRACT. For any semigroup $(S,\cdot)$ the relation: $a \leq b$ if an only if $a = xb = by$, $xa = a$ for some $x, y \in S^1$ is called the natural partial order of S (see [8]). The relationship between the structure of S and certain properties of its natural partial order is investigated: trivial or total order, principal order ideals defined by $\mathscr{D}$-related elements, compatibility with multiplication, primitive or completely simple or group congruences, retract extensions of regular semigroups, and strong semilattices of semigroups.

1. Introduction

For any semigroup $(S,\cdot)$ a partial order was defined in [8] by

$$a \leq b \iff a = xb = by\,, \quad xa = a \qquad \text{for some} \quad x, y \in S^1\,,$$

the so called *natural partial order of* S. Its restriction to the subset E_S of all idempotents of S (if it exists) coincides with the usual ordering: $e \leq f$ if and only if $e = ef = fe$. If S is a regular semigroup, then the relation $\leq$ on S is equal to the natural partial order found by H a r t w i g [5] and N a m b o o r i p a d [10], independently:

$$a \leq b \iff a = eb = bf \qquad \text{for some} \quad e, f \in E_S\,.$$

If S is group-bound (in particular, periodic), then its natural partial order has the same form as that for regular semigroups (H i g g i n s [7; 1.4.6]).

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By [8; Theorem 4, Corollary] for an arbitrary semigroup S and its natural partial order the following are equivalent:

(i) $a \leq b$.
(ii) $a = xb = by$, $ay = a$ for some $x, y \in S^1$.
(iii) $a = xb = by$, $xa = a = ay$ for some $x, y \in S^1$.

In the following, $\leq$ will always denote the natural partial order of the semigroup S.

It is the purpose of this note to study the natural partial order with respect to the structure of the semigroup and to provide some applications of this concept. In Section 2, the semigroups totally ordered by $\leq$ are characterized. Further, it is shown that for every semigroup S the principal order ideals defined by any two $\mathscr{D}$-related elements of S are order-isomorphic. This yields an extension of the Corollary of Green's lemma on the equipotency of the $\mathscr{H}$-classes of such elements (enlarged by the corresponding order ideals). Section 3 deals with the problem of (right-, left-) compatibility of the natural partial order with multiplication. After two criteria for compatibility, some classes of semigroups are shown to have this property in general: commutative or centric semigroups, inflations of rectangular, periodic groups, or monoids which are strong semilattices of trivially ordered semigroups. Section 4 contains three applications. First, the least primitive congruence is investigated, showing that for certain semigroups this relation can be defined by means of the natural ordering (for regular semigroups, see N a m b o o r i p a d [10]). Then retract extensions of regular semigroups are characterized by an order-theoretical property (generalizing a result of P e t r i c h [11]). Finally, order properties of strong semilattices of semigroups S_α satisfying some mild conditions are proved (for the case that each S_α is regular and simple, see P e t r i c h [12]).

2. General properties

First we make some elementary observations concerning idempotents and regular elements (if they exist) – see also H i g g i n s [7: 1.4.4].

LEMMA 2.1. *For every semigroup S the following hold:*

(i) $a \leq e$, $a \in S$, $e \in E_S$ *imply* $a \in E_S$;
(ii) $a \leq b$, $b \in S$, b *regular imply* a *regular;*
(iii) $a \leq b$, $a, b \in S$, a *regular imply* $a = eb = bf$, $e, f \in E_S$, $e \mathscr{D} f$.

P r o o f.

(i) $a \leq e$, $a \in S$, $e \in E_S \implies a = xe = ey$, $xa = a = ay$ $(x, y \in S^1)$ $\implies a^2 = xe \cdot ey = xe \cdot y = a \cdot y = a \implies a \in E_S$.

(ii) $a \leq b$, b regular $\implies a = xb = by$, $a = ay \implies$ for every inverse element b' of b: $a = ay = xb \cdot y = x \cdot bb'b \cdot y = xb \cdot b' \cdot by = ab'a \implies a$ regular.

(iii) $a \leq b$, a regular $\implies a = xb = by$, $xa = a = ay$ $(x, y \in S^1) \implies$ for every inverse element a' of a: $a = aa'a = aa' \cdot xb = eb$ for $e = aa'x \in E_S$; similarly, $a = bf$ for $f = ya'a \in E_S$. Finally, $e \mathscr{R} a$ since $e = a \cdot a'x$, $a = e \cdot b$, and $a \mathscr{L} f$ since $a = b \cdot f$, $f = ya' \cdot a$; hence $e \mathscr{D} f$.

R e m a r k. The natural partial order is *trivial* if all elements are incomparable (note that the zero, if it exists, is the least element). For example, this occurs if

(1) S is *weakly cancellative* (i.e. $ax = bx$, $xa = xb$ imply $a = b$);

$$a \leq b \implies a = xb = by, \quad xa = a = ay \quad (x, y \in S, \text{ otherwise } a = b)$$
$$\implies yx \cdot a = ya = yx \cdot b, \quad a \cdot yx = ax = b \cdot yx \implies a = b.$$

(2) S is *right-* (*left-*) *simple*;

$$a \leq b, \ a \mathscr{R} b \implies a = xb = by, \quad xa = a, \quad b = az \quad (x, y, z \in S^1)$$
$$\implies a = xb = x \cdot az = xa \cdot z = a \cdot z = b.$$

(3) S is *right-* (*left-*) *stratified* (i.e. $a \in abS$ for all $a, b \in S$);
if S contains an idempotent, then by C l i f f o r d - P r e s t o n [2; Theorem 8.14], S is completely simple, hence by P e t r i c h [13; IV.2.4], weakly cancellative, thus $(S, \leq)$ is trivially ordered by (1);
if S is idempotent-free, then by C l i f f o r d - P r e s t o n [2; Lemma 8.15], the equation $xa = a$ cannot hold for $a, x \in S$; thus $a < b$ is impossible in S. (Note that by [2; Theorem 6.36], S is right-stratified if and only if S is simple and contains a minimal right ideal.)

The class of semigroups having a trivial natural partial order is not known yet. But in the class of all *E-inversive* semigroups S without zero (i.e. for every $a \in S$ there is $x \in S$ such that $ax \in E_S$), the trivially ordered semigroups are exactly the completely simple ones (see M i t s c h [9; Proposition 3]). Note that every regular and every periodic (in particular, finite) semigroup is E-inversive. Thus we have

PROPOSITION 2.2. *A periodic semigroup is trivially ordered if and only if it is completely simple. In particular, a finite semigroup is trivially ordered if and only if it is simple.*

For the other extreme, where all elements of the semigroup S are comparable. i.e. $(S,\leq)$ is a chain, we have the following

THEOREM 2.3. *A semigroup S is totally ordered with respect to its natural partial order if and only if S is one of the following:*

(i) $S = E_S$ *and* $(E_S,\leq)$ *is a chain,*

or

(ii) $S = E_S \cup \{a\}$ *for some* $a \notin E_S$ *such that* $ea = ae = e$ *for every* $e \in E_S$, *and* E_S *is a chain with a greatest element* a^2.

P r o o f. If S is of type (i) or (ii), then $(S,\leq)$ is a chain since in the second case $e < a$ for every $e \in E_S$.

Conversely, suppose that $(S,\leq)$ is totally ordered. Then we have:

(1) $ef = fe$ for all $e, f \in E_S$:
$e \leq f$ or $f \leq e$ implies that $e = ef = fe$ or $f = fe = ef$;
thus, $ef = fe$.

(2) $a^2 \in E_S$ for every $a \in S$:
if $a \leq a^2$, then $a = xa^2$, $xa = a$ $(x \in S^1)$ and $a^2 = xa \cdot a = xa^2 = a$;
if $a^2 \leq a$, then $a^2 = xa$, $xa^2 = a^2$ $(x \in S^1)$ and $a^2 \cdot a^2 = xa \cdot a^2 = xa^2 \cdot a = a^2 \cdot a = xa \cdot a = xa^2 = a^2$.

(3) $e < a$ for every $e \in E_S$, $a \notin E_S$:
if $a \leq e$, then $a \in E_S$ by Lemma 2.1.(i).

(4) $ea = ae = e$ for all $e \in E_S$, $a \notin E_S$:
$e < a$ by (3), hence $e = fa = ag$ for some $f, g \in E_S$ by Lemma 2.1.(iii);
thus, $fe = e$, and by (1), $ef = e$; consequently, $e = e^2 = e \cdot fa = ea$, and similarly, $ae = e$.

(5) $|S \setminus E_S| \leq 1$:
if $a, b \notin E_S$ and $a < b$, then $a = xb = by$, $xa = a$ $(x, y \in S)$;
thus $x^2a = xa = a$; since $x^2 \in E_S$ by (2), it follows that $x^2a = x^2$ by (4), and $a = x^2a = x^2 \in E_S$, which is a contradiction.

Now if $|S\setminus E_S| = 0$, then $S = E_S$, and $(E_S,\leq)$ is totally ordered. If $|S\setminus E_S| = 1$, then there is only one $a \in S$ such that $a \notin E_S$; by (4), it follows that $ea = ae = e$ for every $e \in E_S$. Finally, $a^2 \in E_S$ by (2), and a^2 is the greatest element of E_S because $ea = e$ implies $ea^2 = ea = e$ and $a^2e = e$ for all $e \in E_S$.

For any semigroup S the partially ordered set $(S,\leq)$ shows some remarkably strong symmetries; for any $a \in S$ the *principal order ideal* defined by a is the set $(a] = \{x \in S \mid x \leq a\}$.

PROPOSITION 2.4. *Let S be a semigroup, and $a, b \in S$ such that $a \mathscr{D} b$; then there is a bijection from $(a]$ onto $(b]$.*

P r o o f. We will not give the details of calculations here, since the result follows from Theorem 2.6 below. But we will indicate a bijection because we will need it later. Since $a \mathscr{D} b$, there is some $c \in S$ such that $a \mathscr{R} c$ and $c \mathscr{L} b$. Hence there exist $s, t, u, v \in S^1$ such that $as = c$, $ct = a$, $uc = b$, $vb = c$. Then the mappings

$$\varphi\colon (a] \to (b]\,, \quad x\varphi = uxs \qquad \text{and} \qquad \psi\colon (b] \to (a]\,, \quad x\psi = vxt$$

are mutually inverse functions.

The last result leads to the following extension of a corollary of Green's lemma. Recall that this states that for any semigroup S and every pair of $\mathscr{D}$-related elements $a, b \in S$ there is a bijection between the $\mathscr{H}$-classes H_a and H_b in S. Enlarging the $\mathscr{H}$-class H_a by $(a]$ we observe that $H_a \cap (a] = \{a\}$ for any $a \in S$. In fact, by the Remark (2) above, all elements of an arbitrary $\mathscr{R}$- or $\mathscr{L}$-class, hence of any $\mathscr{H}$-class of S, are incomparable. Thus, extending the bijection $f\colon H_a \to H_b$ given by the Corollary of Green's lemma (see for example [2; Theorem 2.3]) by the bijection $\varphi\colon (a] \to (b]$ specified in the proof of Proposition 2.4, we obtain

THEOREM 2.5. *For every semigroup S and all $a, b \in S$ such that $a \mathscr{D} b$ there is a bijection between $H_a \cup (a]$ and $H_b \cup (b]$.*

R e m a r k. It is interesting to note that f and φ above are given by the same elements by which each $x \in H_a$ or $x \in (a]$ is multiplied on the left and on the right, respectively.

By Proposition 2.4, the principal order ideals defined by two $\mathscr{D}$-related elements of a semigroup are equipotent. Since these ideals are also partially ordered sets, it is natural to ask if they are even order-isomorphic. In case that e, f are $\mathscr{D}$-related idempotents of a semigroup S, F a r è s [3] showed that indeed, $(e]$ and $(f]$ are order-isomorphic. For arbitrary $\mathscr{D}$-related elements of a regular semigroup this was proved by H i c k e y [6]. P. R. Jones generalized this property of principal order ideals to arbitrary semigroups, in fact proving a more general result. I am grateful to him for providing me with the following proof.

THEOREM 2.6. (P. R. Jones) *Let S be a semigroup, and $a, b \in S$ such that $a \mathscr{D} b$. Then there exists $c \in S$ such that $a \mathscr{R} c$, $c \mathscr{L} b$, and there are $s, t, u, v \in S^1$ such that $as = c$, $ct = a$, $uc = b$, $vb = c$. Let $\varrho_s\colon S^1a \to S^1c$, $x\varrho_s = xs$, $\varrho_t\colon S^1c \to S^1a$, $x\varrho_t = xt$; then*

(i) *ϱ_s and ϱ_t are mutually inverse order-isomorphisms; dually for the left-translations $\lambda_u\colon cS^1 \to bS^1$, $\lambda_v\colon bS^1 \to cS^1$;*

(ii) *the partially ordered sets $\bigcup_{a' \in H_a} (a']$ and $\bigcup_{b' \in H_b} (b']$ are order-isomorphic; in particular, $(a]$ and $(b]$ are order-isomorphic.*

Proof.

(i) By Green's lemma [4], ϱ_s and ϱ_t are mutually inverse bijections between the principal ideals S^1a and S^1c of S. These mappings are also order-preserving; in fact:

Let $x, y \in S^1a$ such that $x \leq y$. Then $x = py = yq$, $px = x$ $(p, q \in S^1)$. Thus, $xs = p \cdot ys = yqs$, where $yqs = yst \cdot qs$, since $y \in S^1a$ implies $yst = y(\varrho_s \circ \varrho_t) = y$. It follows that $xs = p \cdot ys = ys \cdot z$ $(z \in S^1)$, and $p \cdot xs = xs$. That is, $xs \leq ys$, and $x\varrho_s \leq y\varrho_s$. Similarly, for ϱ_t, λ_u, λ_v.

(ii) Let $d \in \bigcup_{a' \in H_a} (a']$; then $d \in (a']$ for some $a' \in H_a$. Thus, $d \leq a'$ and $S^1a' = S^1a$. Hence $d \in S^1a' = S^1a$, so that by (i) – with d instead of x and with a' instead of y – $ds = d\varrho_s \leq a'\varrho_s = a's$.

We next show that $a's \in H_c$ for $c = as$ (specified in the statement):

$$\begin{aligned} &a' \mathscr{H} a \implies a' = xa = ay \ (x, y \in S^1) \implies a's = xas = ays, \\ &\text{where } ays = ast \cdot ys = as \cdot z \ (z \in S^1) \implies a's = xc = cz; \end{aligned}$$

similarly,

$$\begin{aligned} &a \mathscr{H} a' \implies a = wa' = a'r \ (w, r \in S^1) \implies as = wa's = a'rs, \\ &\text{where } a'rs = a'st \cdot rs = a'sm \ (m \in S^1) \implies c = w \cdot a's = a's \cdot m; \end{aligned}$$

consequently,

$$a's \mathscr{H} c.$$

Thus we have proved that $d\varrho_s \in (c']$ for $c' = a's \in H_c$; it follows that $\Big(\bigcup_{a' \in H_a} (a']\Big)\varrho_s \subseteq \bigcup_{c' \in H_c} (c']$. Similarly, $\Big(\bigcup_{c' \in H_c} (c']\Big)\varrho_t \subseteq \bigcup_{a' \in H_a} (a']$.

Applying ϱ_s to the latter inclusion we obtain by (i) equality in the first inclusion. Dually, using the mappings λ_u, λ_v we get

$$\Big(\bigcup_{c' \in H_c} (c']\Big)\lambda_u = \bigcup_{b' \in H_b} (b'].$$

The composed function $\varrho_s \circ \lambda_u$ thus yields the desired order-isomorphism. These arguments applied to a single order-ideal $(a]$ (related with $(c]$) and $(b]$ (related with $(c]$) give again the order-isomorphism $\varrho_s \circ \lambda_u$ since $as = c$ and $uc = b$.

3. Compatibility

A useful property of the natural partial order would be its compatibility with multiplication:

$$a \leq b \implies ac \leq bc \text{ and } ca \leq cb \quad \text{for all } c \in S.$$

But already for regular semigroups these implications do not hold, in general. Nambooripad [10] proved that for a regular semigroup S the natural partial order is compatible (on both sides) if and only if S is locally inverse (that is, each local submonoid eSe, $e \in E_S$, of S is an inverse semigroup). The regular semigroups satisfying the one-sided implication were characterized by Blyth-Gomes [1] as those S for which in each local submonoid eSe, $e \in E_S$, every $\mathscr{L}$-($\mathscr{R}$-)class contains exactly one idempotent. – For general semigroups we give the following characterization of compatibility which is quite close to the definition.

PROPOSITION 3.1. *For a semigroup S the natural partial order is compatible on the right with multiplication if and only if for all $a, b, x, y \in S$ such that $x^2a = xa = ay$ there is some $z \in S^1$ such that $ayb = abz$.*

Proof.

Sufficiency: Let $a < b$ in S; then $a = xb = by$, $xa = a$ $(x, y \in S)$. Thus, $x^2b = xa = a = xb = by$, and by hypothesis, for each $c \in S$ there is some $z \in S^1$ such that $byc = bcz$. Hence, $ac = x \cdot bc = byc = bc \cdot z$ and $xac = ac$; that is, $ac \leq bc$.

Necessity: Let $a, b, x, y \in S$ such that $x^2a = xa = ay$. Then the element $c = xa = ay$ satisfies $c \leq a$ (since $xc = x^2a = xa = c$). Thus $cb \leq ab$ by hypothesis, and $cb = ab \cdot z$ for some $z \in S^1$. But $c = ay$ implies $cb = ayb$, so that $ayb = abz$.

COROLLARY 3.2. *For every commutative semigroup the natural partial order is compatible with multiplication.*

COROLLARY 3.3. *For every semigroup S which is an inflation of a rectangular band, the natural partial order is compatible with multiplication. More precisely, $a < b$ in S implies $ac = bc$ and $ca = cb$ for all $c \in S$.*

Proof. By Petrich [13; III.4.10.4], S satisfies the identity $xyz = xz$ (for all $x, y, z \in S$). Consequently, $a < b$ in S implies $a = xb = by$ $(x, y \in S)$ and $ac = by \cdot c = bc$; similarly, $ca = c \cdot xb = cb$.

R e m a r k. Note that $\leq$ is *not* trivial if the inflation S of the rectangular band B is not trivial. In fact: let $S = \left(\bigcup Z_\alpha, *\right)$, Z_α the inflation of $\alpha \in B$: then $\alpha < a$ for each $a \in Z_\alpha$, $a \neq \alpha$, because $\alpha \in E_S$ and $\alpha = \alpha\alpha = \alpha * a = a * \alpha$.

For periodic semigroups, Proposition 3.1 takes the following form:

PROPOSITION 3.4. *Let S be a periodic (in particular, finite) semigroup. Then the natural partial order of S is compatible on the right if and only if for all $a, b, x, y \in S$ such that $xa = ay$ there is some $n \in \mathbb{N}$, $z \in S^1$ such that $ay^n b = abz$.*

P r o o f.

Sufficiency: Let $a < b$ in S; then $a = xb = by$, $xa = a = ay$ $(x, y \in S)$. Hence $ac = x \cdot bc = byc$ for every $c \in S$. Also $a = ay = by^2 = by^k$, and $ac = by^k c$ for every $k \in \mathbb{N}$. Since $xb = by$, by hypothesis there is $n \in \mathbb{N}$ and $z \in S^1$ such that $by^n c = bcz$. Thus, $ac = x \cdot bc = bc \cdot z$ and $x \cdot ac = ac$; hence $ac \leq bc$.

Necessity: Let $a, b, x, y \in S$ such that $xa = ay$. Then $x^2 a = xxa = xay = xa \cdot y = ay^2$ and $x^k a = ay^k$ for every $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ such that $x^n = e \in E_S$. Then the element $c = ea = ay^n$ satisfies $c \leq a$. Thus, by hypothesis, $cb \leq ab$. Hence $cb = ab \cdot z$ for some $z \in S^1$. But $c = ay^n$ implies that $cb = ay^n b$; it follows that $ay^n b = abz$.

R e m a r k. Note that the condition in 3.4 is sufficient for right compatibility in *every* semigroup. Furthermore, it can be formulated in the following way: for all $a, b, x, y \in S$ such that $xa = ay$ there is some $n \in \mathbb{N}$, $z \in S^1$ such that $y^n = e \in E_S$ and $aeb = abz$. This form is more appropriate for its application to the more general inflations of rectangular groups:

COROLLARY 3.5. *Let S be a semigroup which is an inflation of a rectangular, periodic group. Then the natural partial order of S is compatible with multiplication; more precisely, $a < b$ implies that $ac = bc$ and $ca = cb$ for every $c \in S$.*

P r o o f. First note that S is a periodic semigroup since the group G in the rectangular group $T = R \times G$ (R a rectangular band) is periodic: $a \in S = \bigcup Z_\alpha \implies a \in Z_\alpha$ for some $\alpha = (e, g) \in T \implies$ if $g^n = 1$, $a^n = \alpha^n = (e, g^n) = (e, 1) \in E_T = E_S$. Let $a < b$ in S; then by Remark in Section 1, $a = eb = bf$ for some $e, f \in E_S$. Now, by P e t r i c h [13; IV.3.12.3], S satisfies the identity $xey = xy$ (for all $x, y \in S$, $e \in E_S$). It follows that $ac = bfc = bc$ and $ca = ceb = cb$.

Remark. Again, $\leq$ on S is *not* trivial if the inflation of the rectangular group $T = R \times G$ is not trivial: $\alpha < a$ for every $\alpha \in T$, $a \in Z_\alpha$, $a \neq \alpha$, because $\alpha = (e,g) = (e,1)(e,g) = (e,g)(e,1) \implies \alpha = (e,1)\alpha = \alpha(e,1) \implies \alpha = (e,1) * a = a * (e,1)$ with $(e,1) \in E_S \implies \alpha < a$.

In the following, some further classes of semigroups are specified for which $\leq$ is compatible on the right or left or both.

PROPOSITION 3.6. *Let S be a semigroup such that $Sa \subseteq aS$ for every $a \in S$. Then the natural partial order of S is compatible on the right with multiplication.*

Proof. $a < b \implies a = xb = by$, $xa = a$ $(x, y \in S) \implies ac = x \cdot bc = by \cdot c = b \cdot yc = bcz$ $(z \in S)$, $x \cdot ac = ac \implies ac \leq bc$.

COROLLARY 3.7. *For every centric semigroup S (i.e. $aS = Sa$ for every $a \in S$) the natural partial order is compatible with multiplication.*

Examples of semigroups satisfying the condition of 3.6 are given by semilattices of right-simple semigroups (see Petrich [13; II.4.9]). Note that by Remark (2) in Section 2, the natural partial order on a right-simple semigroup is trivial. (For further properties of centric semigroups see Clifford-Preston [2; Theorem 10.29]). Considering *strong* semilattices of trivially ordered semigroups we first show

THEOREM 3.8. *Let S be a strong semilattice Y of monoids S_α $(\alpha \in Y)$. Then the natural partial order on S is compatible on the right with multiplication if and only if the natural partial order $\leq_\alpha$ is compatible on the right in each S_α $(\alpha \in Y)$.*

Proof. We first prove: $a \leq b$ $(a \in S_\alpha,\ b \in S_\beta)$ if and only if $\alpha \leq \beta$, $a \leq_\alpha b\varphi_{\beta,\alpha}$, where $\varphi_{\beta,\alpha}\colon S_\beta \to S_\alpha$ is the given structure homomorphism.

If $a = b$, then $\alpha = \beta$ and $a = b = b\varphi_{\alpha,\alpha}$. Let $a < b$ in S; then $a = xb = by$, $xa = a = ay$ for some $x \in S_\gamma$, $y \in S_\delta$. Since $a = xb \in S_{\gamma\beta}$, we have $\alpha = \gamma\beta$; similarly, $\alpha = \beta\delta$; thus, $\alpha \leq \beta$ and $\alpha \leq \gamma$. It follows that

$$\begin{aligned}
a &= xb = (x\varphi_{\gamma,\gamma\beta})(b\varphi_{\beta,\gamma\beta}) = (x\varphi_{\gamma,\alpha})(b\varphi_{\beta,\alpha}) = w(b\varphi_{\beta,\alpha}) && \text{with} \quad w \in S_\alpha\,, \\
a &= by = (b\varphi_{\beta,\beta\delta})(y\varphi_{\delta,\beta\delta}) = (b\varphi_{\beta,\alpha})(y\varphi_{\delta,\alpha}) = (b\varphi_{\beta,\alpha})z && \text{with} \quad z \in S_\alpha\,, \\
wa &= (x\varphi_{\gamma,\alpha})a = (x\varphi_{\gamma,\alpha})(a\varphi_{\alpha,\alpha}) = (x\varphi_{\gamma,\alpha\gamma})(a\varphi_{\alpha,\alpha\gamma}) = xa = \ a\,.
\end{aligned}$$

Hence $\alpha \leq \beta$, and $a \leq_\alpha b\varphi_{\beta,\alpha}$ in S_α. Conversely, let $\alpha \leq \beta$ in Y, and $a \leq_\alpha b\varphi_{\beta,\alpha}$. Then $a = w(b\varphi_{\beta,\alpha}) = (b\varphi_{\beta,\alpha})z$, $wa = a$ for some $w, z \in S_\alpha$ (since S_α is a monoid). Thus, $a = (w\varphi_{\alpha,\alpha})(b\varphi_{\beta,\alpha}) = (w\varphi_{\alpha,\alpha\beta})(b\varphi_{\beta,\alpha\beta}) = wb$; similarly, $a = bz$. Since $wa = a$, it follows that $a \leq b$ in S.

By the argument above, the restriction of $\leq$ on S to S_α coincides with $\leq_\alpha$ on S_α for each $\alpha \in Y$: for, if $a, b \in S_\alpha$, then $a \leq b$ in S if and only if $a \leq_\alpha b\varphi_{\alpha,\alpha} = b$ in S_α. Thus for the proof of the statement, we only have to show sufficiency.

Let $a \leq b$ in S and $c \in S$. Then $a \in S_\alpha$, $b \in S_\beta$, $c \in S_\gamma$, say, and thus $\alpha \leq \beta$ and $a \leq b\varphi_{\beta,\alpha}$ in S_α. Applying the homomorphism $\varphi_{\alpha,\alpha\gamma}$ to the last inequality we obtain:

$$a\varphi_{\alpha,\alpha\gamma} \leq (b\varphi_{\beta,\alpha})\varphi_{\alpha,\alpha\gamma} = b\varphi_{\beta,\alpha\gamma} \qquad \text{in} \quad S_{\alpha\gamma}\,,$$

and by hypothesis,

$$(a\varphi_{\alpha,\alpha\gamma})(c\varphi_{\gamma,\alpha\gamma}) \leq (b\varphi_{\beta,\alpha\gamma})(c\varphi_{\gamma,\alpha\gamma})\,.$$

Thus, since $\alpha\gamma \leq \beta\gamma$:

$$\begin{aligned} ac &\leq \big[(b\varphi_{\beta,\beta\gamma})\varphi_{\beta\gamma,\alpha\gamma}\big]\big[(c\varphi_{\gamma,\beta\gamma})\varphi_{\beta\gamma,\alpha\gamma}\big] \\ &= \big[(b\varphi_{\beta,\beta\gamma})(c\varphi_{\gamma,\beta\gamma})\varphi_{\beta\gamma,\alpha\gamma} = (bc)\varphi_{\beta\gamma,\alpha\gamma}\,. \end{aligned}$$

Since $ac \in S_{\alpha\gamma}$, $bc \in S_{\beta\gamma}$ and $\alpha\gamma \leq \beta\gamma$, we conclude that $ac \leq bc$.

As a consequence, this result implies for the case of a trivial partial order on each S_α (which evidently is compatible with multiplication) the following

COROLLARY 3.9. *Let S be a strong semilattice of trivially ordered monoids. Then the natural partial order on S is compatible with multiplication.*

Remarks.

1) If the semilattice Y has at least two elements, $\alpha > \beta$, say, then the natural partial order on S is *not* trivial. Indeed, for any $a \in S_\alpha$ we trivially have $a\varphi_{\alpha\beta} \geq a\varphi_{\alpha,\beta}$, which by the proof of 3.8, means that $a > a\varphi_{\alpha,\beta}$ ($a\varphi_{\alpha,\beta} \in S_\beta$).

2) The result 3.9 can be seen as an order-theoretical generalization of Clifford-semigroups, that is, strong semilattices of groups. In fact, every group is a trivially ordered monoid with respect to its natural partial order (see Remark (1) in Section 2). But a strong semilattice of groups is an inverse semigroup, and the natural partial order on every inverse semigroup is compatible with multiplication (see Clifford-Preston [2; Corollary 7.53 and Lemma 7.2]).

3) Theorem 3.8 and Corollary 3.9 also hold for *monoids S which are strong semilattices of semigroups*. To see this, let S be a semigroup with identity e which is the strong semilattice Y of semigroups S_α ($\alpha \in Y$). Then $e \in S_\gamma$, say. The proof of 3.8 runs through up to: Conversely, let $\alpha \leq \beta$, $a \leq_\alpha b\varphi_{\beta,\alpha}$. If S_α

is not a monoid, then a problem arises in the case when $\alpha < \beta$ and $a = b\varphi_{\beta,\alpha}$. But $a = ea = e(b\varphi_{\beta,\alpha})$ implies that $\alpha \le \gamma$ and

$$\begin{aligned} a &= (e\varphi_{\gamma,\alpha\gamma})\big[(b\varphi_{\beta,\alpha})\varphi_{\alpha,\alpha\gamma}\big] = (e\varphi_{\gamma,\alpha})(b\varphi_{\beta,\alpha}) = e_\alpha(b\varphi_{\beta,\alpha}) \\ &= (e_\alpha\varphi_{\alpha,\alpha})(b\varphi_{\beta,\alpha}) = (e_\alpha\varphi_{\alpha,\alpha\beta})(b\varphi_{\beta,\alpha\beta}) = e_\alpha b\,, \end{aligned}$$

where $e_\alpha \in E_S$ denotes the image of $e \in S_\gamma$ under the homomorphism $\varphi_{\gamma,\alpha}$.

Similarly, $a = be_\alpha$; consequently, $a \le b$ in S. Thus the characterization of the natural partial order on S given in the proof of 3.8 also holds in this case.

Our last result concerning the compatibility of $\le$ on a semigroup S is related with the so called *$\mathscr{L}$-majorization in E_S*: if $e, f, g \in E_S$ are such that $e > f$, $e > g$ and $f \mathscr{L} g$, then $f = g$. For *regular* semigroups this property of idempotents is equivalent to the fact that in every local submonoid eSe, $e \in E_S$, of S each $\mathscr{L}$-class contains exactly one idempotent. But this property of S is equivalent to right-compatibility of $\le$ on S by the result of B l y t h - G o m e s [1] mentioned at the beginning of this section. Now, M . P e t r i c h pointed out that for regular semigroups S, $\mathscr{L}$-majorization in E_S is equivalent to $\mathscr{L}$-majorization in S: if $a, b, c \in S$ are such that $a > b$, $a > c$ and $b \mathscr{L} c$, then $b = c$. Following his ideas, we will show that for certain non-regular semigroups this last property is equivalent to right-compatibility of the natural partial order, too.

THEOREM 3.10. *Let S be a semigroup such that S^2 is regular. Then the natural partial order on S is compatible on the right with multiplication if and only if S satisfies $\mathscr{L}$-majorization.*

P r o o f. Let $\le$ on S be right-compatible, and let $a, b, c \in S$ such that $a > b$, $a > c$, $b \mathscr{L} c$. Then $b = xa = ay$, $xb = b = by$, $cy \le ay$ and $c = zb$ for some $x, y, z \in S^1$. Thus $c = zb = z \cdot by = zb \cdot y = cy \le ay = b$. Together with $b \mathscr{L} c$, this implies by Remark (2) in Section 2 that $b = c$.

Conversely, suppose that S satisfies $\mathscr{L}$-majorization, and let $a < b$, $c \in S$. Then $a = xb = by$, $xa = a = ay$ for some $x, y \in S$. Since $ac \in S^2$, there is some $z \in S$ such that $ac \cdot z \cdot ac = ac$ and $z \cdot ac \cdot z = z$. Consequently,

$acza \le b$	because	$acza = acz \cdot xb = by \cdot cza$ and
		$aczx \cdot acza = acz \cdot a \cdot cza = aczac \cdot za = acza$;
$bcza \le b$	because	$bcza = bcz \cdot xb = b \cdot cza$ and
		$bczx \cdot bcza = bcza \cdot cza = bc \cdot zacz \cdot a = bcza$;
$acza \mathscr{L} bcza$	because	$acza = xb \cdot cza$, $bcza = bc \cdot zacz \cdot a = bcz \cdot acz \cdot a$.

By hypothesis, it follows that $acza = bcza$. Thus,

$$ac = x \cdot bc\,, \qquad ac = acza \cdot c = bcza \cdot c = bc \cdot w \;\; (w \in S^1) \qquad \text{and} \qquad xac = ac\,.$$

This means that $ac \leq bc$.

Remark. Note that without regularity of S^2, for any semigroup S right-compatibility of $\leq$ implies $\mathscr{L}$-majorization.

4. Applications

In this section, we will make use of the concept of a natural partial order to show how structural properties of a semigroup can be described by its natural order.

The first application concerns the following binary relation, which can be defined on every semigroup S:

$$\beta = \{(a,b) \in S \times S \mid \; c \leq a \text{ and } c \leq b \text{ for some } c \in S\}\,.$$

For an inverse semigroup S, β is the least group congruence on S as described by V. V. Vagner [14]. For regular semigroups S, this relation was studied by Nambooripad [10], who showed, in particular, that for locally inverse semigroups (see Section 3) β is the least primitive, and hence completely simple congruence. For general semigroups S, similar to [10; 4.2] we have

LEMMA 4.1. *Let S be a semigroup. Then β is an equivalence on S if and only if each principal order ideal $(a]$ of S is directed downwards.*

A sufficient condition for β to be an equivalence is the compatibility of $\leq$ on S with multiplication. This also ensures that β is a congruence:

LEMMA 4.2. *Let S be a semigroup such that the natural partial order on S is compatible with multiplication. Then β is a congruence on S.*

Proof. Note that if S has a zero, then β is the universal relation. Trivially, β is reflexive and symmetric. In order to show transitivity, let $a\,\beta\,b$ and $b\,\beta\,c$. Then there are $s, t \in S$ such that $s \leq a$, $s \leq b$ and $t \leq b$, $t \leq c$. Thus, $s = xb = by$, $t = wb = bz$ for some $x, y, w, z \in S^1$. Consequently, $sz = xb \cdot z = x \cdot bz = xt$; let $u = sz = xt$ (this idea is due to S. Reither). Then by hypothesis, $s \leq b$ implies that $u = sz \leq bz = t$, and $t \leq b$ implies that $u = xt \leq xb = s$. From $u \leq s \leq a$ and $u \leq t \leq c$ it follows that $a\,\beta\,c$.

Together with Lemma 4.1, this result implies the following

COROLLARY 4.3. *If the natural partial order of a semigroup S is compatible, then each principal order ideal of S is directed downwards.*

Remark. The converse of 4.3 (4.2) does not hold: let $S = (T_2, \circ)$, the (regular) transformation semigroup on two elements, and consider S^0. Then β is the universal relation on S^0, hence by 4.1, each principal order ideal of S^0 is directed downwards, but $\leq$ on S^0 is not compatible (on the left).

Nambooripad [10] called a mapping $f\colon X \to Y$ of a quasi-ordered set $(X, \leq)$ into a quasi-ordered set $(Y, \preccurlyeq)$ *reflecting* if for all $y, y' \in Xf$ such that $y' \preccurlyeq y$ and $x \in X$ with $xf = y$ there is some $x' \in X$ such that $x' \leq x$ and $x'f = y'$. An essential property of homomorphisms of *regular* semigroups is that they reflect the natural partial order (see [10; Theorem 1.8]).

THEOREM 4.4. *Let S be a semigroup such that β is a congruence and the natural homomorphism for β is reflecting the natural partial order. Then β is the least primitive congruence on S, that is, such that $(S/\beta, \preccurlyeq)$ is trivially ordered.*

Proof. Let φ denote the natural homomorphism defined by β, i.e. $\varphi\colon S \to T = S/\beta$, $a\varphi = a\beta$. Suppose that $s \preccurlyeq t$ in T. Since φ reflects the natural partial order, there exist $a \leq b$ in S such that $a\varphi = s$, $b\varphi = t$. But $a \leq b$ implies that $a \,\beta\, b$, thus $s = a\varphi = a\beta = b\beta = b\varphi = t$. Hence $\preccurlyeq$ on T is trivial. Let ϱ be any congruence on T such that $(S/\varrho, \preccurlyeq)$ is trivially ordered. If ψ denotes the natural homomorphism corresponding to ϱ, then we have:

$$a \,\beta\, b \implies c \leq a,\ c \leq b \text{ for some } c \in S \implies c\psi \preccurlyeq a\psi,\ c\psi \preccurlyeq b\psi \text{ in } S/\varrho$$
$$\implies c\psi = a\psi = b\psi \implies a\varrho = b\varrho \implies a\,\varrho\, b, \text{ i.e. } \beta \subseteq \varrho.$$

For *E-inversive* semigroups (see Section 2) β gives the following type of congruence:

COROLLARY 4.5. *Let S be an E-inversive semigroup such that β is a congruence whose natural homomorphism reflects the natural partial order. Then β is the least completely simple congruence on S.*

Proof. Since S is E-inversive, S/β is E-inversive, too. By 4.4, the natural partial order on S/β is trivial. Hence, by [9; Proposition 3], S/β is completely simple. The relation β is the least such congruence because for each such congruence ϱ on S, $(S/\varrho, \preccurlyeq)$ is trivially ordered (see [9; Proposition 3]), and thus $\beta \subseteq \varrho$ by 4.4.

For *E-dense monoids*, that is, E-inversive monoids S with commuting idempotents (see [9]), β is of the following form:

THEOREM 4.6. *Let S be an E-dense monoid such that β is a congruence: then β is a group congruence on S. If, furthermore, S is group-bound, then β is the least group congruence on S.*

P r o o f. First we will show that S/β is a group. By hypothesis, we have for $g = ef = fe$ ($e, f \in E_S$) that $g \leq e$ and $g \leq f$; thus $e \beta f$ for all $e, f \in E_S$. Consequently, the β-class of S containing the identity 1 of S contains all the idempotents of S and, further, is the identity of S/β. Since S is E-inversive, for every $a \in S$ there exists $x \in S$ such that $ax = e \in E_S$. It follows that $(a\beta)(x\beta) = (ax)\beta = e\beta = 1\beta$, and S/β is a group.

Let, furthermore, S be group-bound and suppose that ϱ is any group congruence on S. If $a \beta b$ for some $a, b \in S$, then there exists $c \in S$ such that $c \leq a$, $c \leq b$. Since S is group-bound, there are $e, f \in E_S$ such that $c = ea = fb$ (H i g g i n s [7; 1.4.6]). Now ϱ being a group congruence, $e\varrho = f\varrho$ is the identity of S/ϱ. Consequently, $(e\varrho)(a\varrho) = (f\varrho)(b\varrho)$ implies that $a\varrho = b\varrho$, hence $a \varrho b$ and $\beta \subseteq \varrho$.

By Lemma 4.2 and Corollary 3.2, the conditions of 4.6 are satisfied for every periodic (finite), commutative monoid; for example, the residue class semigroups $(\mathbb{Z}_n, \cdot)$, $n \in \mathbb{N}$. Thus we have

COROLLARY 4.7. *For every periodic, commutative monoid S the least group congruence is given by $\beta = \{(a,b) \in S \times S \mid c \leq a,\ c \leq b \text{ for some } c \in S\}$.*

The second application deals with an order-theoretical characterization of retract extensions of regular semigroups. Recall that a semigroup T is called a *ideal extension* of a semigroup S if S is an ideal of T. Further, an ideal extension T of S is called a *retract extension* if there is a (surjective) homomorphism $\varphi\colon T \to S$ whose restriction to S is the identity mapping. The following characterization involves the principal order ideals of T. Restricted to principal order ideals generated by idempotents, this result was proved by P e t r i c h [11] imposing some mild restriction on T: for each $a \in T$ there are $e, f \in E_T$ such that $ea = a = af$. Note that such a semigroup T is necessarily *weakly reductive* (i.e. for any $a, b \in T$, $ax = bx$ and $xa = xb$ for all $x \in T$ imply $a = b$). Indeed, we have more generally:

$a = ra = as$ ($r, s \in T$) implies that $a = rb = bs$, $ra = a$, thus $a \leq b$; similarly, $b \leq a$, and the equality follows.

THEOREM 4.8. *Let T be an ideal extension of a regular semigroup S such that for each $a \in T$ there are $e, f \in E_T$ with $ea = a = af$. Then T is a retract extension of S if and only if $(a] \cap S$ admits a greatest element for each $a \in T \setminus S$.*

Proof. Sufficiency is clear by Petrich [11] (see also Petrich [13; III.4.7]). Conversely, let T be a retract extension of S. Then by Petrich [11], $(e] \cap S$ admits a greatest element e' for each $e \in E_{T \setminus S}$. Note that by Lemma 2.1.(i), every element in $(e] \cap S$, in particular e', is an idempotent of S. Let $a \in T \setminus S$; then there are $e, f \in E_T$ such that $ea = a = af$. Let $x \in S$; then $ax \in S$ since S is an ideal of T. Since S is regular, $ax = ax \cdot u \cdot ax$ for some $u \in S$. Put $t = axue$; then $t \in S$ (S is an ideal). But $axu \in E_S$; thus $t = axu \cdot e = e \cdot axue$ implies that $t \leq e$. Since $e \notin S$ (otherwise, $a = ea \in S$), it follows that $t < e$; hence $t \leq e'$, the greatest element of $(e] \cap S$. Now we will show:

(1) $ax = e'ax$:

$$ax = axuax = axu \cdot ea \cdot x = tax = e't \cdot ax = e' \cdot axue \cdot ax = e' \cdot axuax = e'ax\,;$$

(2) $xa = xe'a$:

Since $x \in S$, $xe \in S$ and $xe = xevxe$ for some $v \in S$; then $s = evxe$ satisfies $s \in S$, $s < e$, and $s \leq e'$; consequently, since $e' \leq e$, it follows that $xa = x \cdot ea = xevxe \cdot a = x \cdot sa = x \cdot se' \cdot a = x \cdot evxe \cdot e'a = xe \cdot e'a = xe'a$;

(3) $ax = af'x$, $xa = xaf'$, where f' denotes the greatest element of $(f] \cap S$:

The proof of (3) is similar to that of (1) and (2).

Put $x = f' \in S$ in (1); then $af' = e'af'$. Put $x = e' \in S$ in (3); then $e'a = e'af'$. It follows that $af' = e'a = e'af'$ and $e'af' \leq a$. Since $e' \in S$, we have $e'af' \in S$. Thus, $a \in T \setminus S$ implies that $e'af' < a$ and $e'af' \in (a] \cap S$.

Finally, we show that $e'af'$ is the greatest element of $(a] \cap S$. Let $b \in (a] \cap S$; then $b \in S$ and $b < a$ ($b = a$ implies $b = a \in T \setminus S$). Hence $b = xa = ay$, $xb = b = by$ for some $x, y \in T$. We can choose $x, y \in S$ because:

$b = gb = bh$ for $g = bb'$, $h = b'b \in E_S$ (S is regular) implies that

$$b = gb = g \cdot xa = x'a \qquad \text{for} \quad x' = gx \in S \quad (\text{since } g = bb' \in S)\,,$$

$$b = bh = ay \cdot h = ay' \qquad \text{for} \quad y' = yh \in S \quad (\text{since } h = b'b \in S)\,;$$

furthermore, $x'b = gx \cdot b = gb = b$. Consequently, $b = xa = ay$ implies by (2) that $b = x \cdot e'a$, and by (1) that $b = e'a \cdot y$. Together with $xb = b$ it follows that $b \leq e'a = e'af'$.

The third application describes order-theoretical properties of *strong semilattices of semigroups*. For semigroups S which are strong semilattices of regular and simple semigroups S_α, Petrich [12; 3.5] found two such properties concerning the $\mathcal{J}$-classes of S (note that for such semigroups S, Green's relation

$\mathcal{J}$ yields the semilattice decomposition of S into the subsemigroups S_α: see [12; 3.2]). We will show these properties for strong semilattices of semigroups satisfying a condition weaker than that of Theorem 4.8 (note that every regular semigroup fulfills this condition).

THEOREM 4.9. *Let S be a strong semilattice Y of semigroups S_α $(\alpha \in Y)$ such that for every $a \in S_\alpha$ there are $r, s \in S_\alpha$ with $ra = a = as$ $(\alpha \in Y)$: then*

(i) *for any $a \in S_\alpha$ and all $\beta < \alpha$ in Y, the partially ordered set $(a] \cap S_\beta$ admits a greatest element;*

(ii) *for all $\alpha > \beta > \gamma$ in Y and any $a \in S_\alpha$, $c \in S_\gamma$ such that $a > c$, if $\bar{a}$ is the greatest element of $(a] \cap S_\beta$, then $\bar{a} > c$.*

Proof.

(i) Let $a \in S_\alpha$; then there are $r, s \in S_\alpha$ such that $ra = a = as$. Let $x \in S_\beta$ with $\beta < \alpha$; then we have for $r' = r\varphi_{\alpha,\beta} \in S_\beta$:

$$\begin{aligned} xa &= (x\varphi_{\beta,\alpha\beta})(a\varphi_{\alpha,\alpha\beta}) = x(a\varphi_{\alpha,\beta}) = x \cdot (ra)\varphi_{\alpha,\beta} = x(r\varphi_{\alpha,\beta})(a\varphi_{\alpha,\beta}) \\ &= xr'(a\varphi_{\alpha,\beta}), \\ xr'a &= (x\varphi_{\beta,\alpha\beta})(r'\varphi_{\beta,\alpha\beta})(a\varphi_{\alpha,\alpha\beta}) = xr'(a\varphi_{\alpha,\beta}); \end{aligned}$$

it follows that

(1) $xa = xr'a$ for every $x \in S_\beta$ with $\beta < \alpha$;
(2) similarly: $ax = r'ax$ for every $x \in S_\beta$ with $\beta < \alpha$;
(3) similarly: for $s' = s\varphi_{\alpha,\beta} \in S_\beta$

$$xa = xas' \quad \text{and} \quad ax = as'x \qquad \text{for every} \quad x \in S_\beta \quad \text{with} \quad \beta < \alpha .$$

Put $x = s'$ in (2); then $as' = r'as'$. Put $x = r'$ in (3); then $r'a = r'as'$. Hence, $as' = r'a = r'as' \in S_\beta$; since $r' \cdot as' = as'$, it follows that $r'as' \leq a$ and $r'as' \in (a] \cap S_\beta$.

We show that $r'as'$ is the greatest element of $(a] \cap S_\beta$. Let $b \in (a] \cap S_\beta$: then $b \in S_\beta$ and $b < a$ ($b = a$ implies $b \in S_\alpha$). Hence, $b = ua = av$, $ub = b = bv$ for some $u, v \in S$. We can choose $u, v \in S_\beta$. Indeed, by hypothesis, for $b \in S_\beta$ there are $p, q \in S_\beta$ such that $pb = b = bq$; thus, $b = pb = p \cdot ua = za$ for $z = pu$; whence $zb = pu \cdot b = pb = b$ and $z \in S_\beta$ (because $p \in S_\beta$, $u \in S_\gamma$, say, $\implies$ $z = pu \in S_{\beta\gamma}$ and $b = zb \in S_{\beta\gamma\beta} = S_{\beta\gamma}$ $\implies$ $\beta = \beta\gamma$). Similarly, for $v \in S$. Thus, $b = ua = av$ implies by (1) that $b = u \cdot r'a$ and by (2) that $b = r'a \cdot v$. Together with $ub = b$, it follows that $b \leq r'a = r'as'$.

(ii) Let $\alpha > \beta > \gamma$ in Y, $a \in S_\alpha$, $c \in S_\gamma$, $a > c$. Then by (i), $(a] \cap S_\beta$ has a greatest element $\bar{a} = r'a = as' = r'as'$, where $r' = r\varphi_{\alpha,\beta}$, $s' = s\varphi_{\alpha,\beta}$. We have to show that $c < \bar{a}$:

Since $c < a$, there are $x, y \in S$ such that $c = xa = ay$, $xc = c = cy$. Consequently,

$$c = xa = x \cdot as = cs = (c\varphi_{\gamma,\alpha\gamma})(s\varphi_{\alpha,\alpha\gamma}) = c(s\varphi_{\alpha,\gamma}) = (c\varphi_{\gamma,\beta\gamma})\big[(s\varphi_{\alpha,\beta})\varphi_{\beta,\beta\gamma}\big]$$
$$= c \cdot (s\varphi_{\alpha,\beta}) = cs' = xa \cdot s' = x\bar{a}\,.$$

Similarly, $c = \bar{a}y$. Since $xc = c$, it follows that $c \leq \bar{a}$. But $c \in S_\gamma$ and $\bar{a} = r'a \in S_{\beta\alpha} = S_\beta$ with $\gamma < \beta$; whence $c < \bar{a}$.

Note added in proof. Corollary 3.5 can be generalized as follows:

If S is an inflation of a semigroup T, then the natural partial order of S is (right-, left-) compatible if and only if that of T is (right-, left-) compatible. Since a rectangular group T is trivially ordered, every inflation S of T has a compatible natural partial order.

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