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ON OSCILLATORY FOURTH ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS II

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ABSTRACT. Oscillatory and asymptotic behaviour of solutions of a class of nonlinear fourth order neutral differential equations of the form

$$\big(r(t)\big(y(t)+p(t)y(t-\tau)\big)''\big)''+q(t)G\big(y(t-\sigma)\big)=0$$

and

$$\big(r(t)\big(y(t)+p(t)y(t-\tau)\big)''\big)''+q(t)G\big(y(t-\sigma)\big)=f(t) \qquad (*)$$

are studied under the assumption $\int\limits_0^\infty t/r(t)\,\mathrm{d}t=\infty$ for various ranges of $p(t)$. Sufficient conditions are obtained for the existence of bounded positive solutions of $(*)$.

1. Introduction

In a recent paper [6], we have studied oscillation of solutions of fourth order nonlinear neutral differential equations of the form

$$\big[r(t)\big(y(t)+p(t)y(t-\tau)\big)''\big]''+q(t)G\big(y(t-\sigma)\big)=0 \tag{1}$$

and the associated forced equations

$$\big[r(t)\big(y(t)+p(t)y(t-\tau)\big)''\big]''+q(t)G\big(y(t-\sigma)\big)=f(t) \tag{2}$$

under the assumption

$$\int\limits_0^\infty \big(t/r(t)\big)\,\mathrm{d}t<\infty\,, \tag{3}$$

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where $r \in C([0,\infty),(0,\infty))$, $p \in C([0,\infty),\mathbb{R})$, $q \in C([0,\infty),[0,\infty))$, $f \in C([0,\infty),\mathbb{R})$, $G \in C(\mathbb{R},\mathbb{R})$ is nondecreasing and $uG(u) > 0$ for $u \neq 0$, $\tau > 0$ and $\sigma > 0$. Further, sufficient conditions are obtained for the existence of bounded positive solutions of (2). In this paper, we continue the study of equations (1) and (2) under the assumption

(H_1) $\int_0^\infty (t/r(t))\, \mathrm{d}t = \infty$.

If $r(t) \equiv 1$, then (H_1) is satisfied and equations (1) and (2) reduced. respectively, to

$$\big(y(t) + p(t)y(t-\tau)\big)'''' + q(t)G\big(y(t-\sigma)\big) = 0 \tag{4}$$

and

$$\big(y(t) + p(t)y(t-\tau)\big)'''' + q(t)G\big(y(t-\sigma)\big) = f(t)\,. \tag{5}$$

In recent papers [4], [5], Parhi and Rath have studied oscillation of solutions of nth order neutral differential equations

$$\big(y(t) + p(t)y(t-\tau)\big)^{(n)} + q(t)G\big(y(t-\sigma)\big) = 0 \tag{6}$$

and

$$\big(y(t) + p(t)y(t-\tau)\big)^{(n)} + q(t)G\big(y(t-\sigma)\big) = f(t)\,. \tag{7}$$

Clearly, equations (4) and (5) are particular cases of equations (6) and (7) respectively. However, equations (1) and (2) cannot be termed, in general, as particular cases of equations (6) and (7) in view of (H_1). Therefore, it is interesting to study equations (1) and (2) under (H_1). Moreover, most of the results in [5] hold for n even. The motivation for the work in [6] and the present work came from the work of Kusano and Naito [2], [3], where they studied oscillatory behaviour of solutions of fourth order nonlinear differential equations of the form

$$\big(r(t)y(t)''\big)'' + y(t)F(y^2(t),t) = 0\,,$$

where r and F are continuous and positive on $[0,\infty)$ and $(0,\infty)\times[0,\infty)$ respectively, under the assumption (3) or (H_1). It is interesting to observe that the nature of the function r influences the behaviour of solutions of (1) or (2). This influence is more explicit in case of unforced equation (1).

By a *solution* of equation (2) we understand a function $y \in C([-\rho,\infty),\mathbb{R})$ such that $y(t) + p(t)y(t-\tau)$ is twice continuously differentiable, $r(t)\big(y(t) + p(t)y(t-\tau)\big)''$ is twice continuously differentiable and (2) is satisfied for $t \geq 0$, where $\rho = \max\{\tau,\sigma\}$, and $\sup\{|y(t)| : t \geq t_0\} > 0$ for every $t_0 \geq 0$. A solution of (2) is said to be *oscillatory* if it has arbitrarily large zeros; otherwise, it is called *nonoscillatory*.

2. Sufficient conditions for oscillation

In this section, sufficient conditions are obtained for oscillation of solutions of (1) and (2). We need the following lemmas for our work in the sequel:

LEMMA 2.1. *Let* (H_1) *hold. Let* u *be a twice continuously differentiable function on* $[0,\infty)$ *such that* $r(t)u''(t)$ *is twice continuously differentiable and* $\big(r(t)u''(t)\big)'' \le 0$ *for large* t. *If* $u(t) > 0$ *ultimately, then one of the cases* (a) *and* (b) *holds for large* t, *and if* $u(t) < 0$ *ultimately, then one of the cases* (b), (c), (d) *and* (e) *holds for large* t, *where*

(a) $u'(t) > 0$, $u''(t) > 0$ *and* $\big(r(t)u''(t)\big)' > 0$,
(b) $u'(t) > 0$, $u''(t) < 0$ *and* $\big(r(t)u''(t)\big)' > 0$,
(c) $u'(t) < 0$, $u''(t) < 0$ *and* $\big(r(t)u''(t)\big)' > 0$,
(d) $u'(t) < 0$, $u''(t) < 0$ *and* $\big(r(t)u''(t)\big)' < 0$,
(e) $u'(t) < 0$, $u''(t) > 0$ *and* $\big(r(t)u''(t)\big)' > 0$.

Proof. Since $\big(r(t)u''(t)\big)'' \le 0$, then $u(t)$, $u'(t)$, $r(t)u''(t)$ and $\big(r(t)u''(t)\big)'$ are monotonic and hence there are eight possibilities. Let $u(t) > 0$ for $t \ge t_0 > 0$. It is enough to show that (c), (d), (e) and the following cases, viz.,

(f) $u'(t) < 0$, $u''(t) > 0$ and $\big(r(t)u''(t)\big)' < 0$,
(g) $u'(t) > 0$, $u''(t) > 0$ and $\big(r(t)u''(t)\big)' < 0$,
(h) $u'(t) > 0$, $u''(t) < 0$ and $\big(r(t)u''(t)\big)' < 0$

do not hold. Indeed, in each of the cases (c) and (d), $u(t) < 0$ for large t, which is a contradiction. In case (e), $u''(t) > \big(r(t_1)u''(t_1)\big)/r(t)$ for $t \ge t_1 > t_0$. Multiplying the inequality through by t and then integrating it we obtain $u'(t) > 0$ for large t due to (H_1), a contradiction. Since $\big(r(t)u''(t)\big)'$ is monotonic decreasing, then in each of the cases (f) and (g), $u''(t) < 0$ for large t, which is a contradiction. In case (h), integrating $\big(r(t)u''(t)\big)'' \le 0$ twice we obtain $r(t)u''(t) < (ru'')'(t_1)(t-t_1)$, $t_1 > t_0$, and hence $r(t)u''(t) < -kt$ for $t \ge t_2 > t_1$, where $k > 0$. Consequently, $u'(t) < 0$ for large t in view of (H_1), which is a contradiction. Next suppose that $u(t) < 0$ for $t \ge t_0 > 0$. The case (a) does not occur because in this case $u(t) > 0$ ultimately. In each of the cases (f) and (g), $u''(t) < 0$ for large t, which is a contradiction. Proceeding as above we obtain a contradiction in the case (h). Thus the lemma is proved. □

LEMMA 2.2. *Let the conditions of Lemma* 2.1 *hold. If* $u(t) > 0$ *ultimately, then* $u(t) > R_T(t)\big(r(t)u''(t)\big)'$ *for* $t \ge T \ge 0$, *where*

$$R_T(t) = \int_T^t \frac{(t-s)(s-T)}{r(s)}\,\mathrm{d}s\,.$$

Proof. Let $u(t) > 0$ for $t \geq t_0 > 0$. From Lemma 2.1 it follows that one of the cases (a) and (b) holds. Suppose that (a) holds. Since $(ru'')'$ is decreasing, then we have

$$r(t)u''(t) > \int_T^t \big(r(s)u''(s)\big)' \, \mathrm{d}s > (t-T)\big(r(t)u''(t)\big)'$$

for $t \geq T > t_0$. Hence

$$u'(t) > \big(r(t)u''(t)\big)' \int_T^t \big((s-T)/r(s)\big) \, \mathrm{d}s .$$

Consequently the integration by parts yields

$$\begin{aligned} u(t) &> \big(r(t)u''(t)\big)' \int_T^t \bigg(\int_T^\theta \big((s-T)/r(s)\big) \, \mathrm{d}s \bigg) \, \mathrm{d}\theta \\ &= \big(r(t)u''(t)\big)' \int_T^t \big((t-s)(s-T)/r(s)\big) \, \mathrm{d}s \\ &= \big(r(t)u''(t)\big)' R_T(t) . \end{aligned}$$

Next suppose that (b) holds. For $t \geq T > t_0$, we integrate $R_T(t)\big(r(t)u''(t)\big)'' \leq 0$ by parts to obtain

$$\begin{aligned} 0 &\geq \int_T^t R_T(s)\big(r(s)u''(s)\big)'' \, \mathrm{d}s \\ &> R_T(t)\big(r(t)u''(t)\big)' + \int_T^t (s-T)u''(s) \, \mathrm{d}s \\ &> R_T(t)\big(r(t)u''(t)\big)' - u(t) \end{aligned}$$

since $R'_T(t) > 0$ and $R'_T(T) = 0$. Hence $u(t) > \big(r(t)u''(t)\big)' R_T(t)$. Thus the lemma is proved. □

Lemma 2.3. ([1; p. 46]) *If $q \in C\big([0,\infty),[0,\infty)\big)$ and*

$$\liminf_{t\to\infty} \int_{t-\tau}^t q(s) \, \mathrm{d}s > 1/\mathrm{e} ,$$

then $x'(t) + q(t)x(t-\tau) \leq 0$, $t \geq 0$, cannot have an eventually positive solution.

THEOREM 2.4. *Let $0 \le p(t) \le p < \infty$, $\tau \le \sigma$ and (H_1) hold. Suppose that*

(H_2) *there exists $\lambda > 0$ such that $G(u) + G(\nu) \ge \lambda G(u+\nu)$ for $u > 0$ and $\nu > 0$;*

(H_3) *$G(u)G(\nu) = G(u\nu)$ for $u, \nu \in \mathbb{R}$;*

(H_4) *$\int_0^c \mathrm{d}u/G(u) < \infty$ for all $c > 0$;*

(H_5) *$\int_{T+\rho}^{\infty} G\big(R_T(t-\sigma)\big)Q(t)\,\mathrm{d}t = \infty$, $T \ge 0$,*
where $Q(t) = \min\{q(t), q(t-\tau)\}$ for $t \ge \tau$.

Then every solution of (1) *oscillates.*

Remark. (H_3) implies that $G(-u) = -G(u)$, $u \in \mathbb{R}$. Indeed, $G(1)G(1) = G(1)$ and $G(1) > 0$ imply that $G(1) = 1$. Further, $G(-1)G(-1) = G(1) = 1$ and $G(-1) < 0$ imply that $G(-1) = -1$. Hence $G(-u) = G(-1)G(u) = -G(u)$, $u \in \mathbb{R}$. On the other hand, $G(u\nu) = G(u)G(\nu)$ for $u > 0$ and $\nu > 0$ and $G(-u) = -G(u)$, $u \in \mathbb{R}$, imply that $G(xy) = G(x)G(y)$ for $x, y \in \mathbb{R}$.

Proof of Theorem 2.4. If possible, let $y(t)$ be a nonoscillatory solution of (1). Hence $y(t) > 0$ or < 0 for $t \ge t_0 > 0$. Let $y(t) > 0$ for $t \ge t_0$. Setting

$$z(t) = y(t) + p(t)y(t-\tau)\,, \tag{8}$$

we obtain

$$0 < z(t) \le y(t) + py(t-\tau) \tag{9}$$

and

$$\big(r(t)z''(t)\big)'' = -q(t)G\big(y(t-\sigma)\big) \le 0 \tag{10}$$

for $t \ge t_0 + \rho$. Then one of the cases (a) and (b) of Lemma 2.1 holds. The use of (H_2) and (H_3) yields

$$\begin{aligned} 0 &= \big(r(t)z''(t)''\big)'' + q(t)G\big(y(t-\sigma)\big) + G(p)\big(r(t-\tau)z''(t-\tau)\big)'' \\ &\qquad\qquad + G(p)q(t-\tau)G\big(y(t-\tau)\big) \\ &\ge \big(r(t)z''(t)\big)'' + G(p)\big(r(t-\tau)z''(t-\tau-\sigma)\big)'' \\ &\qquad + \lambda Q(t)G\big(y(t-\sigma) + py(t-\tau-\sigma)\big) \\ &\ge \big(r(t)z''(t)\big)'' + G(p)\big(r(t-\tau)z''(t-\tau)\big)'' + \lambda Q(t)G\big(z(t-\sigma)\big) \end{aligned}$$

for $t \ge t_1 \ge t_0 + 2\rho$. Hence, by Lemma 2.2, we obtain

$$\begin{aligned}
0 &\geq \big(r(t)z''(t)\big)'' + G(p)\big(r(t-\tau)z''(t-\tau)\big)'' \\
&\qquad + \lambda Q(t)G\big(R_T(t-\sigma)\big(r(t-\sigma)z''(t-\sigma)\big)'\big) \\
&= \big(r(t)z''(t)\big)'' + G(p)\big(r(t-\tau)z''(t-\tau)\big)'' \\
&\qquad + \lambda Q(t)G\big(R_T(t-\sigma)\big)G\big(\big(r(t-\sigma)z''(t-\sigma)\big)'\big)
\end{aligned}$$

for $t \geq T + \rho > t_1$. Hence

$$\begin{aligned}
\lambda Q(t)G\big(R_T(t-\sigma)\big) &\leq -\big[G\big(\big(r(t-\sigma)z''(t-\sigma)\big)'\big)\big]^{-1}\big(r(t)z''(t)\big)'' \\
&\qquad - G(p)\big[G\big(\big(r(t-\sigma)z''(t-\sigma)\big)'\big)\big]^{-1}\big(r(t-\tau)z''(t-\tau)\big)'' \\
&\leq -\big[G\big(\big(r(t)z''(t)\big)'\big)\big]^{-1}\big(r(t)z''(t)\big)'' \\
&\qquad - G(p)\big[G\big(\big(r(t-\tau)z''(t-\tau)\big)'\big)\big]^{-1}\big(r(t-\tau)z''(t-\tau)\big)''.
\end{aligned}$$

Since $\lim\limits_{t\to\infty}\big(r(t)z''(t)\big)'$ exists, then the use of (H_4) yields

$$\int\limits_{T+\rho}^{\infty} Q(t)G\big(R_T(t-\sigma)\big)\,\mathrm{d}t < \infty,$$

which contradicts (H_5). Hence $y(t) < 0$ for $t \geq t_0$. Putting $x(t) = -y(t)$ for $t \geq t_0$, we obtain $x(t) > 0$ for $t \geq t_0$ and

$$\big[r(t)\big(x(t) + p(t)x(t-\tau)\big)''\big]'' + q(t)G\big(x(t-\sigma)\big) = 0.$$

Proceeding as above we arrive at a contradiction. Thus the proof of the theorem is complete. □

THEOREM 2.5. *Let $0 \leq p(t) \leq p < \infty$. Suppose that (H_1) and (H_2) hold. If*

(H_3') $G(u)G(\nu) \geq G(u\nu)$ *for* $u > 0$, $\nu > 0$,

(H_6) $G(-u) = -G(u)$, $u \in \mathbb{R}$,

and

(H_7) $\int\limits_{\tau}^{\infty} Q(t)\,\mathrm{d}t = \infty$,

then every solution of (1) *oscillates.*

P r o o f. Let $y(t)$ be a nonoscillatory solution of (1). Let $y(t) > 0$ for $t \geq t_0 > 0$. The proof for the case $y(t) < 0$, $t \geq t_0$, is similar. Setting $z(t)$ as in (8) we obtain (9) and (10) for $t \geq t_0 + \rho$. From Lemma 2.1 it follows that one of the cases (a) and (b) holds. Hence $z(t) > k > 0$ for $t \geq t_1 > t_0 + \rho$. Proceeding as in the proof of Theorem 2.4 we obtain

$$\begin{aligned} 0 &\geq \big(r(t)z''(t)''\big)'' + G(p)\big(r(t-\tau)z''(t-\tau)\big)'' + \lambda Q(t)G\big(z(t-\sigma)\big) \\ &\geq \big(r(t)z''(t)\big)'' + G(p)\big(r(t-\tau)z''(t-\tau)\big)'' + \lambda Q(t)G(k) \end{aligned}$$

for $t \geq t_2 > t_1 + 2\rho$. Hence $\int_{t_2}^{\infty} Q(t)\, \mathrm{d}t < \infty$, which is a contradiction to (H_7). Thus the theorem is proved. □

Remark. (H_3') and (H_6) need not imply (H_3). Indeed, if

$$G(u) = \big(\alpha + \beta|u|^{\lambda}\big)|u|^{\mu} \operatorname{sgn} u\,, \qquad \lambda > 0\,,\ \ \mu > 0\,,\ \ \alpha \geq 1\,,\ \ \beta \geq 1\,,$$

then (H_3') and (H_6) are satisfied but (H_3) fails to hold.

Remark. The prototype of G satisfying (H_2), (H_3') and (H_6) is

$$G(u) = \big(a + b|u|^{\lambda}\big)|u|^{\mu} \operatorname{sgn} u\,, \qquad \text{where} \quad a \geq 1\,,\ \ b \geq 1\,,\ \ \lambda \geq 0\,,\ \ \mu \geq 0\,.$$

Remark. In Theorem 2.5, G could be superlinear, sublinear or linear. However, (H_7) implies (H_5) because $R_T'(t) > 0$ for $t \geq T_1 > T$.

Theorem 2.6. *Let $0 \leq p(t) \leq p < 1$. Suppose that (H_1) and (H_3) hold. If*

(H_8) $\liminf\limits_{|x|\to 0}\big(G(x)/x\big) \geq \gamma > 0$

and

(H_9) $\liminf\limits_{t\to\infty} \int_{t-\sigma}^{t} G\big(R_T(s-\sigma)\big)q(s)\, \mathrm{d}s > \big(\mathrm{e}\,\gamma G(1-p)\big)^{-1}$,

then all solutions of (1) *oscillate.*

Remark. (H_9) implies that

(H_{10}) $\int_{T+\sigma}^{\infty} G\big(R_T(s-\sigma)\big)q(s)\, \mathrm{d}s = \infty$.

Indeed, if

$$\int_{T+\sigma}^{\infty} G\big(R_T(s-\sigma)\big)q(s)\, \mathrm{d}s = \alpha < \infty\,,$$

then, for $t > T + 2\sigma$,

$$\int_{t-\sigma}^{t} G\big(R_T(s-\sigma)\big)q(s)\, \mathrm{d}s = \left(\int_{T+\sigma}^{t} - \int_{T+\sigma}^{t-\sigma}\right) G\big(R_T(s-\sigma)\big)q(s)\, \mathrm{d}s$$

implies that

$$\liminf_{t\to\infty} \int\limits_{t-\sigma}^{t} G\big(R_T(s-\sigma)\big)q(s)\ \mathrm{d}s \le \alpha - \alpha = 0\,,$$

which is a contradiction to (H_9).

Proof of Theorem 2.6. Suppose that $y(t)$ is a nonoscillatory solution of (1). Let $y(t) > 0$ for $t \ge t_0 > 0$. The case $y(t) < 0$ for $t \ge t_0$ may similarly be dealt with. Setting $z(t)$ as in (8) we obtain $z(t) > 0$ and (10) for $t \ge t_0 + \rho$. Then one of the cases (a) and (b) of Lemma 2.1 holds. In each case, $z(t)$ is increasing. Hence, for $t \ge t_0 + 2\rho$

$$\big(1-p(t)\big)z(t) < z(t) - p(t)z(t-\tau) = y(t) - p(t)p(t-\tau)y(t-2\tau) < y(t)$$

and $\big(1-p(t)\big)z(t) \ge (1-p)z(t)$. Thus $y(t) > (1-p)z(t)$. From (10) we obtain, for $t \ge T+\sigma > t_0 + 2\rho + \sigma$,

$$\begin{aligned} 0 &\ge \big(r(t)z''(t)\big)'' + q(t)G\big((1-p)z(t-\sigma)\big) \\ &\ge \big(r(t)z''(t)\big)'' + q(t)G\Big((1-p)R_T(t-\sigma)\big(r(t-\sigma)z''(t-\sigma)\big)'\Big) \\ &= \big(r(t)z''(t)\big)'' + G(1-p)q(t)G\big(R_T(t-\sigma)\big)G\Big(\big(r(t-\sigma)z''(t-\sigma)\big)'\Big) \end{aligned} \tag{11}$$

using Lemma 2.2 and (H_3). Let $\lim\limits_{t\to\infty}\big(r(t)\,z''(t)\big)' = \alpha$. If $0 < \alpha < \infty$, then $\big(r(t)z''(t)\big)' > \beta > 0$ for $t \ge t_1 > T+\sigma$. From (11) we obtain, for $t \ge t_2 > t_1+\sigma$,

$$G(1-p)q(t)G\big(R_T(t-\sigma)\big)G(\beta) \le -\big(r(t)z''(t)\big)''\,.$$

Integrating the above inequality yields

$$\int\limits_{t_2}^{\infty} q(t)G\big(R_T(t-\sigma)\big)\ \mathrm{d}t < \infty\,,$$

a contradiction to (H_{10}). Hence $\alpha = 0$. Consequently, (H_8) implies that $G\Big(\big(r(t)z''(t)\big)'\Big) \ge \gamma\big(r(t)z''(t)\big)'$ for $t \ge t_3 > t_2$. Using this in (11) we obtain, for $t \ge t_3 + \sigma$,

$$\big(r(t)z''(t)\big)'' + \gamma G(1-p)q(t)G\big(R_T(t-\sigma)\big)\big(r(t-\sigma)z''(t-\sigma)\big)' \le 0\,.$$

Hence the inequality $u'(t)+\gamma G(1-p)q(t)G\big(R_T(t-\sigma)\big)u(t-\sigma) \le 0$ admits a positive solution $\big(r(t)z''(t)\big)'$, which is a contradiction due to (H_9) and Lemma 2.3. This completes the proof of the theorem. □

THEOREM 2.7. *Let $0 \le p(t) \le p < \infty$, $\tau \le \sigma$ and (H_1), (H_2) and (H_3) hold. Suppose that*

(H_{11}) $G(x_1)/x_1^\alpha \ge G(x_2)/x_2^\alpha$ *for* $x_1 \ge x_2 > 0$ *and* $\alpha \ge 1$

and

(H_{12}) $\int\limits_{T+\sigma}^{\infty} R_T^\alpha(t-\sigma)Q(t)\,\mathrm{d}t = \infty$.

Then every solution of (1) *oscillates.*

P r o o f. Proceeding as in the proof of Theorem 2.4 we obtain

$$\big(r(t)z''(t)\big)'' + G(p)\big(r(t-\tau)z''(t-\tau)\big)'' + \lambda Q(t)G\big(z(t-\sigma)\big) \le 0 \tag{12}$$

for $t \ge t_1 > t_0 + 2\rho$. Since $z(t)$ is increasing, then $z(t) > k > 0$ for $t \ge t_2 > t_1$. Using (H_{11}) and Lemma 2.2 we obtain, for $t \ge T + \sigma > t_2 + \sigma$,

$$\begin{aligned} G\big(z(t-\sigma)\big) &= \Big(G\big(z(t-\sigma)\big)/z^\alpha(t-\sigma)\Big)z^\alpha(t-\sigma) \\ &\ge \big(G(k)/k^\alpha\big)z^\alpha(t-\sigma) \\ &> \big(G(k)/k^\alpha\big)R_T^\alpha(t-\sigma)\Big(\big(r(t-\sigma)z''(t-\sigma)\big)'\Big)^\alpha. \end{aligned}$$

Hence (12) yields

$$\begin{aligned} &\lambda\big(G(k)/k^\alpha\big)R_T^\alpha(t-\sigma)Q(t) \\ &\quad< -\Big(\big(r(t-\sigma)z''(t-\sigma)\big)'\Big)^{-\alpha}\Big[\big(r(t)z''(t)\big)'' + G(p)\big(r(t-\tau)z''(t-\tau)\big)''\Big] \\ &\quad< -\Big(\big(r(t)z''(t)\big)'\Big)^{-\alpha}\big(r(t)z''(t)\big)'' \\ &\qquad - G(p)\Big(\big(r(t-\tau)z''(t-\tau)\big)'\Big)^{-\alpha}\big(r(t-\tau)z''(t-\tau)\big)''. \end{aligned}$$

Since $\lim\limits_{t\to\infty}\big(r(t)z''(t)\big)'$ exists, then proceeding as in the proof of Theorem 2.4 we obtain

$$\int\limits_{T+\sigma}^{\infty} R_T^\alpha(t-\sigma)Q(t)\,\mathrm{d}t < \infty,$$

a contradiction to (H_{12}). Thus the theorem is proved. □

THEOREM 2.8. *Let $-1 < p \le p(t) \le 0$. If (H_1), (H_3), (H_4) hold and if*

(H_{13}) $\int\limits_0^{\infty} q(t)\,\mathrm{d}t = \infty$,

then every solution of (1) *oscillates or tends to zero as $t \to \infty$.*

P r o o f . Let $y(t)$ be a nonoscillatory solution of (1). In view of (H_3) it is enough to consider $y(t) > 0$ for $t \geq t_0 > 0$. Setting $z(t)$ as in (8) we obtain (10) for $t \geq t_0 + \rho$. Hence $z(t) > 0$ or < 0 for $t \geq t_1 > t_0 + \rho$. Let $z(t) > 0$ for $t \geq t_1$. From Lemma 2.1 it follows that one of the cases (a) and (b) holds. Hence $z(t) > R_T(t)\big(r(t)z''(t)\big)'$ for $t \geq T > t_1$ by Lemma 2.2. Since $z(t) \leq y(t)$ and $\big(r(t)z''(t)\big)'$ is monotonic decreasing, then (10) yields, for $t \geq t_2 > T + \sigma$,

$$\big(r(t)z''(t)\big)'' \leq -q(t)G\big(R_T(t-\sigma)\big)G\Big(\big(r(t)z''(t)\big)'\Big).$$

Hence

$$\int_{t_2}^{\infty} q(t)G\big(R_T(t-\sigma)\big)\,\mathrm{d}t < \infty.$$

Since $R_T'(t) > 0$ and $R_T(t) > 0$, then $\int_{t_2}^{\infty} q(t)\,\mathrm{d}t < \infty$, which is a contradiction to (H_{13}). Hence $z(t) < 0$ for $t \geq t_1$. This implies that $y(t) < -p(t)y(t-\tau) < y(t-\tau)$ and hence $y(t)$ is bounded. Consequently, $z(t)$ is bounded. One of the cases (b)–(e) holds by Lemma 2.1. Let the case (b) hold. If $\lim_{t\to\infty} z(t) = \alpha$, then $-\infty < \alpha \leq 0$. Suppose that $-\infty < \alpha < 0$. Hence $z(t) < \beta < 0$ for $t \geq t_3 > t_2$. Further, $z(t) > py(t-\tau)$ for $t \geq t_0$ and hence $y(t-\sigma) > p^{-1}\beta > 0$ for $t \geq t_3+\rho$. Consequently, (10) yields

$$q(t)G(p^{-1}\beta) \leq -\big(r(t)z''(t)\big)''.$$

Integrating we obtain $\int_{t_3+\rho}^{\infty} q(t)\,\mathrm{d}t < \infty$, a contradiction. Hence $\alpha = 0$. Consequently,

$$\begin{aligned} 0 = \limsup_{t\to\infty} z(t) &\geq \limsup_{t\to\infty}\big(y(t) + py(t-\tau)\big) \geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}\big(py(t-\tau)\big) \\ &= \limsup_{t\to\infty} y(t) + p\limsup_{t\to\infty} y(t-\tau) = (1+p)\limsup_{t\to\infty} y(t). \end{aligned}$$

Since $1 + p > 0$, then $\lim_{t\to\infty} y(t) = 0$. In each of the cases (c) and (d), $\lim_{t\to\infty} z(t) = -\infty$, which contradicts the boundedness of $z(t)$. Suppose the case (e) holds. Since $z(t)$ is bounded, then $\lim_{t\to\infty} z(t)$ exists. Further, $t > t_1$ implies that $z''(t) > \big(r(t_1)z''(t_1)\big)/r(t)$. Multiplying the inequality through by t and then integrating it we obtain $z'(t) > 0$ for large t due to (H_1). This contradicts the fact that $z'(t) < 0$ in case (e). Thus the proof is complete. □

THEOREM 2.9. *Let $-\infty < p_1 \leq p(t) \leq p_2 \leq -1$. If ($\mathrm{H}_1$) and ($\mathrm{H}_{13}$) hold, then every bounded solution of (1) oscillates or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a bounded nonoscillatory solution of (1). Then $y(t) > 0$ or < 0 for $t \ge t_0 > 0$. Let $y(t) > 0$ for $t \ge t_0$. Setting $z(t)$ as in (8) we obtain (10) for $t \ge t_0 + \rho$ and hence $z(t) > 0$ or < 0 for $t \ge t_1 > t_0 + \rho$. If $z(t) > 0$ for $t \ge t_1$, then one of the cases (a) and (b) of Lemma 2.1 holds and $y(t) > -p(t)y(t-\tau) > y(t-\tau)$. Hence $\liminf\limits_{t\to\infty} y(t) > 0$. From (10) it follows that $\int\limits_{t_2}^{\infty} q(t)\,\mathrm{d}t < \infty$, $t_2 > t_1$, which is a contradiction to (H_{13}). Hence $z(t) < 0$ for $t \ge t_1$. Since $y(t)$ is bounded, then $z(t)$ is bounded and hence none of the cases (c), (d), (e) of Lemma 2.1 occurs. Suppose that the case (b) of Lemma 2.1 holds. If $-\infty < \lim\limits_{t\to\infty} z(t) < 0$, then proceeding as in the proof of Theorem 2.8 we arrive at a contradiction. Hence $\lim\limits_{t\to\infty} z(t) = 0$. Consequently,

$$
\begin{aligned}
0 = \liminf_{t\to\infty} z(t) &\le \liminf_{t\to\infty}\big(y(t) + p_2 y(t-\tau)\big) \le \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}\big(p_2 y(t-\tau)\big) \\
&= (1+p_2)\limsup_{t\to\infty} y(t)\,.
\end{aligned}
$$

Since $1 + p_2 < 0$, then $\lim\limits_{t\to\infty} y(t) = 0$. If $y(t) < 0$ for $t \ge t_0$, then we set $x(t) = -y(t)$ to obtain $x(t) > 0$ for $t \ge t_0$ and

$$
\Big(r(t)\big(x(t) + p(t)x(t-\tau)\big)''\Big)'' + q(t)\tilde{G}\big(x(t-\sigma)\big) = 0\,,
$$

where $\tilde{G}(u) = -G(-u)$. Proceeding as above we obtain $\lim\limits_{t\to\infty} x(t) = 0$ and hence $\lim\limits_{t\to\infty} y(t) = 0$. Thus the theorem is proved. □

In the following, we obtain sufficient conditions for oscillation of solutions of forced equation (2). Let

(H_{14}) there exists $F \in C^2\big([0,\infty),\mathbb{R}\big)$ such that $F(t)$ changes sign, $rF'' \in C^2\big([0,\infty),\mathbb{R}\big)$ and $(rF'')'' = f$;

(H_{15}) there exists $F \in C^2\big([0,\infty),\mathbb{R}\big)$ such that $F(t)$ changes sign with $-\infty < \liminf\limits_{t\to\infty} F(t) < 0 < \limsup\limits_{t\to\infty} F(t) < \infty$, $rF'' \in C^2\big([0,\infty),\mathbb{R}\big)$ and $(rF'')'' = f$;

(H_{16}) there exists $F \in C^2\big([0,\infty),\mathbb{R}\big)$ such that $F(t)$ does not change sign, $\lim\limits_{t\to\infty} F(t) = 0$, $rF'' \in C^2\big([0,\infty),\mathbb{R}\big)$ and $(rF'')'' = f$;

(H'_{16}) there exists $F \in C^2\big([0,\infty),\mathbb{R}\big)$ such that $\lim\limits_{t\to\infty} F(t) = 0$, $rF'' \in C^2\big([0,\infty),\mathbb{R}\big)$ and $(rF'')'' = f$.

Remark. If $\lim_{t\to\infty} F(t) = \alpha \neq 0$ in (H_{16}), then we may proceed as follows: We set $\tilde{F}(t) = F(t) - \alpha$ to obtain, $\tilde{F} \in C^2([0,\infty),\mathbb{R})$, $\tilde{F}''(t) = F''(t)$ and $\lim_{t\to\infty} \tilde{F}(t) = 0$. If $\tilde{F}(t)$ changes sign, then it comes under (H_{14}). If $\tilde{F}(t)$ does not change sign, then it comes under (H_{16}).

THEOREM 2.10. *Let $0 \le p(t) \le p < \infty$. Suppose that (H_1), (H_2), (H_3'), (H_6) and (H_{14}) hold. If*

(H_{17}) $\int_\sigma^\infty Q(t)G(F^+(t-\sigma))\,\mathrm{d}t = \infty$ *and* $\int_\sigma^\infty Q(t)G(F^-(t-\sigma))\,\mathrm{d}t = \infty$, *where* $F^+(t) = \max\{F(t),0\}$ *and* $F^-(t) = \max\{-F(t),0\}$,

then all solutions of (2) *oscillate.*

Proof. Let $y(t)$ be a nonoscillatory solution of (2). Hence $y(t) > 0$ or < 0 for $t \ge t_0 > 0$. Suppose that $y(t) > 0$ for $t \ge t_0$. Setting $z(t)$ as in (8) we obtain (9) for $t \ge t_0 + \rho$. Let

$$w(t) = z(t) - F(t). \tag{13}$$

Hence

$$\big(r(t)w''(t)\big)'' = -q(t)G\big(y(t-\sigma)\big) \le 0 \tag{14}$$

for $t \ge t_0 + \rho$. Thus $w(t) > 0$ or < 0 for $t \ge t_1 > t_0 + \rho$. Since $F(t)$ changes sign, then $w(t) > 0$ for $t \ge t_1$ by (13). Hence one of the cases (a) and (b) of Lemma 2.1 holds for large t and $z(t) \ge F^+(t)$. For $t \ge t_2 > t_1$, we have

$$\begin{aligned} 0 &= \big(r(t)w''(t)\big)'' + G(p)\big(r(t-\tau)w''(t-\tau)\big)'' + q(t)G\big(y(t-\sigma)\big) \\ &\qquad + G(p)q(t-\tau)G\big(y(t-\tau-\sigma)\big) \\ &\ge \big(r(t)w''(t)\big)'' + G(p)\big(r(t-\tau)w''(t-\tau)\big)'' \\ &\qquad + \lambda Q(t)G\big(y(t-\sigma) + py(t-\tau-\sigma)\big) \\ &\ge \big(r(t)w''(t)\big)'' + G(p)\big(r(t-\tau)w''(t-\tau)\big)'' + \lambda Q(t)G\big(z(t-\sigma)\big) \\ &\ge \big(r(t)w''(t)\big)'' + G(p)\big(r(t-\tau)w''(t-\tau)\big)'' + \lambda Q(t)G\big(F^+(t-\sigma)\big). \end{aligned} \tag{15}$$

Hence

$$\int_{t_2+\sigma}^{\infty} Q(t)G\big(F^+(t-\sigma)\big)\,\mathrm{d}t < \infty,$$

which is a contradiction to (H_{17}). If $y(t) < 0$ for $t \ge t_0$, we set $x(t) = -y(t)$ to obtain $x(t) > 0$ for $t \ge t_0$ and

$$\Big(r(t)\big(x(t) + p(t)x(t-\tau)\big)''\Big)'' + q(t)\tilde{G}\big(x(t-\sigma)\big) = \tilde{f}(t),$$

where $\tilde{f}(t) = -f(t)$. If $\tilde{F}(t) = -F(t)$, then $\tilde{F}(t)$ changes sign, $\tilde{F}^+(t) = F^-(t)$ and $\big(r(t)\tilde{F}''(t)\big)'' = f(t)$. Proceeding as above we obtain a contradiction. Thus the theorem is proved. □

EXAMPLE. Consider

$$\left[\mathrm{e}^{-t}\big(y(t)+(1+\mathrm{e}^{-t})y(t-\pi)\big)''\right]''+\mathrm{e}^{t}\,y(t-2\pi)=2\,\mathrm{e}^{-t}\cos t\,,\tag{16}$$

$t\ge 0$. Hence $0<p(t)=1+\mathrm{e}^{-t}<2$ and $Q(t)=\min\{\mathrm{e}^{t},\mathrm{e}^{t-\pi}\}=\mathrm{e}^{t-\pi}$. Further, $F(t)=\sin t$ implies that $\big(r(t)F''(t)\big)''=\big(-\mathrm{e}^{-t}\sin t\big)''=2\,\mathrm{e}^{-t}\cos t$. As $F(t-2\pi)=\sin t$, then

$$F^{+}(t-2\pi)=\begin{cases}\sin t\,, & 2n\pi\le t\le(2n+1)\pi\,,\\ 0\,, & (2n+1)\pi\le t\le 2(n+1)\pi\end{cases}$$

and

$$F^{-}(t-2\pi)=\begin{cases}0\,, & 2n\pi\le t\le(2n+1)\pi\,,\\ -\sin t\,, & (2n+1)\pi\le t\le 2(n+1)\pi\,,\end{cases}$$

$n=0,1,2,\dots$. Thus

$$\begin{aligned}\int_{2\pi}^{\infty}Q(t)G\big(F^{+}(t-2\pi)\big)\,\mathrm{d}t&=\mathrm{e}^{-\pi}\int_{2\pi}^{\infty}\mathrm{e}^{t}\,F^{+}(t-2\pi)\,\mathrm{d}t\\&=\mathrm{e}^{-\pi}\sum_{n=1}^{\infty}\int_{2n\pi}^{(2n+1)\pi}\mathrm{e}^{t}\sin t\,\mathrm{d}t\\&=\frac{\mathrm{e}^{-\pi}}{2}\sum_{n=1}^{\infty}\big[\mathrm{e}^{t}(\sin t-\cos t)\big]_{2n\pi}^{(2n+1)\pi}\\&=\frac12\,\mathrm{e}^{-\pi}(\mathrm{e}^{\pi}+1)\sum_{n=1}^{\infty}\mathrm{e}^{2n\pi}=\infty\end{aligned}$$

and

$$\begin{aligned}\int_{2\pi}^{\infty}Q(t)G\big(F^{-}(t-2\pi)\big)\,\mathrm{d}t&=-\,\mathrm{e}^{-\pi}\sum_{n=0}^{\infty}\int_{(2n+1)\pi}^{2(n+1)\pi}\mathrm{e}^{t}\sin t\,\mathrm{d}t\\&=\frac12\,\mathrm{e}^{-\pi}\big(\mathrm{e}^{2\pi}+\mathrm{e}^{\pi}\big)\sum_{n=0}^{\infty}\mathrm{e}^{2n\pi}=\infty\,.\end{aligned}$$

From Theorem 2.10 it follows that all solutions of (16) oscillate. Equation (16) may be written as

$$\begin{aligned}y^{(4)}(t)+\big(1+\mathrm{e}^{-t}\big)y^{(4)}(t-\pi)-2y'''(t)-2\big(1+3\,\mathrm{e}^{-t}\big)y'''(t-\pi)&\\+y''(t)+\big(1+13\,\mathrm{e}^{-t}\big)y''(t-\pi)-12\,\mathrm{e}^{-t}\,y'(t-\pi)&\\+4\,\mathrm{e}^{-t}\,y(t-\pi)+\mathrm{e}^{2t}\,y(t-2\pi)=2\cos t\,.&\end{aligned}\tag{17}$$

However, (17) cannot be written in the form

$$\left[y(t)+p(t)y(t-\tau)\right]^{(4)}+\sum_{i=1}^{m} Q_i(t)G\left(y(t-\sigma_i)\right)=f(t) \tag{18}$$

due to the presence of the terms $\left(1+\mathrm{e}^{-t}\right)y^{(4)}(t-\pi)$ and $y'''(t)$. If we take $p(t)=\left(1+\mathrm{e}^{-t}\right)$, then we obtain

$$\begin{aligned}\left[y(t)+(1+\mathrm{e}^{-t})y(t-\pi)\right]^{(4)} &= y^{(4)}(t)+\left(1+\mathrm{e}^{-t}\right)y^{(4)}(t-\pi)-4\,\mathrm{e}^{-t}\,y'''(t-\pi)\\ &\quad +6\,\mathrm{e}^{-t}\,y''(t-\pi)-4\,\mathrm{e}^{-t}\,y'(t-\pi)+\mathrm{e}^{-t}\,y(t-\pi)\,.\end{aligned}$$

If $p(t)$ is a term other than $1+\mathrm{e}^{-t}$, then we cannot get $\left(1+\mathrm{e}^{-t}\right)y^{(4)}(t-\pi)$. Hence the results valid for (18) cannot hold for (16). Thus the results in [4], [5] cannot be applied to equation (2). However, the results in this paper can be applied to (18) for $r(t)\equiv 1$ in (2) because (H_1) holds for $r(t)\equiv 1$. As there are a few results in [4], [5] for even n, the present work may be viewed as the complement of the work in [4], [5].

THEOREM 2.11. *Let $-1<p\le p(t)\le 0$. Suppose that (H_1) and (H_{15}) hold. If*

(H_{18}) $\int_\sigma^\infty q(t)G\left(F^+(t-\sigma)\right)\,\mathrm{d}t=\infty$ *and* $\int_\sigma^\infty q(t)G\left(F^-(t+\tau-\sigma)\right)\,\mathrm{d}t=\infty$

and

(H_{19}) $\int_\sigma^\infty q(t)G\left(F^-(t-\sigma)\right)\,\mathrm{d}t=\infty$ *and* $\int_\sigma^\infty q(t)G\left(F^+(t+\tau-\sigma)\right)\,\mathrm{d}t=\infty$,

then every solution of (2) *oscillates.*

Proof. Proceeding as in the proof of Theorem 2.10 we obtain $w(t)>0$ or <0 for $t\ge t_1>t_0+\rho$ when $y(t)>0$ for $t\ge t_0$. Let $w(t)>0$ for $t\ge t_1$. Hence one of the cases (a) and (b) of Lemma 2.1 holds. Further, $w(t)>0$ implies that $y(t)\ge y(t)+p(t)y(t-\tau)>F(t)$ and hence $y(t)\ge F^+(t)$. From (14) we obtain $\int_{t_1+\sigma}^\infty q(t)G\left(F^+(t-\sigma)\right)\,\mathrm{d}t<\infty$, a contradiction. Hence $w(t)<0$ for $t\ge t_1$. Then one of the cases (b)–(e) of Lemma 2.1 holds. Let the case (b) hold. Since $w(t)<0$, then $y(t)>F^-(t+\tau)$ for $t\ge t_1$. From (14) we get

$$\int_{t_1+\sigma}^{\infty} q(t)G\left(F^-(t+\tau-\sigma)\right)\,\mathrm{d}t<\infty\,,$$

a contradiction. If $y(t)$ is unbounded, then there exists an increasing sequence $\{\sigma_n\}_{n=1}^\infty$ such that $\sigma_n\to\infty$ and $y(\sigma_n)\to\infty$ as $n\to\infty$ and $y(\sigma_n)=\max\{y(t):t_1\le t\le\sigma_n\}$. We may choose n large enough such that $\sigma_n-\tau>t_1$. Hence

$w(\sigma_n) \geq y(\sigma_n) + py(\sigma_n - \tau) - F(\sigma_n) \geq (1+p)y(\sigma_n) - F(\sigma_n)$. Since $F(t)$ is bounded and $(1+p) > 0$, then $w(\sigma_n) > 0$ for large n, which is a contradiction. Hence $y(t)$ is bounded. Consequently, $w(t)$ is bounded. This implies that the cases (c) and (d) of Lemma 2.1 fail to hold. The boundedness of $w(t)$ and (H_1) imply that the case (e) of Lemma 2.1 does not hold. If $y(t) < 0$ for $t \geq t_0$, then we set $x(t) = -y(t)$ to obtain $x(t) > 0$ for $t \geq t_0$,

$$\big(r(t)\big(x(t) + p(t)x(t-\tau)\big)''\big)'' + q(t)\tilde{G}\big(x(t-\sigma)\big) = \tilde{f}(t),$$

where $\tilde{G}(u) = -G(-u)$ and $\tilde{f}(t) = -f(t)$. If $\tilde{F}(t) = -F(t)$, then $\tilde{F}(t)$ changes sign with $-\infty < \liminf\limits_{t\to\infty} \tilde{F}(t) < 0 < \limsup\limits_{t\to\infty} \tilde{F}(t) < \infty$, $\tilde{F}^{+}(t) = F^{-}(t)$, $\tilde{F}\ \ (t) = F^{+}(t)$ and $\big(r(t)\tilde{F}''(t)\big)'' = \tilde{f}(t)$. Proceeding as above a contradiction is obtained. Thus the theorem is proved. □

Remark. If $q(t)$ is nonincreasing or τ-periodic, then

$$\int_{\sigma}^{\infty} q(t)G\big(F^{+}(t-\sigma)\big)\,\mathrm{d}t = \infty \implies \int_{\sigma}^{\infty} q(t)G\big(F^{+}(t+\tau-\sigma)\big)\,\mathrm{d}t = \infty$$

and

$$\int_{\sigma}^{\infty} q(t)G\big(F^{-}(t-\sigma)\big)\,\mathrm{d}t = \infty \implies \int_{\sigma}^{\infty} q(t)G\big(F^{-}(t+\tau-\sigma)\big)\,\mathrm{d}t = \infty.$$

THEOREM 2.12. *Let $-\infty < p \leq p(t) \leq 0$. Suppose that (H_1), (H_3), (H_{15}), (H_{18}) and (H_{19}) hold. Then every solution of (2) oscillates or tends to $\pm\infty$ as $t \to \infty$.*

Proof. Proceeding as in the proof of Theorem 2.11 we obtain a contradiction if $w(t) > 0$ for $t \geq t_1 > t_0 + \rho$. Hence $w(t) < 0$ for $t \geq t_1$. One of the cases (b)–(e) of Lemma 2.1 holds. Suppose that the case (b) holds. Since $w(t) < 0$, then $py(t-\tau) < F(t)$, that is, $y(t) \geq (-p^{-1})F^{-}(t+\tau)$ for $t \geq t_1$. Integrating (14) and using (H_3) we obtain $\int\limits_{t_1+\sigma}^{\infty} q(t)G\big(F^{-}(t+\tau-\sigma)\big)\,\mathrm{d}t < \infty$, a contradiction. In each of the cases (c) and (d), $\lim\limits_{t\to\infty} w(t) = -\infty$. If, in case (e), $-\infty < \lim\limits_{t\to\infty} w(t) < 0$, then we obtain a contradiction due to (H_1). Thus $\lim\limits_{t\to\infty} w(t) = -\infty$ in each of the cases (c)–(e). Consequently, $py(t-\tau) < w(t) + F(t)$ implies that $\limsup\limits_{t\to\infty}\big(py(t-\tau)\big) \leq \lim\limits_{t\to\infty} w(t) + \limsup\limits_{t\to\infty} F(t)$, that is, $p\liminf\limits_{t\to\infty} y(t) = -\infty$ due to (H_{15}). Hence $\lim\limits_{t\to\infty} y(t) = \infty$. The proof for the case $y(t) < 0$ for $t \geq t_0$ is similar. Thus the proof of the theorem is complete. □

COROLLARY 2.13. *If the conditions of Theorem* 2.12 *are satisfied, then every bounded solution of* (2) *oscillates.*

THEOREM 2.14. *Let* $0 \le p(t) \le p < \infty$ *and let* (H_1), (H_2), (H_3'), (H_6) *and* (H_{16}) *hold. If* $\int_\sigma^\infty Q(t)G\bigl(|F(t-\sigma)|\bigr)\,\mathrm{d}t = \infty$, *then every bounded solution of* (2) *oscillates or tends to zero as* $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 2.10 we obtain $w(t) > 0$ or < 0 for $t \ge t_1 > t_0 + \rho$. Let $w(t) > 0$ for $t \ge t_1$. Hence $z(t) > F(t)$. Suppose that $F(t) > 0$ for $t \ge t_2 > t_1$. From (15) it follows by Lemma 2.1, that $\int_{t_2+\sigma}^\infty Q(t)G\bigl(F(t-\sigma)\bigr)\,\mathrm{d}t < \infty$, which is a contradiction. Hence $F(t) < 0$ for $t \ge t_2$. From (14) we obtain $\int_{t_2+\sigma}^\infty Q(t)G\bigl(y(t-\sigma)\bigr)\,\mathrm{d}t < \infty$ due to Lemma 2.1. Hence $\liminf_{t\to\infty} y(t) = 0$ because $\int_\sigma^\infty Q(t)G\bigl(|F(t-\sigma)|\bigr)\,\mathrm{d}t = \infty$ implies that $\int_\sigma^\infty q(t)\,\mathrm{d}t = \infty$. Since $w(t)$ is bounded and monotonic, then $\lim_{t\to\infty} w(t)$ exists and hence $\lim_{t\to\infty} z(t)$ exists. Thus $\lim_{t\to\infty} z(t) = 0$ (see [1; Lemma 1.5.2]). As $z(t) \ge y(t)$, then $\lim_{t\to\infty} y(t) = 0$. Suppose that $w(t) < 0$ for $t \ge t_1$. Hence $y(t) \le z(t) < F(t)$. Consequently, $\lim_{t\to\infty} y(t) = 0$. Thus the theorem is proved. □

THEOREM 2.15. *Let* $-1 < p \le p(t) \le 0$. *If* (H_1) *and* (H_{16}) *hold and if* $\int_0^\infty q(t)\,\mathrm{d}t = \infty$, *then every solution of* (2) *oscillates or tends to zero as* $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 2.10, we have $w(t) > 0$ or < 0 for $t \ge t_1 > t_0 + \rho$. Let $w(t) > 0$ for $t \ge t_1$. From (14) we obtain, due to Lemma 2.1 that

$$\int_{t_2+\sigma}^\infty q(t)G\bigl(y(t-\sigma)\bigr)\,\mathrm{d}t < \infty, \tag{19}$$

where $t_2 > t_1$. Hence $\liminf_{t\to\infty} y(t) = 0$. On the other hand, $\lim_{t\to\infty} w(t) = \infty$ in the case (a) of Lemma 2.1. Hence $\lim_{t\to\infty} z(t) = \infty$. However, $y(t) \ge z(t)$ implies that $\lim_{t\to\infty} y(t) = \infty$, which is a contradiction. In the case (b) of Lemma 2.1, $\lim_{t\to\infty} w(t) = \alpha$, where $0 < \alpha \le \infty$. If $\alpha = \infty$, we obtain a contradiction as above. Hence $0 < \alpha < \infty$. Consequently, $\lim_{t\to\infty} z(t) = \alpha$. From [1; Lemma 1.5.2] it follows that $\alpha = 0$, which is a contradiction. Hence $w(t) < 0$ for $t \ge t_1$. We claim that $y(t)$ is bounded. Indeed, if $y(t)$ is unbounded, then there exists an increasing sequence $\{\sigma_n\}_{n=1}^\infty$ such that $\sigma_n \to \infty$ and $y(\sigma_n) \to \infty$ as $n \to \infty$ and $y(\sigma_n) = \max\{y(t) :\ t_1 \le t \le \sigma_n\}$. Hence $w(\sigma_n) \ge y(\sigma_n) + py(\sigma_n - \tau)$

$- F(\sigma_n) \geq (1+p)y(\sigma_n) - F(\sigma_n)$. Consequently, $w(\sigma_n) > 0$ for large n, which is a contradiction. Thus $w(t)$ is bounded. In each of the cases (c) and (d) of Lemma 2.1, $\lim_{t\to\infty} w(t) = -\infty$, which is a contradiction. In each of the cases (b) and (e) of Lemma 2.1, (19) holds and hence $\liminf_{t\to\infty} y(t) = 0$ and $\lim_{t\to\infty} w(t)$ exists. Consequently, $\lim_{t\to\infty} z(t)$ exists. From [1; Lemma 1.5.2] it follows that $\lim_{t\to\infty} z(t) = 0$. Hence

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty}\big[y(t) + p(t)y(t-\tau)\big] \geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty} py(t-\tau) \\ &= (1+p)\limsup_{t\to\infty} y(t)\,. \end{aligned}$$

As $1+p>0$, then $\lim_{t\to\infty} y(t) = 0$. The proof of the theorem is complete. □

Example. Consider

$$\Big[\mathrm{e}^{-t}\big(y(t) + \mathrm{e}^{-t}(\mathrm{e}^{-t}-1)y(t-1)\big)''\Big]'' + 4\,\mathrm{e}^{-6}\,y^3(t-2) = 40\,\mathrm{e}^{-3t}\,, \tag{20}$$

$t \geq 1$. Here $-1 < -\mathrm{e}^{-1} < p(t) < 0$ and $q(t) = 4\,\mathrm{e}^{-6}$. If $F(t) = (10/9)\,\mathrm{e}^{-2t}$, then $\big(\mathrm{e}^{-t} F''(t)\big)'' = 40\,\mathrm{e}^{-3t}$ and $\lim_{t\to\infty} F(t) = 0$. As all the conditions of Theorem 2.15 are satisfied, then every solution of (20) oscillates or tends to zero as $t \to \infty$. In particular, $y(t) = \mathrm{e}^{-t}$ is a solution of (20) which goes to zero as $t \to \infty$. Equation (20) may be written as

$$\begin{aligned} y^{(4)}(t) &- \big(\mathrm{e}^{-(t+1)} - \mathrm{e}^{-1}\big)y^{(4)}(t-1) - 2y'''(t) + \big(2\,\mathrm{e}^{-1} - 6\,\mathrm{e}^{-(t+1)}\big)y'''(t-1) \\ &+ y''(t) - \big(3\,\mathrm{e}^{-(t+1)} + \mathrm{e}^{-1}\big)y''(t-1) - 12\,\mathrm{e}^{-(t+1)}\,y'(t-1) \\ &\qquad + 4\,\mathrm{e}^{-(t+1)}\,y(t-1) + 4\,\mathrm{e}^{(t-6)}\,y^3(t-2) = 40\,\mathrm{e}^{-2t}\,. \end{aligned}$$

Explanation given in the example following Theorem 2.10 also holds here.

Theorem 2.16. *Let* $-\infty < p(t) \leq 0$. *If* (H_1) *and* (H_{16}) *hold and if* $\int_0^\infty q(t)\,\mathrm{d}t = \infty$, *then every bounded solution of* (2) *oscillates or tends to zero as* $t \to \infty$.

The proof is similar to that of Theorem 2.16 and hence is omitted.

Corollary 2.17. *Suppose that the conditions of Theorem* 2.16 *are satisfied. Then every nonoscillatory solution of* (2) *which does not tend to zero as* $t \to \infty$ *is unbounded.*

Remark. Theorems 2.10–2.12, 2.14 and Corollary 2.13 do not hold for equation (1). However, Theorems 2.15 and 2.16 hold for equation (1).

3. Existence of positive solutions

Sufficient conditions are obtained for the existence of bounded positive solutions of equation (2).

THEOREM 3.1. *Let* $0 \le p(t) \le p < 1$. *Suppose that* (H_{15}) *holds with* $-\frac{1}{8}(1-p) < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \frac{1}{2}(1-p)$ *and* G *is Lipschitzian on intervals of the form* $[a,b]$, $0 < a < b < \infty$. *If*

$$\int_0^\infty \frac{t}{r(t)}\left(\int_t^\infty sq(s)\,\mathrm{d}s\right)\mathrm{d}t < \infty, \tag{21}$$

then (2) *admits a positive bounded solution.*

Proof. It is possible to choose $t_0 > 0$ sufficiently large such that

$$L\int_{t_0}^\infty \frac{t}{r(t)}\left(\int_t^\infty sq(s)\,\mathrm{d}s\right)\mathrm{d}t < \frac{1}{4}(1-p),$$

where $L = \max\{L_1, G(1)\}$ and L_1 is the Lipschitz constant of G on $\left[\frac{1}{8}(1-p), 1\right]$. Let $X = BC\big([t_0,\infty),\mathbb{R}\big)$. Then X is a Banach space with respect to the supremum norm. Let

$$S = \left\{x \in X:\ \tfrac{1}{8}(1-p) \le x(t) \le 1,\ \ t \ge t_0\right\}.$$

Hence S is a complete metric space. For $y \in S$, we define

$$Ty(t) = \begin{cases} Ty(t_0+\rho) & \text{for } t \in [t_0, t_0+\rho], \\[2ex] -p(t)y(t-\tau) + \frac{1+p}{2} + F(t) & \\ \quad - \int_t^\infty \left(\frac{s-t}{r(s)}\int_s^\infty (u-s)q(u)G\big(y(u-\sigma)\big)\,\mathrm{d}u\right)\mathrm{d}s & \text{for } t \ge t_0+\rho. \end{cases}$$

Hence $Ty(t) < \frac{1+p}{2} + \frac{1-p}{2} = 1$ and $Ty(t) > -p + \frac{1+p}{2} - \frac{1-p}{8} - \frac{1-p}{4} = \frac{1-p}{8}$ for $t \ge t_0 + \rho$. Consequently, $Ty \in S$, that is, $T\colon S \to S$. Further, for $x, y \in S$,

$$|Ty(t) - Tx(t)| \le p\|x-y\| + \frac{1-p}{4}\|x-y\| = \frac{1+3p}{4}\|x-y\|.$$

Hence $\|Ty - Tx\| \le \frac{1+3p}{4}\|x-y\|$ for every $x, y \in S$. Thus T is a contraction. Consequently, T has a unique fixed point y_0 in S. Then $y_0(t)$ is a solution of (2) with $\frac{1}{8}(1-p) \le y_0(t) \le 1$. Thus the theorem is proved. □

THEOREM 3.2. *Let $0 \le p(t) \le p < 1$. Suppose that* (H'_{16}) *and* (21) *hold. If G is Lipschitzian on intervals of the form $[a,b]$, $0 < a < b < \infty$, then* (2) *admits a bounded positive solution.*

Proof. We may choose $t_0 > 0$ sufficiently large such that $|F(t)| < \frac{1-p}{10}$ and

$$L \int_{t_0}^{\infty} \frac{t}{r(t)} \left(\int_t^{\infty} sq(s)\, \mathrm{d}s \right) \mathrm{d}t < \frac{1-p}{20},$$

where $L = \max\{L_1, G(1)\}$ and L_1 is the Lipschitz constant of G on $\left[\frac{1-p}{20}, 1\right]$. We set $X = BC\big([t_0,\infty),\mathbb{R}\big)$ and

$$S = \left\{ x \in X :\ \tfrac{1-p}{20} \le x(t) \le 1,\ \ t \ge t_0 \right\}.$$

For $y \in S$, we set

$$Ty(t) = \begin{cases} Ty(t_0+\rho) & \text{for } t \in [t_0, t_0+\rho], \\[2ex] -\,p(t)y(t-\tau) + \dfrac{1+4p}{5} + F(t) & \\ -\displaystyle\int_t^{\infty} \Big(\tfrac{s-t}{r(s)} \int_s^{\infty} (u-s)q(u)G\big(y(u-\sigma)\big)\, \mathrm{d}u \Big)\, \mathrm{d}s & \text{for } t \ge t_0+\rho. \end{cases}$$

Proceeding as in the proof of Theorem 3.1 we may show that T has a unique fixed point y_0 in S and it is the required solution of (2). The proof of the theorem is complete. □

Remark. Theorems similar to Theorems 3.1 and 3.2 can be proved in other ranges of $p(t)$.

Summary

In [6], equations (1) and (2) are studied under the assumption (3). As (H_1) holds for $r(t) \equiv 1$, then the results of this paper can be compared to some of the results in [5]. Indeed, Theorem 2.4 is better than [5; Theorem 3.6] for $n = 4$. Theorems 2.5 2.7 are new. Theorem 2.8 is comparable with [5; Theorem 3.12]. As equation (1) is not, in general, a particular case of equation (6), then we obtain every solution of (1) oscillates or tends to zero as $t \to \infty$ in Theorem 2.8. Similar situation occurs when Theorem 2.9 is compared with [5; Theorem 3.14]. Theorems 2.10 and 2.11 are similar to [5; Theorems 2.3, 2.1] respectively for $n = 4$. Theorem 2.12 can be compared with [5; Theorem 2.2] for $n = 4$. Theorems 2.14 2.16 are new. It would be interesting to study neutral differential equations with quasi-derivatives of the form

$$\Big(r_3(t)\Big(r_2(t)\Big(r_1(t)\big(y(t)+p(t)y(t-\tau)\big)'\Big)'\Big)'\Big)' + q(t)G\big(y(t-\sigma)\big) = f(t).$$

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