Mathematica Slovaca

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Mathematica Slovaca, Vol. 31 (1981), No. 2, 183--185

Persistent URL: http://dml.cz/dmlcz/132403

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THE INTERSECTION OF VALUATION RINGS

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Let K be a field, G an additive abelian totally ordered group. By a valuation we mean a mapping v: $K' = K - \{0\} \to G$ such that $v(x \cdot y) = v(x) + v(y)$ and $v(x + y) \geqslant \min \{v(x), v(y)\}$. The set of all $x \in K$ such that $v(x) \geqslant 0$ or $x = 0$ is naturally a ring V. This ring is said to be the valuation ring of v. We shall say that $A \subset K$ is a valuation ring of K if A is the valuation ring for some valuation of K.

If A, B are subrings of K (having the unit element), $A \vee B$ denotes the smallest subring containing A and B.

We first note: If V is a valuation ring and B any subring of K, then

$$V \vee B = V \cdot B \tag{1}$$

holds (Hereby $V \cdot B = \{a \in V, b \in B\}$). Clearly $V \cdot B \subset V \vee B$ holds, therefore it is sufficient to prove only the converse inclusion.

Let $x \in V \vee B$. Then $x = a_1 b_1 + \ldots + a_m b_m$, where $a_i \in V$, $b_i \in B$, $1 \leqslant i \leqslant m$. Let v be such a valuation of the field K for which $V - \{0\} = \{x \in K: v(x) \geqslant 0\}$. Choose $b_j \in \{b_1, \ldots, b_m\}$ such that $v(b_j) = \min \{v(b_i): i = 1, 2, \ldots, m\}$. Then $x = x b_j^{-1} b_j$ and $v(x b_j^{-1}) \geqslant 0$, hence $x b_j^{-1} \in V$ and $b_j \in B$, therefore $x \in V \cdot B$.

In this paper we show that (1) can be generalized if we suppose moreover that the group of values of valuations is archimedean. Henceforth we suppose that all valuation rings considered below have the property that the group of values of their valuations is archimedean. We shall also suppose that to the valuation ring V_i there belongs the valuation v_i and that each subring of K has a unit element. For the basic notion of the present paper see [1] and [2].

We prove

Proposition. *Let K be a field, let V_i ($i = 1, 2, \ldots, n$) be valuation rings contained in K. Then for each subring B of the field K*

$$B \vee \bigcap_{i=1}^{n} V_i = B \cdot \bigcap_{i=1}^{n} V_i \tag{2}$$

holds.

To prove this proposition the following lemma will be needed.

Lemma. *Let V_i $(i=1,2,\ldots,n)$ be valuation rings of the field K. Let B be a subring of the field K such that $B\subset\bigcup_{i=1}^{n}V_i$. Then there exists an index j, $1\leq j\leq n$ such that $B\subset V_j$.*

Proof of the lemma. We shall proceed by induction with respect to n.

Let $n=2$; then suppose that $B\subset V_1\cup V_2$. Suppose for an indirect proof that there exist elements $b_1,\ b_2\in B$ such that $b_1\in V_1-V_2$, $b_2\in V_2-V_1$. Since $b_1+b_2\in B$, we have $b_1+b_2\in V_1\cup V_2$. Therefore either $b_1+b_2\in V_1$ or $b_1+b_2\in V_2$. Suppose, e.g., $b_1+b_2\in V_1$. Then $b_1+b_2-b_1\in V_1$, i.e., $b_2\in V_1$, a contradiction. Analogically for the case $b_1+b_2\in V_2$. Hence we have either $B\subset V_1$ or $B\subset V_2$. The lemma is true for $n=2$.

Suppose that $n>2$ and the assertion of the Lemma holds for all $m<n$. Let $B\subset\bigcup_{i=1}^{n}V_i$. We can suppose that for each i $(1\leq i\leq n)$ $B\cap V_i\not\subset\bigcup_{\substack{j=1\\ j\neq i}}^{n}V_j$ holds, since otherwise we would have $B\subset\bigcup_{\substack{j=1\\ j\neq i}}^{n}V_j$ and then by the assumption there is an index j, $1\leq j\leq n$ such that $B\subset V_j$.

Under our assumption there exist n elements b_i $(i=1,\ldots,n)$ such that $b_i\in(B\cap V_i)-V_j$ for all $j,\ j\neq i$. We prove that this leads to a contradiction. Let us consider the sequence $\{b_1+k\cdot b_n\}_{k=0}^{\infty}$. Since $b_1\notin V_n$, we have $b_1+k\cdot b_n\notin V_n$ for all k. Therefore $b_1+k\cdot b_n\in V_{j_k}-V_n$ for $0\leq k<n$ and for an index $j_k\neq n$. Hence there exist integers k_1, k_2 and an index $j_s\neq n$ such that $0\leq k_1<k_2<n$, $b_1+k_1b_n\in V_{j_s}$ and $b_1+k_2b_n\in V_{j_s}$. Denote $k_2-k_1=l_1$. Then $l_1b_n=(b_1+k_2b_n)\ -\ (b_1+k_1b_n)\in V_{j_s}$.

Now, let us consider the sequence $\{b_1+k\cdot b_n^r\}_{k=0}^{\infty}$. Similarly as above for each natural r there exists an integer l_r, $0<l_r<n$, and an index $j_r\neq n$ such that $l_r\cdot b_n^r\in V_{j_r}$. Since $0<l_r<n$ for each r and there is only a finite number of different rings V_i, we conclude that there exist two integers $l,\ j,\ 0<l,\ j<n$ such that for infinitely many values of r we have $l\cdot b_n^r\in V_j$. But from $l\cdot b_n^r\in V_j$ it follows that $v_j(l\cdot b_n^r)\geq 0$. Since $v_j(b_n)<0$, this contradicts the fact that the group of values of the valuation v_j is archimedean. Our assumption cannot hold. So there exists an index i_1 such that $B\cap V_{i_1}\subset\bigcup_{\substack{j=1\\ j\neq i_1}}^{n}V_j$, which implies $B\subset\bigcup_{\substack{j=1\\ j\neq i_1}}^{n}V_j$ and by the inductive assumption there exists an index i such that $B\subset V_i$. This proves our Lemma.

Proof of the Proposition. Clearly $B\cdot\bigcap_{i=1}^{n}V_i\subset B\vee\bigcap_{i=1}^{n}V_i$. We shall prove the converse inclusion. Let $x\in B\vee\bigcap_{i=1}^{n}V_i$. Then $x=\sum_{j=1}^{m}a_jb_j$ where $a_j\in\bigcap_{i=1}^{n}V_i$, $b_j\in B$.

We try to find an element $y\in B$ such that $xy^{-1}\in\bigcap_{i=1}^{n}V_i$, since then $x=(xy^{-1})y$, $y\in B$, hence $x\in B\cdot\bigcap_{i=1}^{n}V_i$.

If $B \not\subset \bigcup_{i=1}^{n} V_i$, then there exists an element $z \in B$ such that $z \notin \bigcup_{i=1}^{n} V_i$. This implies $v_i(z) < 0$ (for $i = 1, 2, \ldots, n$), hence there is $k \in N$ such that $v_i(b_j z^{-k}) \geq 0$ (for $i = 1, 2, \ldots, n$). In this case it is sufficient to put $y = z^k$.

If $B \subset \bigcup_{i=1}^{n} V_i$, then by our Lemma there exists an index j, $1 \leq j \leq n$ such that $B \subset V_j$. Without loss of generality we may suppose that $B \subset V_1$.

If $B \subset \bigcap_{i=1}^{n} V_i$, then (2) clearly holds. Let $B \subset \bigcap_{i=1}^{s} V_i$ for $s < n$ and $B \not\subset V_i$ for $i > s$. Then by our Lemma there exist an element $b \in B$ such that $b \in V_i$ for $i \leq s$ and $b \notin V_i$ for $i > s$.

Now it is sufficien to prove that there exist an element $z \in B$ such that $v_i(z) = 0$ for $i \leq s$ and $v_i(z) < 0$ for $i > s$, since then there is an integer k such that $x \cdot z^{-k} \in \bigcap_{i=1}^{n} V_i$ and in this case it is sufficient to put $y = z^k$.

If $v_i(b) = 0$ for all $i \leq s$, we can put $z = b$. Suppose therefore that there is an $i \leq s$ such that $v_i(b) > 0$. Define the sequence $\{b_r\}_{r=1}^{\infty}$ as follous: $b_1 = b$, $b_{r+1} = (1 + b_r) \cdot b_r$. Let $K_r = \{V_i\colon\ i \leq s,\ v_i(b_r) = 0\}$. Obviously $K_{r+1} \subset K_r$. If there is j, $1 \leq j < \infty$ such that $K_{j+1} = K_j$, then $v_i(1 + b_j) = 0$ for all $i \leq s$. Since there is only a finite number of rings V_i, $i \leq s$, such an index j certainly exists. Clearly $b_j \notin V_i$ for $i > s$, hence we can put $z = 1 + b_j$ and our Proposition is proved.

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Received August 24, 1979

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ПЕРЕСЕЧЕНИЕ КОЛЕЦ НОРМИРОВАНИЯ

Юрай Кострa

Резюме

В статье доказано, что наименшее кольцо, порожденное пересечением колец нормирования с группой значений нормирования ранга 1 и любого подкольца (в данном поле), равна наименшей полугруппе, порожденной этими объектами, взятыми только с операцией умножения.