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Mathematica Slovaca, Vol. 54 (2004), No. 1, 69--85

Persistent URL: http://dml.cz/dmlcz/132390

Mathematica Slovaca
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Mathematical Institute
Slovak Academy of Sciences

Math. Slovaca, **54** (2004), No. 1, 69–85

Dedicated to Professor Sylvia Pulmannová on the occasion of her 65th birthday

THE BASE-NORMED SPACE OF A UNITAL GROUP

THURLOW A. COOK — DAVID J. FOULIS

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. One of the more elegant approaches to the mathematical foundations of the experimental sciences is the linear-duality formalism featuring an order-unit space U in order duality with a base-normed space V. The unit interval E in U is the set of effects, and the cone base Ω in V is the set of states. For various reasons, some of which we indicate, it is useful to replace the order-unit space U by a partially ordered abelian group G with order unit. One can still associate a base-normed space $V(G)$ with G, and much of the articulation between U and V is still available in this more general context.

1. Effects and states

All approaches to the mathematical foundations of the experimental sciences necessarily feature representations for observables (or random variables) and states (or probability models). Different authors approach foundational questions from widely varying philosophical perspectives, and it is hardly surprising that a number of disparate mathematical representations for observables and states can be found in the literature. The primacy accorded to observables on the one hand, and states on the other, varies from author to author. Heuristics and interpretations are often sketchy and ambiguous. For an intuitive understanding of the various approaches, one often must rely on what can be gleaned from contemplation of the postulated mathematical structures and from a study of just how these structures are employed in practical applications. Fortunately, there is some common ground.

An observable a is understood to be something that can be measured, at least in principle if not in fact. Thus, associated with an observable a is a

2000 Mathematics Subject Classification: Primary 81P10, 06F20, 46A40.
Keywords: unital group, quantum logic, effect, base-normed space, order-unit normed space.

class of measurement procedures that test it. Each measurement, or test, of a is supposed to yield an *outcome*, and to say that a is a Ξ-*valued* observable means that all possible outcomes of measurements of a belong to the nonempty set Ξ. For the development of statistical notions, it may be necessary to equip the outcome set Ξ with a σ-field $\mathcal{B}$ of subsets, thus forming a so-called measurable space $(\Xi, \mathcal{B})$.

It is usually possible to identify a set, call it E, of "elementary observables" from which more general observables are built up in one way or the other. In most accounts, the elementary observables $e \in E$ take values in a two-element set such as $\Xi = \{\text{yes, no}\}$. Following G. Ludwig [15], the elementary observables $e \in E$ are often called *effects*. For instance, according to the authoritative paper [13; p. 360], "One may think of a quantum effect as an elementary yes-no measurement that may be unsharp or imprecise." Thus, effects, like the events employed in the theory of probability, are two-valued and are to be revealed by experiments, measurements, or tests. However, unlike events, effects can have an unsharp, imprecise, or fuzzy character. To say that an effect is *observed* presumably means that it is tested and the *yes* outcome is obtained.

A *state* is often thought of as an equivalence class of preparation procedures. To say that a measurement is made "in state ω" can be interpreted to mean that it is made on a system or an ensemble that has been prepared in state ω. The set of all states, call it Ω, is usually assumed to be convex. Thus, if $\omega_1, \omega_2 \in \Omega$ and $0 \le t \le 1$, then $\omega := t\omega_1 + (1-t)\omega_2 \in \Omega$ can be thought of as the state of an ensemble a fraction t of which is in state ω_1 and a fraction $1-t$ of which is in state ω_2. The state space Ω may be endowed with suitable topological structure to allow for the notion of convergence of a sequence of states to a limit state. We formalize the notion of a state space as follows.

1.1. DEFINITION. A *state space* is a nonempty convex set Ω with a Hausdorff topology such that the mapping $[0,1] \times \Omega \times \Omega \to \Omega$ given by $(t, \omega_1, \omega_2) \mapsto t\omega_1 + (1-t)\omega_2$ is continuous.

The states $\omega \in \Omega$ are supposed to determine the probabilities associated with outcomes of measurements. An intuitively appealing way to introduce a linkage between the set of effects E and the state space Ω is to assume that E and Ω are in "statistical duality" under a "probability mapping" $\mathrm{Prob}\colon E \times \Omega \to [0,1] \subseteq \mathbb{R}$. The idea is that, if $e \in E$ and $\omega \in \Omega$, then $\mathrm{Prob}(e, \omega)$ is the probability of observing the effect e when it is tested in state ω ([2]). Here is a formal definition.

1.2. DEFINITION. The nonempty set E and the state space Ω are in *statistical duality* under the mapping $\mathrm{Prob}\colon E \times \Omega \to [0,1]$ if and only if, for each $e \in E$, the mapping $\mathrm{Prob}(e, \cdot)\colon \Omega \to [0,1]$ is affine (i.e., preserves convex combinations) and continuous.

The statistical duality Prob: $E \times \Omega \to [0,1]$ is said to *separate effects* if and only if, for all $e_1, e_2 \in E$, the condition $\mathrm{Prob}(e_1,\omega) = \mathrm{Prob}(e_2,\omega)$ for all $\omega \in \Omega$ implies that $e_1 = e_2$. It is said to *separate states* if and only if, for all $\omega_1, \omega_2 \in \Omega$, the condition $\mathrm{Prob}(e,\omega_1) = \mathrm{Prob}(e,\omega_2)$ for all $e \in E$ implies that $\omega_1 = \omega_2$.

In the so-called *convexity approach*, the state space Ω is awarded primacy and all other concepts, including effects and observables, are formulated in terms of Ω. An authoritative account of the convexity approach can be found in [14]. In the so-called *effect-algebra approach* ([8]), E (with the structure of an effect algebra) is awarded primacy, states and observables being defined in terms of E.

2. The linear-duality approach

Owing to the attractions of the highly developed theory of partially ordered linear spaces, investigators have sought realizations of the set E of effects and of the state space Ω as sets of vectors in linear spaces endowed with suitable topological and order-theoretic structures and in order duality under a nondegenerate bilinear form. Apart from forging a connection with functional analysis, such a *linear-duality approach* has the advantage that E and Ω are treated on an equal footing.

One of the most successful formulations of the linear-duality approach is largely based on the original pioneering work of Ludwig [15]. The idea is to realize E as the order-unit interval $E = \{e \in U : 0 \le e \le u\}$ in a real partially ordered linear topological space U with order unit u, and to realize Ω as a cone base for the positive cone $V^+ = \{\nu \in V : 0 \le \nu\}$ in a real partially ordered linear topological space $V = V^+ - V^+$. The convex and topological structures on Ω are understood to be inherited from the linear and topological structures on V. The linear spaces U and V are assumed to be in order duality under a nondegenerate continuous bilinear form $\langle\cdot,\cdot\rangle\colon U \times V \to \mathbb{R}$ in such a way that, for $e \in E$ and $\omega \in \Omega$, $\langle e,\omega\rangle \in [0,1] \subseteq \mathbb{R}$. Consequently, under the restriction Prob of $\langle\cdot,\cdot\rangle$ to $E \times \Omega$, the effect set E and the state space Ω are in statistical duality as in Definition 1.2. The nondegeneracy of $\langle\cdot,\cdot\rangle$ implies that Prob: $E \times \Omega \to [0,1]$ separates both effects and states.

The basic linear-duality formalism outlined above may be enhanced by various additional hypotheses, e.g., U is an order-unit Banach space [1; p. 69], V is a base-normed Banach space [1; p. 77], V is the Banach dual of U, and so on. For an elegant exposition of an up-to-date version of the linear-duality formalism, see [13]. Here are two important examples.

2.1. EXAMPLE. Let $(\Xi, \mathcal{B})$ be a measurable space, say, the phase space of a classical mechanical system. An observable for the phase space Ξ is a measurable function $f\colon \Xi \to \mathbb{R}$, E is the set of all observables e such that $0 \le e(x) \le 1$ for all $x \in \Xi$, and U is the partially ordered normed linear space of all bounded observables with the constant function $u(x) \equiv 1$ as the order unit and with the supremum norm. (If one prefers to have an order-unit Banach space, one replaces U by the space $\overline{U}$ of all uniform limits of functions in U, in which case the order-unit norm is the supremum norm.) The partially ordered normed linear space V is the space of all countably additive, bounded, $\mathbb{R}$-valued measures on $(\Xi, \mathcal{B})$ with the variation norm, and the cone base Ω is the subset of V consisting of all the countably additive probability measures on $(\Xi, \mathcal{B})$. (If one prefers to have a base-normed Banach space, one replaces V by the space $\overline{V}$ of all finitely additive, bounded $\mathbb{R}$-valued measures on $(\Xi, \mathcal{B})$ and Ω by the convex subset $\overline{\Omega} \subseteq \overline{V}$ of all finitely additive probability measures, in which case the base norm is the variation norm.) The spaces U and V (as well as $\overline{U}$ and $\overline{V}$) are in order duality under $\langle f, \mu \rangle := \int_{\Xi} f \, \mathrm{d}\mu$.

In Example 2.1, the characteristic set functions χ_M of sets $M \in \mathcal{B}$ belong to E, and may be regarded as the "sharp effects" ([11]). The remaining functions $e \in E$ map Ξ into $[0,1]$ and may be considered to be "fuzzy versions" of the sharp effects ([12]).

2.2. EXAMPLE. Let $\mathcal{H}$ be a separable Hilbert space, say the Hilbert space associated with a quantum-mechanical system. In orthodox quantum mechanics, an observable for $\mathcal{H}$ is a (possibly unbounded) self-adjoint operator on $\mathcal{H}$, or what is the same thing via the spectral theorem, a projection-valued measure on the σ-field of real Borel sets. Let U be the linear space over $\mathbb{R}$ of all bounded self-adjoint operators on $\mathcal{H}$, partially ordered as usual, and let E be the set of all operators $A \in U$ with $\mathbf{0} \le A \le \mathbf{1}$. Then U is an order-unit Banach space with $\mathbf{1}$ as the order unit, and the order-unit norm is the usual uniform operator norm on U. Let V be the linear subspace of U consisting of the bounded, self-adjoint, trace-class operators, and let Ω be the set of all operators $W \in V$ such that $\mathbf{0} \le W$ and $\operatorname{trace}(W) = 1$. Then V is a base-normed Banach space and the base-norm on V is the trace norm. The spaces U and V are in order duality under $\langle A, W \rangle := \operatorname{trace}(AW)$.

In Example 2.2, the projection operators P_M onto closed linear subspaces M of $\mathcal{H}$ belong to E, have spectrum contained in $\{0,1\}$, and may be regarded as the "sharp effects." The remaining operators $A \in E$ have spectrum contained in $[0,1]$ and may be considered to be "fuzzy versions" of the sharp effects. (In a more up-to-date and more realistic theory of quantum measurement, observables are represented by effect-valued measures on a σ-field of sets, whence the effect operators $A \in E$ play an even more prominent role ([4]).)

In the linear-duality formalism, the fact that $E = \{e \in U : 0 \le e \le u\} \subseteq U$ bestows considerable structure on the set E. To begin with, there are two special elements 0 and u in E, and E is a bounded poset under the restriction of the partial order $\le$ on U. Also there is a natural involution $' \colon E \to E$ defined for all $e \in E$ by $e' := u - e$, and $e \le f \implies f' \le e'$ holds for all $e, f \in E$. The partial binary operation $\oplus$ obtained by restriction to E of $+$ on U is an important feature of E, although its phenomenological interpretation is a bit problematic. According to [13], "The element $a \oplus b$ represents a statistical combination of a and b whose probability of occurrence equals the sum of the probabilities that a and b occur individually." In other words, if $a, b \in E$ and $a \oplus b$ is defined, then $\mathrm{Prob}(a \oplus b, \omega) = \mathrm{Prob}(a, \omega) + \mathrm{Prob}(b, \omega)$ for all $\omega \in \Omega$.

Another important consequence of the linear-duality formalism is that it becomes possible to multiply an effect $a \in E$ by a scalar λ, and if $0 \le \lambda \le 1$, the product is again an effect $\lambda a \in E$. Turning again to [13] for intuition, we find that, for a quantum effect (Example 2.2), "λa represents the effect a attenuated by a factor λ", and that, "a similar interpretation is given for fuzzy events" (Example 2.1). The notion of λ as an attenuating factor seems resistant to phenomenological interpretation, at least as long as we subscribe to the two-valuedness of effects. Presumably, an effect is either observed or it isn't. What meaning shall we give to an attenuated observation? Again we have to resort to a *statistical* interpretation, namely, λa is an effect such that, for every state ω, $\mathrm{Prob}(\lambda a, \omega) = \lambda\,\mathrm{Prob}(a, \omega)$.

The possibility of multiplying effects by scalars in the linear formalism implies that if a and b are effects and $0 \le \lambda \le 1$, we can form the convex combination $\lambda a \oplus (1 - \lambda) b \in E$. Again, a statistical interpretation of convex combinations of effects is available, but a suitable phenomenological interpretation appears to be problematic.

3. The unital-group approach

The linear-duality formalism, as compelling and elegant as it may be, gives rise to questions of interpretation, some of which we have alluded to in Section 2. Also, there are circumstances in which this formalism is much more elaborate than needed for practical applications. In the analysis of a dice game, it is probably not necessary to deal with effects such as $(\sqrt{2}/2)$ *hard-eight*, even if it is possible to make sense of such a thing. There are situations in which we do not care to avail ourselves of the possibility of multiplying effects by scalars. The solution is simple — we just "remove the scalars" from the linear space U. If we disregard multiplication by scalars, U forms a partially ordered abelian group under addition, and many of the notions discussed above are readily formulated in that context.

We begin by reviewing some terminology. Details, as well as concepts not elucidated in this brief review, can be found in [1] and [10]. If P is a real linear space, then a subset $C \subseteq P$ is called a *wedge* if and only if $0 \in C$, C is closed under addition, and C is closed under multiplication by nonnegative real numbers. A *cone* in P is a wedge C such that $p, -p \in C \implies p = 0$. (Caution: Some authors, e.g., [1], call such a C a *proper* or a *strict* cone.) If C is a cone in P, then the relation $\leq$ defined on P by $p \leq q$ if and only if $q - p \in C$ is a partial order, it is translation invariant (i.e., for all $r \in P$, $p \leq q \implies p + r \leq q + r$), it is invariant under multiplication by nonnegative scalars (i.e., if $0 \leq \lambda \in \mathbb{R}$, then $p \leq q \implies \lambda p \leq \lambda q$), and it determines C according to $C = \{p \in P : \ 0 \leq p\}$.

A linear space P, equipped with a partial order relation $\leq$ that is translation invariant and invariant under multiplication by nonnegative scalars, is called a *partially ordered linear space*, and $P^+ := \{p \in P : \ 0 \leq p\}$ is called the *positive cone* in P. We note that $0 \in P^+$ in spite of the terminology "positive cone". The positive cone P^+ is in fact a cone in P, and for $p, q \in P$, $p \leq q \iff q - p \in P^+$. Thus, the partial order $\leq$ determines the cone P^+ and the cone P^+ determines the partial order $\leq$. The partially ordered linear space P is said to be *directed* if and only if P is the linear span of P^+. Evidently P is directed if and only if $P = P^+ - P^+$. A subset D of P^+ is called a *cone base* for P^+ if and only if every nonzero element $p \in P^+$ can be written in the form $p = \lambda d$ for a uniquely determined positive real number λ and a uniquely determined element $d \in D$.

The abelian groups that we consider will all be additively written. For partially ordered abelian groups, the story is much the same as for partially ordered linear spaces, except that multiplication by (arbitrary) real scalars may no longer make sense. (Of course, such a group is a module over the ring $\mathbb{Z}$ of integers, so one can still multiply its elements by integers and form integer linear combinations of its elements.) If G is an abelian group, then a subset $C \subseteq G$ is called a *cone* if and only if $0 \in C$, C is closed under addition, and $g, -g \in C \implies g = 0$. (Caution: Again, some authors, e.g., [10], call such a C a *proper* or a *strict* cone.) If C is a cone in G, then the relation $\leq$ defined on G by $g \leq h$ if and only if $h - g \in C$ is a translation invariant partial order on G. A *partially ordered* abelian group is an abelian group G equipped with a translation invariant partial order relation $\leq$, and the corresponding *positive cone* is $G^+ := \{g \in G : \ 0 \leq g\}$. The positive cone G^+ is in fact a cone in G, and for $g, h \in G$, $g \leq h \iff h - g \in G^+$. If G^+ generates G, i.e., if $G = G^+ - G^+$, then G is said to be *directed*.

If P is a partially ordered linear space, and if we neglect multiplication by (non-integer) scalars and consider P as an abelian group under addition, then P forms a partially ordered abelian group. With this understanding, all concepts that we define for partially ordered abelian groups are applicable to partially ordered linear spaces as well. For instance, a partially ordered abelian group G is said to be *archimedean* if and only if, for all $g, h \in G$, the condition $ng \le h$ holds for all positive integers n only if $-g \in G^+$. The same definition carries over directly to partially ordered linear spaces.

If G is a partially ordered abelian group and $u \in G^+$, we define the *interval* $G^+[0,u] := \{e \in G : 0 \le e \le u\}$ and regard it as a bounded partially ordered set under the restriction of the partial order on G. A set $A \subseteq G^+$ is said to be *cone generating* if and only if every element $g \in G^+$ is a finite linear combination of elements of A with nonnegative integer coefficients. If each element in G is dominated by a suitably large positive integer multiple of u, then u is called an *order unit* ([10; p. 4]). A *unital group* is defined to be a partially ordered abelian group G with a specified order unit $u \in G^+$, called the *unit*, such that the interval $E := G^+[0,u]$, called the *unit interval*, is cone generating [5; Definition 2.5]. Since a unital group G has an order unit, it is directed [10; p. 4].

The notion of a unital group is really quite general. Indeed, suppose H is any partially ordered abelian group and u is any element in H^+. Define C to be the subset of H consisting of all finite linear combinations of elements of $H^+[0,u]$ with nonnegative integer coefficients. Then $u \in H^+[0,u] \subseteq C \subseteq H^+$, C is a cone in H, $G := C - C$ is a subgroup of H, and C is a cone in G. Partially order G by taking $G^+ := C$. Then G is a unital group with unit u and $H^+[0,u] = G^+[0,u]$ is the unit interval in G.

The ordered additive abelian group $\mathbb{R}$ of real numbers with the standard positive cone $\mathbb{R}^+ = \{x \in \mathbb{R} : 0 \le x\}$ forms an archimedean unital group with unit 1, and its unit interval is the standard unit interval $[0,1]$. Let G be a unital group with unit u and unit interval E. The set of all group homomorphisms from G to the additive group $\mathbb{R}$ is denoted by $\hom(G,\mathbb{R})$. Under pointwise operations, $\hom(G,\mathbb{R})$ forms a real linear space, and $\hom(G,\mathbb{R})^+ := \{\nu \in \hom(G,\mathbb{R}) : \nu(G^+) \subseteq \mathbb{R}^+\}$ is a cone in $\hom(G,\mathbb{R})$. In what follows, we understand that $\hom(G,\mathbb{R})$ *is organized into a partially ordered linear space with positive cone* $\hom(G,\mathbb{R})^+$. The elements of $\hom(G,\mathbb{R})^+$ are the order-preserving group homomorphisms $\nu\colon G \to \mathbb{R}$. We also understand that $\hom(G,\mathbb{R})$ *carries the relative topology it inherits as a closed linear subspace of the locally convex linear topological space* $\mathbb{R}^G$ *with the topology of pointwise convergence.*

3.1. DEFINITION. Let G be a unital group with unit u and unit interval $E = G^+[0,u]$ and let K be an abelian group. A mapping $\phi\colon E \to K$ is called a *K-valued measure* if and only if $e, f, e+f \in E \implies \phi(e+f) = \phi(e) + \phi(f)$.

If every K-valued measure $\phi\colon E \to K$ can be extended to a group homomorphism $\Phi\colon G \to K$, then G is called a *K-unital group* ([5; Definition 2.5]). If G is K-unital for every abelian group K, we say that G is a *unigroup* ([9]).

If K is a partially ordered abelian group, then a mapping $\phi\colon E \to K^+$ is called a *K^+-valued measure* if and only if it is an K-valued measure.

If every K^+-valued measure $\phi\colon E \to K^+$ can be extended to a group homomorphism $\Phi\colon G \to K$, then G is called a *K^+-unital group.*

A (finitely additive) *probability measure* on E is an $\mathbb{R}^+$-valued measure $\mu\colon E \to \mathbb{R}^+$ such that $\mu(u) = 1$.

Suppose G is a unital group with unit interval E, K is a partially ordered abelian group, and $\phi\colon E \to K^+$ is a K^+-valued measure. Because E generates the positive cone G^+, it is clear that an extension of ϕ to a group homomorphism $\Phi\colon G \to K$ necessarily satisfies the condition $\Phi(G^+) \subseteq K^+$, i.e., Φ is an order-preserving group homomorphism.

3.2. DEFINITION. Let G be a unital group with unit u. By definition, a *state* on (or for) G is an element $\omega \in \hom(G,\mathbb{R})^+$ that is *normalized* in the sense that $\omega(u) = 1$ ([1; p. 72], [10; Chap. 4]). We denote the set of all states on G by $\Omega(G)$ and give $\Omega(G)$ the relative topology it inherits as a subset of the linear topological space $\hom(G,\mathbb{R})$.

If $G \neq \{0\}$ is a unital group, then $\Omega(G)$ is nonempty ([10; Corollary 4.4]) and $\Omega(G)$ is a compact convex subset of $\hom(G,\mathbb{R})$ ([10; Proposition 6.2]). In fact, by [10; Proposition 6.5], $\Omega(G)$ is σ-convex. The unit interval E in G and the state space $\Omega(G)$ are in statistical duality (as per Definition 1.2) under $\mathrm{Prob}\colon E \times \Omega(G) \to [0,1]$ given by $\mathrm{Prob}(e,\omega) := \omega(e)$ for $e \in E$, $\omega \in \Omega(G)$. If $\omega \in \Omega(G)$, then the restriction $\omega|_E = \mathrm{Prob}(\cdot,\omega)$ of ω to E is a probability measure on E.

3.3. LEMMA. *Let G be a unital group with unit u and let E be the unit interval in G. Then:*

(i) *$\Omega(G)$ is a cone base for* $\hom(G,\mathbb{R})^+$.

(ii) *If $\omega \in \Omega(G)$, then the restriction $\omega|_E$ of ω to E is a probability measure on E.*

(iii) *G is $\mathbb{R}^+$-unital if and only if every probability measure μ on E can be extended to a state $\omega \in \Omega(G)$.*

(iv) *If G is $\mathbb{R}^+$-unital, then the states $\omega \in \Omega(G)$ are in bijective correspondence with the probability measures μ on E under the mapping $\omega \mapsto \mu = \omega|_E$.*

(v) *G is a unigroup $\implies$ G is $\mathbb{R}$-unital $\implies$ G is $\mathbb{R}^+$-unital.*

Proof.

(i) Let $\nu \in \hom(G,\mathbb{R})^+$. Evidently, $0 \leq \nu(u)$. If $\nu(u) = 0$, then, owing to the fact that ν is order preserving, $\nu(E) = \{0\}$. But E is cone-generating and $G = G^+ - G^+$, whence $\nu(u) = 0 \implies \nu = 0$. Therefore, if $0 \neq \nu \in \hom(G,\mathbb{R})^+$, then $\omega := (1/t)\nu \in \Omega(G)$ where $0 < t := \nu(u)$. Uniqueness is evident.

(ii) Part (ii) is obvious.

(iii) Suppose every probability measure on E can be extended to a state on G and let $\phi\colon E \to \mathbb{R}^+$ be a $\mathbb{R}^+$-valued measure on E. Suppose $\phi(u) = 0$. If $e \in E$, then $u - e \in E$ and we have $\phi(e) + \phi(u - e) = \phi(u) = 0$ with $0 \le \phi(e), \phi(u - e)$, and it follows that $\phi(e) = 0$. Therefore, if $\phi(u) = 0$, then $\phi(e) = 0$ for all $e \in E$, so the zero homomorphism $0 \in \hom(G, \mathbb{R})^+$ is an extension of ϕ. Suppose $\phi(u) \neq 0$ and let $t := \phi(u)$. Then $\mu := (1/t)\phi$ is a probability measure on E, whence it can be extended to a state $\omega \in \Omega(G)$, and $\nu := t\omega \in \hom(G, \mathbb{R})^+$ is an extension of ϕ.

(iv) By (iii), $\omega \mapsto \omega|_E$ maps $\Omega(G)$ surjectively onto the set of probability measures on E. Now E generates G^+, $G = G^+ - G^+$, and each $\omega \in \Omega(G)$ is a group homomorphism $\omega\colon G \to \mathbb{R}$, whence each $\omega \in \Omega$ is uniquely determined by its restriction $\omega|_E$ to E. Therefore, the mapping $\omega \mapsto \omega|_E$ is injective.

(v) Part (v) is obvious. □

3.4. Lemma. *Let U be a partially ordered linear space over $\mathbb{R}$ and let u be an order unit in U^+. Disregarding multiplication by scalars, we can regard U as a partially ordered abelian group under addition. As such, U is a unigroup with unit u and $\hom(U, \mathbb{R})^+$ is the set of linear functionals ν on the linear space U such that $\nu(U^+) \subseteq \mathbb{R}^+$.*

Proof. If we regard U simply as an additive abelian group, it is obviously a partially ordered abelian group with positive cone U^+ and u is an order unit for U, but we have to prove that every element of U^+ is the sum of a sequence of elements in the interval $U^+[0, u]$. Suppose $y \in U^+$ and choose a positive integer m such that $mu - y \in U^+$. Then $u - (1/m)y \in U^+$, whence $(1/m)y$ belongs to $U^+[0, u]$. Therefore $y = \sum_{i=1}^{m} y_i$ with $y_i := (1/m)y$ for $i = 1, 2, \ldots, m$, and it follows that U is a unital group. That the unital group U is a unigroup follows from [3; Corollary 4.6]. That every group homomorphism in $\hom(U, \mathbb{R})^+$ is a linear functional on the linear space U follows from [10; Lemma 6.7]. □

Suppose (U, V) is a dual pair in the linear-duality formalism and u is the order unit in U. Disregarding multiplication by scalars, we can regard U as a partially ordered abelian group under addition as in Lemma 3.4, and thus obtain a unigroup with unit u. Therefore, a unigroup, and more generally a unital group, generalizes the linear space U in the dual pair (U, V). We note that the cone base Ω in V and the state space $\Omega(U)$ as in Definition 3.2 are not necessarily the same thing, although they are closely related. Indeed, if $\omega \in \Omega$, then the mapping $\langle \cdot, \omega \rangle$ belongs to $\Omega(U)$ and the mapping $\omega \mapsto \langle \cdot, \omega \rangle$ is an injective affine mapping of the cone base Ω into the state space $\Omega(U)$.

3.5. EXAMPLE. We revisit Example 2.1 from the perspective of the unital-group approach. The additive group $\mathbb{Z}$ of integers with the standard positive cone $\mathbb{Z}^+ = \{z \in \mathbb{Z} : \ 0 \leq z\}$ forms a lattice-ordered unigroup with unit 1. Given a measurable space $(\Xi, \mathcal{B})$, let G be the abelian group under pointwise addition of all bounded functions $f \colon \Xi \to \mathbb{Z}$ that are measurable in the sense that $f^{-1}(z) \in \mathcal{B}$ for all $z \in \mathbb{Z}$. Partially order G pointwise, so that its positive cone is $G^+ = \{f \in G : \ f(\Xi) \subseteq \mathbb{Z}^+\}$, and let $u \in G$ be the constant function $u(x) \equiv 1$. Then G is a lattice-ordered unigroup with unit u and the unit interval in G is the set E of all characteristic set functions χ_M of sets $M \in \mathcal{B}$. Therefore E has the structure of a σ-complete Boolean algebra isomorphic to the σ-field $\mathcal{B}$. There is a uniquely determined bijective affine mapping $\mu \leftrightarrow \omega$ between finitely additive probability measures μ on $\mathcal{B}$ and states $\omega \in \Omega(G)$ such that $\mu(M) = \omega(\chi_M)$ for all $M \in \mathcal{B}$. Therefore, $\Omega(G)$ is affine isomorphic to the space $\overline{\Omega}$ in Example 2.1.

One of the advantages of the unital-group approach is that it enables one to study finite effect algebras. For instance, in Example 3.5, we can take Ξ to be a finite set with n elements.

3.6. EXAMPLE. Let n be a positive integer. The additive abelian group $\mathbb{Z}^n$ with coordinatewise addition forms a free abelian group of rank n, and every free abelian group of rank n is isomorphic to $\mathbb{Z}^n$. Under the coordinatewise partial order, $\mathbb{Z}^n$ forms a lattice-ordered abelian group with positive cone $(\mathbb{Z}^+)^n$, and as such, it is a so-called *simplicial group* ([10; p. 47]). The vector $\mathbf{u} = (1, 1, \dots, 1)$ is not only an order unit, it is also the smallest order unit in the simplicial group $\mathbb{Z}^n$. In Example 3.5, let $\Xi := \{1, 2, \dots, n\}$ and let $\mathcal{B}$ be the power set of Ξ. Then the unigroup G can be identified with the simplicial group $\mathbb{Z}^n$ in the obvious way, and the order unit is then identified with $\mathbf{u} = (1, 1, \dots, 1)$. Thus, the unit interval E consists of the 2^n vectors $\mathbf{e} \in \mathbb{Z}^n$ having only 0 and 1 as coordinates, hence it is isomorphic to the finite Boolean algebra $\mathbf{2}^n$. Moreover, the state space $\Omega(G)$ is an $(n-1)$-dimensional simplex.

4. The base-normed space $V(G)$

In the linear-duality formalism, the partially ordered linear space U with order unit u is in order duality with a partially ordered linear space V with cone base Ω. By analogy, as we show in this section, if G is a unital group, there is a corresponding "dual" base-normed Banach space $V(G)$ with the state space $\Omega(G)$ as a cone base.

By Lemma 3.3(i), the compact convex set $\Omega(G)$ is a cone base for $\hom(G, \mathbb{R})^+$, but in general, $\hom(G, \mathbb{R})$ will not be directed. For instance, $\hom(\mathbb{R}, \mathbb{R})$ is not directed because $\hom(\mathbb{R},\mathbb{R})^+$ consists of all mappings of the form $x \mapsto \lambda x$, $\lambda \in \mathbb{R}$, and there are discontinuous additive homomorphisms in $\hom(\mathbb{R},\mathbb{R})$. However, we can use the cone $\hom(G,\mathbb{R})^+$ to construct a directed linear space $V(G) := \hom(G,\mathbb{R})^+ - \hom(G,\mathbb{R})^+$ that plays the role of the partially ordered linear space V in the linear-duality approach.

4.1. THEOREM. *Suppose G is a unital group, define $V(G) := \hom(G,\mathbb{R})^+ - \hom(G,\mathbb{R})^+$, and let $V(G)^+ := \hom(G,\mathbb{R})^+$. Then $V(G)$ is a base-normed Banach space with $V(G)^+$ as its positive cone and with $\Omega(G)$ as the cone base.*

Proof. With the topology of pointwise convergence, the cone base $\Omega(G)$ is compact, and it follows that the convex hull B of $\Omega(G) \cup \big(-\Omega(G)\big)$ is compact. Therefore, B is radially compact, and it follows from [1; Proposition II.1.12] that $V(G)$ is a base-normed Banach space with B as its closed unit ball. $\square$

In Example 3.5, $V(G)$ is order isomorphic and isometric to the base-normed Banach space $\overline{V}$ in Example 2.1. In Example 3.6, $V(G)$ is order isomorphic to the lattice-ordered coordinate vector space $\mathbb{R}^n$ with coordinatewise partial order and with the standard $(n-1)$-dimensional simplex as the cone base.

Suppose that V is a base-normed space, Ω is the cone base for V^+, and V^* is the Banach dual space of V. Then V^* can be organized into a directed partially ordered linear space with positive cone $V^{*+} := \{f \in V^* : f(V^+) \subseteq \mathbb{R}^+\}$ and there is a uniquely determined $e_1 \in V^*$ such that $e_1(\omega) = 1$ for all $\omega \in \Omega$. Furthermore, by a theorem of A. J. Ellis [1; Theorem II.1.15], V^* is an order-unit Banach space with order unit e_1 and the order-unit norm on V^* coincides with the usual norm on the Banach dual space V^*.

4.2. DEFINITION. If G is a unital group, we define $V(G) := \hom(G,\mathbb{R})^+ - \hom(G,\mathbb{R})^+$ organized into a base-normed Banach space as in Theorem 4.1. Also, $V(G)^*$ denotes the Banach dual space of $V(G)$ organized into an order-unit Banach space with order unit e_1. Each $g \in G$ determines a linear functional $\widehat{g}\colon V(G) \to \mathbb{R}$ by evaluation, i.e., $\widehat{g}(\nu) := \nu(g)$ for all $\nu \in V(G)$. We define $\widehat{G} := \{\widehat{g} : g \in G\}$, $\widehat{G}^+ := \{\widehat{g} : g \in G^+\}$, and $\widehat{E} := \{h \in \widehat{G}^+ : \widehat{u} - h \in \widehat{G}^+\}$.

Evidently, $e_1 = \widehat{u}$. Caution: In spite of the notation, $\widehat{E}$ is *not necessarily the same as* $\{\widehat{e} : e \in E\}$.

4.3. THEOREM. *Let G be a unital group with unit u and unit interval E. Then:*

(i) *$\widehat{G} \subseteq V(G)^*$ and $g \mapsto \widehat{g}$ is an order-preserving group homomorphism from G into the partially ordered additive group of $V(G)^*$.*

(ii) *$\widehat{G}$ is a unital group with positive cone $\widehat{G}^+ \subseteq V(G)^{*+}$, unit $\widehat{u}$, and unit interval $\widehat{E} = \widehat{G}^+[\widehat{0},\widehat{u}]$.*

(iii) *For* $g \in G$, $\widehat{g} = 0$ *if and only if* $\omega(g) = 0$ *for all* $\omega \in \Omega$.

(iv) *For each* $\nu \in V(G)$ *there is a uniquely determined* $\widehat{\nu} \in V(\widehat{G})$ *such that* $\widehat{\nu}(\widehat{g}) = \nu(g)$ *for all* $g \in G$. *Moreover, the mapping* $\nu \mapsto \widehat{\nu}$ *is a base-normed-space isomorphism from* $V(G)$ *onto* $V(\widehat{G})$.

(v) $\widehat{G}$ *is archimedean if and only if* $\widehat{G}^+$ *is an induced subcone in* $V(G)^*$, *i.e., if and only if* $\widehat{G}^+ = V(G)^{*+} \cap \widehat{G}$.

Proof.

(i) Let $\|\cdot\|$ be the base norm on $V(G)$ and let $g \in G$. To prove that $\widehat{g} \in V(G)^*$ it will be sufficient to show that there exists $K_g \in \mathbb{R}^+$ such that $|\widehat{g}(\nu)| = |\nu(g)| \le K_g\|\nu\|$ for all $\nu \in V(G)$. Suppose first that $g \in G^+$. Then there are integers $c_i \in \mathbb{Z}^+$ and elements $e_i \in E$ such that $g = \sum_{i=1}^{n} c_i e_i$, whence, if $\omega \in \Omega$, we have $0 \le \omega(g) = \sum_{i=1}^{n} c_i\omega(e_i) \le \sum_{i=1}^{n} c_i$. Therefore, there exists $C_g := \sum_{i=1}^{n} c_i \in \mathbb{Z}^+$ such that $0 \le \omega(g) \le C_g$ for all $\omega \in \Omega$. If $\nu \in V(G)^+$, there exist $t \in \mathbb{R}^+$ and $\omega \in \Omega$ such that $\nu = t\omega$, whence $\|\nu\| = t$ and $0 \le \nu(g) = t\omega(g) \le C_g\|\nu\|$. If $\nu \in V(g)$, then by [1; Proposition II.1.14], there exist $\alpha, \beta \in V(G)^+$ such that $\nu = \alpha - \beta$ and $\|\nu\| = \|\alpha\| + \|\beta\|$, whence for $g \in G^+$ we have $|\nu(g)| = |\alpha(g) - \beta(g)| \le \alpha(g) + \beta(g) \le C_g\|\alpha\| + C_g\|\beta\| = C_g\|\nu\|$. Now suppose $g \in G$ and select $g_1, g_2 \in G^+$ such that $g = g_1 - g_2$. Then, for $\nu \in V(G)$, we have $|\nu(g)| \le |\nu(g_1)| + |\nu(g_2)| \le K_g\|\nu\|$, where $K_g := C_{g_1} + C_{g_2}$. Therefore, $g \in G \implies \widehat{g} \in V(G)^*$. Clearly $g \mapsto \widehat{g}$ preserves addition. If $g \in G^+$ and $\nu \in V(G)^+ = \hom(G, \mathbb{R})^+$, then $\widehat{g}(\nu) = \nu(g) \in \mathbb{R}^+$, so $\widehat{g} \in V(G)^{*+}$, and it follows that $g \mapsto \widehat{g}$ is order preserving.

(ii) The proof of (ii) is straightforward.

(iii) Let $g \in G$. Since $V(G)$ is the linear span of $\Omega(G)$, it follows that $\widehat{g} = 0 \iff (\forall\, \omega \in \Omega(G))(\omega(g) = 0)$.

(iv) If $\nu \in V(G)$ and $g_1, g_2 \in G$ with $\widehat{g_1} = \widehat{g_2}$, then $\nu(g_1) = \widehat{g_1}(\nu) = \widehat{g_2}(\nu) = \nu(g_2)$, so we can and do define $\widehat{\nu}\colon \widehat{G} \to \mathbb{R}$ by $\widehat{\nu}(\widehat{g}) := \nu(g)$ for all $g \in G$. By a straightforward argument, $\nu \mapsto \widehat{\nu}$ is a base-normed space isomorphism from $V(G)$ onto $V(\widehat{G})$.

(v) Since $\widehat{G}^+ \subseteq V(G)^{*+}$, it follows that $\widehat{G}^+$ is an induced subcone in $V(G)^*$ if and only if, for all $g \in G$, $0 \le \widehat{g}(\nu)$ for all $\nu \in V(G)^+ \implies \widehat{g} \in \widehat{G}^+$. Because $\Omega(G)$ is a cone base for $V(G)^+$, the latter condition holds if and only if, for all $g \in G$, $0 \le \widehat{g}(\omega)$ for all $\omega \in \Omega(G) \implies \widehat{g} \in \widehat{G}^+$. Therefore, by (iv) above, $\widehat{G}^+$ is an induced subcone in $V(G)^*$ if and only if, for all $g \in G$, $0 \le \widehat{\omega}(\widehat{g})$ for all $\widehat{\omega} \in \Omega(\widehat{G}) \implies \widehat{g} \in \widehat{G}^+$. But, by [10; Theorem 4.14], the latter condition holds if and only if $\widehat{G}$ is archimedean. □

An alternative representation for the order-unit Banach space $V(G)^*$ can be obtained as follows: If G is a unital group with unit u and state space $\Omega(G)$, and if $f \in V(G)^*$, then the restriction $f|_\Omega$ of f to $\Omega(G)$ is a continuous affine function from the compact convex set $\Omega(G)$ to $\mathbb{R}$. The set $\operatorname{aff}(\Omega(G))$ of all continuous affine functions from $\Omega(G)$ to $\mathbb{R}$ forms an order-unit Banach space with $\widehat{u}|_\Omega$ as order unit, and the mapping $f \mapsto f|_\Omega$ is an order-unit-space isomorphism of $V(G)^*$ onto $\operatorname{aff}(\Omega(G))$ ([1; Theorem II.1.8]).

4.4. THEOREM. *Let U be an order-unit space and regard U as a unital group as in Lemma* 3.4. *Then $V(U)$ is the base-normed Banach dual space U^* of U* (*see* [1; Theorem II.1.15]).

P r o o f . By Lemma 3.4, $\hom(U,\mathbb{R})$ is the set of linear functionals $\nu\colon U \to \mathbb{R}$ such that $\nu(U^+) \subseteq \mathbb{R}^+$. Therefore, by [1; Corollary II.1.5], $\hom(U,\mathbb{R})^+ = U^{*+}$, whence $V(U) = U^{*+} - U^{*+} = U^*$. □

5. The passage from G to $\widehat{G}$

By Theorem 4.3, in the passage from the unital group G to the unital group $\widehat{G}$, the state space and the associated base-normed and order-unit Banach spaces remain invariant (up to a homeomorphic affine isomorphism, a base-normed-space isomorphism, and an order-unit-space isomorphism, respectively). In general, $\widehat{G}$ not only retains most of the desirable properties of G, but it acquires some attractive features that G fails to possess. For instance, $\widehat{G}$ may be archimedean although G is not. One of the more important properties that a unital group G may possess is the property of being $\mathbb{R}^+$-unital, as this property allows one to match up the probability measures on the unit interval E of G with the states on G (Lemma 3.3(iii)). Thus, the following result is of interest.

5.1. THEOREM. *If G is an $\mathbb{R}^+$-unital group, then $\widehat{G}$ is $\mathbb{R}^+$-unital.*

P r o o f . Assume the hypothesis, let E be the unit interval in G, and let $\psi\colon \widehat{E} \to \mathbb{R}^+$ be an $\mathbb{R}^+$-valued measure on the unit interval $\widehat{E}$ in $\widehat{G}$. Then $h_1, h_2, h_1+h_2 \in \widehat{E} \implies \psi(h_1 + h_2) = \psi(h_1) + \psi(h_2)$, and by induction, if $h_i \in \widehat{E}$ and $k_i \in \mathbb{Z}^+$ for $i = 1, 2, \dots, n$ with $\sum_{i=1}^{n} k_i h_i \in \widehat{E}$, then $\psi\Big(\sum_{i=1}^{n} k_i h_i\Big) = \sum_{i=1}^{n} k_i \psi(h_i)$. By hypothesis, the $\mathbb{R}^+$-valued measure $\phi\colon E \to \mathbb{R}^+$ defined by $\phi(e) := \psi(\widehat{e})$ for all $e \in E$ can be extended to a group homomorphism $\Phi \in \hom(G,\mathbb{R})^+ = V(G)^+$, and by Theorem 4.3(iv), there is a group homomorphism $\widehat{\Phi} \in \hom(\widehat{G},\mathbb{R})^+ = V(\widehat{G})^+$ such that $\widehat{\Phi}(\widehat{g}) = \Phi(g)$ for all $g \in G$. We claim that

$\widehat{\Phi}$ is an extension of ψ. Indeed, suppose $h \in \widehat{E}$. Then $h \in \widehat{G}^+$, so there exist $e_i \in E$ and $k_i \in \mathbb{Z}^+$ for $i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} k_i \widehat{e_i} = h \in \widehat{E}$. But then, $\widehat{e_i} \in \widehat{E}$ for $i = 1, 2, \ldots, n$, so $\widehat{\Phi}(h) = \sum_{i=1}^{n} k_i \widehat{\Phi}(\widehat{e_i}) = \sum_{i=1}^{n} k_i \Phi(e_i) = \sum_{i=1}^{n} k_i \phi(e_i) = \sum_{i=1}^{n} k_i \psi(\widehat{e_i}) = \psi\Big(\sum_{i=1}^{n} k_i \widehat{e_i}\Big) = \psi(h)$. □

Suppose E is the unit interval in a unital group G with unit u. If the elements of E are to be interpreted as testable effects in some experimental context, then it is desirable to have a reasonably large supply of probability measures on E. Since each state $\omega \in \Omega(G)$ determines a probability measure $\omega|_E$ on E by restriction, one can ensure the existence of a suitably large supply of probability measures on E by requiring that there are sufficiently many states in $\Omega(G)$.

Some of the conditions on $\Omega(G)$ that guarantee the existence of a suitable supply of states on G are as follows:

(i) $\Omega(G)$ is *cone determining* if and only if $G^+ = \{g \in G : \ 0 \le \omega(g)$ for all $\omega \in \Omega(G)\}$.
(ii) $\Omega(G)$ is *ordering* if and only if the condition $\omega(g) \le 1$ for $g \in G^+$ and all $\omega \in \Omega(G)$ implies that $g \in E$.
(iii) $\Omega(G)$ is *ample* if and only if the condition $\omega(g) = 1$ for $g \in G^+$ and all $\omega \in \Omega(G)$ implies that $g = u$.
(iv) $\Omega(G)$ *separates points* in G (respectively, in E) if and only if $g_1, g_2 \in G$ (respectively, $g_1, g_2 \in E$) with $g_1 \ne g_2$ implies that there exists $\omega \in \Omega(G)$ such that $\omega(g_1) \ne \omega(g_2)$.
(v) $\Omega(G)$ is *strictly positive* if and only if, for every $e \in E$, $e \ne 0$ implies that there exists $\omega \in \Omega$ such that $0 < \omega(e)$.

By Theorem 4.3, if $\Omega(G)$ satisfies any one of the conditions (i)–(v), then $\Omega(\widehat{G})$ also satisfies that condition.

Evidently $\Omega(G)$ is cone determining if and only if it determines the partial order on G in the following sense: For $g_1, g_2 \in G$, $\omega(g_1) \le \omega(g_2)$ for all $\omega \in \Omega(G) \iff g_1 \le g_2$. Likewise, $\Omega(G)$ is ordering if and only if it determines the partial order on the unit interval E. By [10; Theorem 4.14], G is archimedean if and only if $\Omega(G)$ is cone determining. Evidently, cone determining $\Longrightarrow$ ordering $\Longrightarrow$ ample $\Longrightarrow$ separates points in E $\Longrightarrow$ strictly positive, and suitable examples show that all of these implications are proper. Also, cone determining $\Longrightarrow$ separates points in G $\Longrightarrow$ ample, and these implications are proper. Finally, separates points in $G \not\Longrightarrow$ ordering, and ordering $\not\Longrightarrow$ separates points in G. Although the states need not separate the points of G, parts (iii) and (iv) of Theorem 4.3 imply that they do separate the points of $\widehat{G}$, and this is one of the main reasons why one might want to pass from G to $\widehat{G}$.

The mapping $e \mapsto \widehat{e}$ from E to $\widehat{E}$ is not necessarily bijective, and the structure of the effect algebra E may be altered in the passage to $\widehat{E}$. Even if $\widehat{E} = \{\widehat{e} : e \in E\}$, E and $\widehat{E}$ need not be isomorphic as effect algebras. However, if $\Omega(G)$ is ample, then E is isomorphic (as an effect algebra) to $\widehat{E}$ under the mapping $e \mapsto \widehat{e}$ ([5; Theorem 2.7(iii)]). The weaker condition that $\Omega(G)$ separates points in E is equivalent to the condition that $e \mapsto \widehat{e}$ maps E injectively into $\widehat{E}$.

As we indicated earlier, one of the attractive aspects of the unital-group approach is that it enables a perspicuous representation for finite effect algebras, and we now focus our attention on the case in which the unital group G has a finite unit interval E.

5.2. LEMMA. *Let G be a unital group with a finite unit interval E. Then:*

(i) *The unit interval $\widehat{E}$ in $\widehat{G}$ is finite.*

(ii) $V(G) = \hom(G, \mathbb{R})$.

Proof.

(i) Since the finite set E generates the positive cone G^+, it follows that the finite set $\{\widehat{e} \mid e \in E\}$ generates the positive cone $\widehat{G}^+ = \{\widehat{g} : g \in G^+\}$. Therefore, by [5; Theorem 2.2], the unit interval $\widehat{E}$ is finite.

(ii) By [6; Corollary 2.3], there is a state $\alpha \in \Omega(G)$ such that, for all $g \in G^+$, $\alpha(g) = 0 \implies g = 0$. Let $\nu \in \hom(G, \mathbb{R})$ and choose $t \in \mathbb{R}^+$ with $t > \max\{\nu(e)/\alpha(e) : 0 \neq e \in E\}$. Then, $0 \neq t\alpha - \nu \in \hom(G, \mathbb{R})$ and $0 \leq (t\alpha - \nu)(e)$ for all $e \in E$. Since E generates G^+, it follows that $0 \leq (t\alpha - \nu)(g)$ for all $g \in G^+$. Therefore, $0 \neq t\alpha - \nu \in \hom(G, \mathbb{R})^+$ and, because $\Omega(G)$ is a cone base for $\hom(G, \mathbb{R})^+$, there exists $\omega \in \Omega(G)$ and $0 \neq s \in \mathbb{R}^+$ with $t\alpha - \nu = s\omega$. Consequently, $\nu = t\alpha - s\omega \in V(G)$. □

If K is an abelian group, we denote the *torsion subgroup* of K, i.e., the subgroup of K consisting of 0 and all elements of K that have finite order, by K_τ. We say that K is *torsion free* if and only if $K_\tau = \{0\}$. If $\nu : K \to \mathbb{R}$ is a group homomorphism from K into the additive group of $\mathbb{R}$, then $\nu(K_\tau) = \{0\}$. Therefore, if G is a unital group and $\Omega(G)$ separates the points in G, it follows that G is torsion free. In particular, $\widehat{G}$ is always torsion free.

5.3. THEOREM. *Let $G \neq \{0\}$ be a unital group with unit u and with a finite unit interval E. Then, G_τ is finite and is a direct summand of the abelian group G. Also, G_τ is the kernel of the surjective group homomorphism $g \mapsto \widehat{g}$ from G onto $\widehat{G}$, whence, as an abelian group, $\widehat{G}$ is isomorphic to the quotient group G/G_τ. Furthermore, the rank of G is a positive integer r, and there is an order-preserving group isomorphism $\theta : \widehat{G} \to \mathbb{Z}^r$ from the unital group $\widehat{G}$ onto the simplicial group $\mathbb{Z}^r$.*

P r o o f . Since the finite set E is a set of generators for the abelian group G, it follows that the torsion subgroup G_τ of G is finite, that it is a direct summand of G, and that there is a surjective group homomorphism $\eta\colon G \to \mathbb{Z}^r$ with $\ker(\eta) = G_\tau$. By [6; Theorem 3.4], r is a positive integer and we can and do choose η in such a way that $\eta(G^+) \subseteq (\mathbb{Z}^+)^r$. By Theorem 4.3(iii), $g \in G_\tau \implies \widehat{g} = 0$, whence there is a surjective group homomorphism $\phi\colon \mathbb{Z}^r \to \widehat{G}$ such that $\widehat{g} = \phi\big(\eta(g)\big)$ for all $g \in G$.

Let $\eta(u) = \big(u_1, u_2, \dots, u_r\big) \in \mathbb{Z}^r$. If $\mathbf{z} \in \mathbb{Z}^r$, there exists $g \in G$ such that $\eta(g) = \mathbf{z}$ and there exists a positive integer n such that $nu - g \in G^+$, whence $n\eta(u) - \mathbf{z} \in \eta(G^+) \subseteq \big(\mathbb{Z}^+\big)^r$. Therefore, $(u_1, u_2, \dots, u_r)$ is an order unit in the simplicial group $\mathbb{Z}^r$, and it follows that $0 < u_1, u_2, \dots, u_r$.

Suppose $g \in G$ and $\big(g_1, g_2, \dots, g_r\big) := \eta(g) \neq \mathbf{0}$. Then there exists $k \in \{1, 2, \dots, r\}$ with $g_k \neq 0$. Let $\pi_k\colon \mathbb{Z}^r \to \mathbb{Z}$ be the projection homomorphism onto the kth component and define $\omega\colon G \to \mathbb{R}$ by $\omega(h) := \pi_k\big(\eta(h)\big)/u_k$ for all $h \in G$. Then $\omega \in \Omega(G)$ and $\omega(g) = g_k/u_k \neq 0$, whence $\widehat{g} \neq 0$ by Theorem 4.3(iii). Consequently, $\phi\big(\eta(g)\big) = \widehat{g} = 0 \iff \eta(g) = \mathbf{0}$, and it follows that ϕ is an isomorphism from the group $\mathbb{Z}^r$ onto the group $\widehat{G}$ and that $G_\tau = \{g \in G : \ \widehat{g} = 0\}$. Hence, $\widehat{G}$ is isomorphic as an abelian group to G/G_τ.

Let $\theta\colon \widehat{G} \to \mathbb{Z}^r$ be the inverse of the group isomorphism $\phi\colon \mathbb{Z}^r \to \widehat{G}$, so that $\theta(\widehat{g}) = \eta(g)$ for all $g \in G$. Then $\theta\big(\widehat{G}^+\big) = \eta\big(G^+\big) \subseteq \big(\mathbb{Z}^+\big)^r$, so θ is order preserving. □

If G is a nonzero unital group with a finite unit interval, then by Theorem 5.3, we may identify $\widehat{G}$, as a group, with $\mathbb{Z}^r$ in such a way that the positive cone $\widehat{G}^+$ is a subcone of the standard positive cone $\big(\mathbb{Z}^+\big)^r$ in the simplicial group $\mathbb{Z}^r$. This throws considerable light on the structure of the unital group $\widehat{G}$ and enables one to deal with it computationally. For instance, the question of whether $\widehat{G}$ is archimedean hinges on whether the subcone $\widehat{G}^+$ of $\big(\mathbb{Z}^+\big)^r$ is determined by a finite set of homogeneous linear inequalities with integer coefficients ([7; Theorem 5.1]).

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Received February 10, 2003

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