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NOTE ON HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

BOGDAN RZEPECKI

1. Introduction

Let f be a continuous function on $[0, a]\times(-\infty, \infty)$ such that $|f(t, x)|\leq M\ t^p$ and $t^r\cdot|f(t, x)-f(t, y)|\leq L\cdot|x-y|^q$, where $M>0$, $L>0$, $p>-1$, $q\geq 1$ and r are constants with $q(p+1)-r=p$ and $(2M)^{q-1}L<(p+1)^q$. Under the above assumption O. Kooi [7] (cf. also [10]) proved the uniqueness of a solution of the equation $x'=f(t, x)$ satisfying the condition $x(0)=x_0$, and the uniform convergence of successive approximation to this solution. (We note that the firs result of the above type was obtained by A. Rosenblatt in [12].) For the Darboux problem for hyperbolic differential equations similar theorems were obtained by J. S. W. Wong [17] and V. Ďurkovič [3]—[6].

The purpose of the present paper is to give some results on the following hyperbolic partial differential equation

$$(+)\qquad \frac{\partial^2 z}{\partial x\,\partial y}=f(x, y, z),$$

where f is a continuous function satisfying the Kooi type conditions.

We consider the questions of the existence of the unique solution (as a limit of successive approximations) and the continuous dependence of the Darboux problem solution on the right-hand side for the above equation. The method used here is hased on the concept due to Luxemburg [9] of the "generalized metric space" (see aldo [10], [11], [16], [3]—[6], [15], [17]). Our results are connected with Bielecki's method ([1], [2]) of norm changing, and extend the facts of [7], [10], [16], [13], [14], [17].

2. Preliminaries

Let X be a non-empty set and let d be a function defined on $X\times X$ with the following conditions:

(L1) $0 \leqslant d(x, y) \leqslant \infty$,
(L2) $d(x, y) = 0$ if and only if $x = y$,
(L3) $d(x, y) = d(y, x)$,
(L4) $d(x, y) \leqslant d(x, z) + d(z, y)$

for all x, y, z in X.

A *generalized metric space* (X, d) is a pair composed with a non-empty set X and a distance function d satisfying the above axioms (L1)—(L4). If further every d-Cauchy sequence in X is d-convergent (i.e., $\lim_{p,q\to\infty} d(x_p, x_q) = 0$ for a sequence (x_n) in X implies the existence of an element $x_0 \in X$ such that $\lim_{n\to\infty} d(x_n, x_0) = 0$), then X is called *a complete generalized metric space.*

Moreover, we shall use the notion of the $\mathscr{L}^*$-space [8]:

The set X is called an *$\mathscr{L}^*$-space* if a certain class of sequences in X (named elements of this class are *convergent sequences*) is distinguished in such a way that for every sequence (p_n) from this class there exists an element $p = \lim_{n\to\infty} p_n$ in X having the following properties:

1° if $\lim_{n\to\infty} p_n = p$ and $k_1 < k_2 < \ldots$, then $\lim_{n\to\infty} p_{k_n} = p$;

2° if $p_n = p$ for all $n \geqslant 1$, then $\lim_{n\to\infty} p_n = p$;

3° if the sequence (p_n) is not convergent to p, then it contains a subsequence (p_{k_n}) in which every subsequence fails to converge to p.

Let X and Y be two $\mathscr{L}^*$-spaces. A mapping f of X into Y is called *continuous at the point* $x_0 \in X$ if for each sequence (x_n) in X converging to x_0 we have $\lim_{n\to\infty} f(x_n) = f(x_0)$. Further, a mapping f is called *continuous* (on the $\mathscr{L}^*$-space X) if it is continuous at each point of X.

Generally, the theorems on the existence of the unique solution to the initial--value problems for the differential equations are proved with the help of either a fixed-point theorem of the Banach type or successive approximations in various forms. The proof of our main result will be based on the following theorem of the Banach fixed-point principle type:

Proposition (cf. [16]). *Let (X, d) be a generalized complete metric space, let T_0 and T_n $(n = 1, 2, \ldots)$ be mappings of X into itself such that $\lim_{n\to\infty} d(T_n x, T_0 x) = 0$ for all x in X. Assume, moreover, that there exist constants $\varepsilon > 0$, $0 \leqslant k < 1$ and an element $z_0 \in X$ such that*

$$d(z_0, T_n z_0) \leq \varepsilon, \quad d(T_n x, T_n y) \leq k \cdot d(x, y)$$

for all $n \geq 1$ *and* $x, y \in X$ *with* $d(x, y) \leq \varepsilon$.

Then the equation $T_m x = x$ $(m = 0, 1, \ldots)$ *has a unique solution* $u_m \in X$ *such that there exists an* ε*-chain joining* z_0 *and* u_m*), *and* $\lim_{n\to\infty} d(u_n, u_0) = 0$. *Further, every sequence of successive approximations* $x_n^{(m)} = T_m x_{n-1}^{(m)}$ $(n = 1, 2, \ldots)$, *where* $x_0^{(m)}$ *is an element such that there exists an* ε*-chain joining* z_0 *and* $x_0^{(m)}$, *is* d*-convergent to this solution* u_m.

3. Assumptions and notations

Let us put $G = (0, a] \times (0, b]$, $P = [0, a] \times [0, b]$ and $Q = P \times (-\infty, \infty)$. Let $\sigma \in C^1[0, a]$ and $\tau \in C^1[0, b]$ be functions such that $\sigma(0) = \tau(0)$. Assume, moreover, that λ is a bounded function on P and $\lambda(x, y) > 0$ for all (x, y) in G.

These assumptions remain valid throughout the paper and will not be repeated in formulations of particular results.

Let us denote:

$\mathfrak{X}$ — the set of all real continuous functions z on P such that $z(x, 0) = \sigma(x)$ for $0 \leq x \leq a$ and $z(0, y) = \tau(y)$ for $0 \leq y \leq b$;

by $\mathfrak{F}$ — the set of real continuous functions f on Q such that

$$|f(x, y, z)| \leq \delta(x, y) \quad \text{for all} \quad (x, y, z) \in Q,$$
$$|f(x, y, u) - f(x, y, v)| \leq L_f(x, y) \cdot |u - v|^{q_f}$$

for all $(x, y) \in G$ and $-\infty < u, v < \infty$, where δ is a non-negative integrable function on P which does not depend on f, $L_f\colon P \to [0, +\infty]$ is a function and $q_f \geq 1$ is a constant which may depend on f.

Let us put:

$$A = \sup_{(x, y) \in G} \frac{1}{\lambda(x, y)} \int_0^x \int_0^y \delta(u, v)\, du\, dv;$$

$$z^0(x, y) = \sigma(x) + \tau(y) - \sigma(0) + \int_0^x \int_0^y \delta(u, v)\, du\, dv \quad \text{for } (x, y) \in P;$$

$$U_f(x, y) = (\lambda(x, y))^{q_f} \cdot L_f(x, y) \quad \text{for } (x, y) \in P;$$

$$B_f = \sup_{(x, y) \in G} \frac{1}{\lambda(x, y)} \int_0^x \int_0^y U_f(u, v)\, du\, dv.$$

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* A finite sequence $x_0, x_1, \ldots, x_q$ of points of X is called ε-chain joining x_0 and x_q if $d(x_{i-1}, x_i) \leq \varepsilon$ for $i = 1, 2, \ldots, q$.

The hyperbolic equation (+) with the initial conditions

$$z(x, 0) = \sigma(x) \quad \text{for} \quad 0 \leqslant x \leqslant a,$$
$$z(0, y) = \tau(y) \text{for} \quad 0 \leqslant y \leqslant b$$

is equivalent to the integral equation

$$z(x, y) = \varphi_0(x, y) + \int_0^x \int_0^y f(u, v, z(u, v))\, du\, dv,$$

where $\varphi_0(x, y) = \sigma(x) + \tau(y) - \sigma(0)$. The successive approximations of the solution to the above problem with $f \in \mathfrak{F}$ are defined by

(*)
$$w_{j+1}(x, y) = \varphi_0(x, y) + \int_0^x \int_0^y f(u, v, w_j(u, v))\, du\, dv$$
$$(j = 0, 1, 2, \ldots),$$

where w_0 is an arbitrary function in $\mathfrak{X}$ such that there exists a $2A$-chain joining z^0 and w_0.

We define on the set $\mathfrak{F}$ a distance funcion d defined in the following way

$$d(z, w) = \sup_{(x, y) \in G} \frac{|z(x, y) - w(x, y)|}{\lambda(x, y)}.$$

We have: $(\sup_{(x, y) \in G} \lambda(x, y))^{-1} \cdot \sup_{(x, y) \in P} |z(x, y) - w(x, y)| \leqslant d(z, w)$. This shows that the d-convergence is generally stronger than the uniform one. Therefore, by a slight modification of the proof from [9] (cf. [11]) we can prove that $(\mathfrak{X}, d)$ is a generalized complete metric space.

4. Main result

Let us put

$$T(f, z)(x, y) = \varphi_0(x, y) + \int_0^x \int_0^y f(u, v, z(u, v))\, du\, dv$$

for $f \in \mathfrak{F}$ and $z \in \mathfrak{X}$, where $\varphi_0(x, y) = \sigma(x) + \tau(y) - \sigma(0)$. The set $\mathfrak{F}$ will be considered as an $\mathcal{S}^*$-space. Moreover, suppose that for every fixed z in $\mathfrak{X}$ the transformation $T(\cdot, z)$ maps continuously the $\mathcal{S}^*$-space $\mathfrak{F}$ into $(\mathfrak{X}, d)$.

Theorem. *Let $A < \infty$ and let $f \in \mathfrak{F}$. Suppose that the function U_f is integrable on P and $(2A)^{q_f - 1} \cdot B_f < 1$. Then there exists the unique function $z_f \in \mathfrak{X}$ satisfying the equation (+) on P. Moreover, there exists a $2A$-chain joining z^0 and z_f, and every sequence of successive approximations (*) is d-convergent to this z_f.*

Next assume that each U_f ($f \in \mathfrak{F}$) is integrable on P and $k = \sup_{f \in \mathfrak{F}} (2A)^{q_f - 1} \cdot B_f < 1$.

Then the function $f \mapsto z_f$ maps continuously the $\mathcal{L}^*$-space $\mathfrak{F}$ into $(\mathfrak{X}, d)$.

Proof. Let $f_m \in \mathfrak{F}$ $(m=0, 1, 2, \ldots)$ be such that $\lim_{n\to\infty} f_n = f_0$. We define a transformation T_m as $z \mapsto T(f_m, z)$. Then this T_m $(m=0, 1, \ldots)$ maps $\mathfrak{X}$ into itself, and for each $z \in \mathfrak{X}$ $d(T_n z, T_0 z) \to 0$ as $n \to \infty$.

Since $|z^0(x, y) - (T_n z^0)(x, y)| \leq 2 \cdot \int_0^x \int_0^y \delta(u, v)\, du\, dv$, and so $d(z^0, T_n z^0) \leq 2A$ for all $n \geqq 1$. We now prove that $d(T_n z, T_n w) \leq k \cdot d(z, w)$ $(n=1, 2, \ldots)$ for all $z, w \in \mathfrak{X}$ such that $d(z, w) \leq 2A$.

Indeed, for $(x, y) \in G$, $n \geq 1$ and $z, w \in \mathfrak{X}$ with $d(z, w) \leq 2A$ we have

$$|(T_n z)(x, y) - (T_n w)(x, y)| \leq \int_0^x \int_0^y L_{f_n}(u, v) |z(u, v) - w(u, v)|^{q_{f_n}}\, du\, dv \leq$$

$$\leq d(z, w) \cdot \int_0^x \int_0^y \lambda(u, v) \cdot L_{f_n}(u, v) \cdot |z(u, v) - w(u, v)|^{q_{f_n}-1}\, du\, dv \leq$$

$$\leq (2A)^{q_{f_n}-1} \cdot d(z, w) \cdot \int_0^x \int_0^y U_{f_n}(u, v)\, du\, dv,$$

whence

$$d(T_n z, T_n w) \leq (2A)^{q_{f_n}-1} \cdot B_{f_n} \cdot d(z, w) \leq k \cdot d(z, w).$$

Consequently, our Proposition is applicable to the mappings T_m and the proof is finished.

Remark. Let $\sup_{(x, y) \in G} (\lambda((x, y))^{-1} \cdot \int_0^x \int_0^y \lambda(u, v)\, du\, dv < \infty$. Assume, moreover, that the set $\mathfrak{F}$ is considered as an $\mathcal{L}^*$-space, where $\lim_{n\to\infty} f_n = f_0$ means that

$$\sup_{\substack{(x, y) \in G \\ z \in \Omega}} \frac{1}{\lambda(x, y)} |f_n(x, y, z) - f_0(x, y, z)| \to 0 \quad \text{as} \quad n \to \infty$$

for every compact Ω in the Euclidean space. Then $T(\cdot, z)$ (z is fixed in $\mathfrak{X}$) maps continuously $\mathfrak{F}$ into $\mathfrak{X}$.

Indeed, fix z in $\mathfrak{X}$ and let (f_n) be a sequence of $\mathfrak{F}$ converging to f_0. Then

$$\left| \int_0^x \int_0^y [f_n(u, v, z(u, v)) - f_0(u, v, z(u, v))]\, du\, dv \right| \leq$$

$$\leq \sup_{\substack{(u, v) \in G \\ u \in z[P]}} \frac{|f_n(u, v, s) - f_0(u, v, s)|}{\lambda(u, v)} \cdot \int_0^x \int_0^y \lambda(u, v)\, du\, dv$$

for (x, y) in G, whence

$$\sup_{(x,y)\in G}\frac{1}{\lambda(x,y)}\left|\int_0^x\int_0^y[f_n(u,v,z(u,v))-f_0(u,v,z(u,v))]\,du\,dv\right|\leqslant$$

$$\leqslant\sup_{(x,y)\in G}\frac{1}{\lambda(x,y)}\int_0^x\int_0^y\lambda(u,v)\,du\,dv\cdot\sup_{\substack{(u,v)\in G\\ s\in z[P]}}\frac{|f_n(u,v,s)-f_0(u,v,s)|}{\lambda(u,v)}$$

and therefore $d(T(f_n, z), T(f_0, z))\to 0$ as $n\to\infty$.

Now we are going to give some corollaries from the above results. Let us denote:

by $\mathfrak{F}_0$ — the set $\mathfrak{F}$ with $q_f\equiv q$ and $\delta(x, y) = M\cdot(x\cdot y)^p$, $L_f(x, y) = L(x\cdot y)^{-r}$ on P, where $M>0$, $L>0$, $p>-1$, $q\geqslant 1$ and r are constants such that $q(p+1)-p=r$;

by $\mathfrak{F}_1$ — the set $\mathfrak{F}$ with $q_f\equiv 1$ and $L_f(x, y)\equiv A_f$ on P, where $A_f>0$ is a constant (depending on a function $f\in\mathfrak{F}_1$);

by $\mathfrak{F}_2$ — the set $\mathfrak{F}_1$ with $\delta(x, y)\equiv C$ on P, where $C>0$ is a constant;

by $\mathfrak{X}_0$ — the generalized metric space $\mathfrak{X}$ with a distance function d generated by $\lambda(x, y) = (x\cdot y)^{p+1}$ on P, where p is a constant from the definition of the set $\mathfrak{F}_0$;

by $C_0(P)$ — the set $\mathfrak{X}$ with the usual supremum metric.

The set $\mathfrak{F}_0$ be considered with the convergence defined as that in the above Remark in the case of $\lambda(x, y) = (x\cdot y)^{p+1}$. Moreover, we shall deal with the sets $\mathfrak{F}_1$, $\mathfrak{F}_2$ as $\mathscr{L}^*$-spaces endowed with the almost uniform convergence and pointwise convergence on Q, respectively.

Corollary 1. *Suppose that $(2M)^{q-1}\cdot L<(p+1)^q$. Then for each $f\in\mathfrak{F}_0$ there exists a unique $z_f\in\mathfrak{X}_0$ satisfying the equation $(+)$ on P and, moreover, the function $f\mapsto z_f$, which maps $\mathfrak{F}_0$ into $\mathfrak{X}_0$, is continuous.*

Proof. Let us put $\lambda(x, y)=(x\cdot y)^{p+1}$ for $(x, y)\in P$, where p is a constant from the definition of the set $\mathfrak{F}_0$. Then

$$A=\sup_G\frac{1}{(x\cdot y)^{p+1}}\cdot\int_0^x\int_0^y M\cdot(u\cdot v)^p\,du\,dv=\frac{M}{(p+1)^2},$$

$$z^0(x, y)=\sigma(x)+\tau(y)-\sigma(0)+\frac{M}{(p+1)^2}(x\cdot y)^{p+1}\ \text{on}\ P.$$

Further, for f in $\mathscr{F}_0$ we have:

$$U_f(x, y)=L(x\cdot y)^{q(p+1)-r}=L(x\cdot y)^p\ \text{on}\ P,$$

$$B_f=\sup_G\frac{1}{(x\cdot y)^{p+1}}\int_0^x\int_0^y L(u\cdot v)^p\,du\,dv=\frac{L}{(p+1)^2}.$$

Since $(2M)^{q-1}\cdot L<(p+1)^{2q}$, and so

$$k=\sup_{f\in\mathfrak{F}_0}(2A)^{q-1}B_f=\frac{(2M)^{q-1}L}{(p+1)^{2q}}<1.$$

The application of our Theorem and Remark completes the proof.

Corollary 2. *For an arbitrary* $f \in \mathfrak{F}_i$ $(i=1, 2)$ *there exists a unique* $z_f \in C_0(P)$ *satisfying the equation* $(+)$ *on* P. *Moreover, if* $\sup \{A_f: f \in \mathfrak{F}_i\} < \infty$, *then* $f \mapsto z_f$ *maps continuously* $\mathfrak{F}_i$ *into* $C_0(P)$.

Proof. Let us put $\lambda(x, y) = \exp(p(x+y))$ for $(x, y) \in P$, where p is a positive constant such that $p^2 > \sup \{A_f: f \in \mathfrak{F}_i\}$.

The distance function d generated by the above λ is equivalent to the original supremum metric of the space of continuous functions on P. For $f \in \mathfrak{F}_i$ $(i=1, 2)$ and $(x, y) \in P$

$$U_f(x, y) = A_f \cdot \exp(p(x+y)) \text{ on } P,$$

$$B_f = \sup_{(x,y)} \exp(-p(x+y)) \int_0^x \int_0^y U_f(u, v)\,du\,dv$$

and

$$k = \sup_{f \in \mathfrak{F}_i} B_f = \sup_{f \in \mathfrak{F}_i} A_f \cdot \sup_{(x,y)} \exp(-p(x+y)) \cdot \int_0^x \int_0^y \exp(p(u+v))\,du\,dv \leq$$

$$\leq p^{-2} \cdot \sup_{f \in \mathfrak{F}_i} A_f < 1.$$

Consequently the case $i=1$ is obvious. Next, let us fix z in $C_0(P)$, let $f_n \in \mathfrak{F}_2$ $(n=1, 2, \ldots)$ and let the sequence (f_n) converge pointwise to f_0. Then the Lebesgue bounded convergence theorem implies that $\lim_{n\to\infty} T(f_n, z)(x, y) = T(f_0, z)(x, y)$ on P. By an equicontinuity of a sequence $(T(f_n, z))$ on the compact P, $\lim_{n\to\infty} T(f_n, z)(x, y) = T(f_0, z)(x, y)$ uniformly on P. Finally, the application of our Theorem completes the proof.

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ЗАМЕТКА ОБ ГИПЕРБОЛИЧЕСКИХ ДИФФЕРЕНЦИАЛЬНЫХ УРАГНЕНИЯХ ВТОРОГО ПОРЯДКА (I)

Б. Жепецки

Резюме

В данной работе рассматривается применение обобщенного принципа Банаха неподвижнои точки к исследованию задачи Дарбу для уравнения вида $\partial^2 z/\partial x\, \partial y = f(x, y, z)$ при условиях типа Коой [7]. Полученные результаты о существовании единственного решения связаны с методом Белецкого о изменении нормы в теории дифференциальных уравнений. Кроме того, мы покажем, что наша задача поставлена корректно. Для этой цели в множествах правых частей и граничных условий введен понятия предела последовательности точек и тем самым наделим их структурной $\mathscr{L}^*$-пространства.