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# ON HAMILTONIAN CYCLES OF COMPLETE $n$-PARTITE GRAPHS

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The notions not defined here will be used in the sense of [1].

We shall show a recursive formula for the number of hamiltonian cycles of the graph $K(m_1, m_2, \ldots, m_n)$. In the case $m_1 = m_2 = \ldots = m_n = 2$ we prove an explicit formula. Further we give an upper estimation for the number of hamiltonian cycles of an arbitrary graph $G$.

A complete $n$-partite graph $G$ is a graph whose vertex set $V$ can be partitioned into $n$ mutually disjoint subsets $V_1, V_2, \ldots, V_n$, whose union is $V$ such that two vertices are connected by an edge if and only if they belong to different parts of the partition.

If $|V_1| = m_1, |V_2| = m_2, \ldots, |V_n| = m_n$, we write $G = K(m_1, m_2, \ldots, m_n)$.

The number of hamiltonian cycles of a graph $G$ will be denote by $H(G)$, in the case of the complete $n$-partite graph $K(m_1, m_2, \ldots, m_n)$ by $H(m_1, m_2, \ldots, m_n, n)$.

The values $H(n, n, 2)$, $H(n, n, n, 3)$, $H(n, n, n, n, 4)$ were found by A. D. Korshunov and J. Ninčák [2] and idependently by A. Vrba [4]:

$$H(n, n, 2) = \frac{(n-1)!\, n!}{2},$$

$$H(n, n, n, 3) = 2^{n-1}(n!)^3 \sum_{i=0}^{[n/2]} \frac{(n+i-1)!}{2^{2i}(n-2i)!\,(i!)^3},$$

$$H(n, n, n, n, 4) = \frac{1}{2} \sum_{j=0}^{n} \sum_{i=0}^{[\frac{2}{3}(n-j)]} (2n-i-j-1)!\, 2^{2i}\, 3^j \frac{(n!)^4}{j!} \times$$

$$\times \sum_{d=\max(0,\, n-j-2i)}^{\min(n-j,\, 2n-3i-2j)} \frac{(3n-3i-3j-2d)!}{(2i+j-n+d)!\,(n-j-d)!\,(2n-3i-2j-d)!} \times$$

$$\times \sum_{e=\max(0,\, i+j-n+d)}^{\min(d,\, 2n-2i-2j-d)} \frac{(2n-2i-2j-2e)!}{e!\,(d-e)!\,(2n-2i-2j-d-e)!\,(n-i-j-d+e)!\,((n-i-j-e)!)^2}$$

In [2] there is presented an asymptotic formula

$$H(s, s, \dots, s, k) \approx \frac{(ks-1)!}{2e^{s-1}},$$

valid for an arbitrary fixed $s$ and $k \to \infty$.
It is clear that $H(2,2,2)=1, H(1,1,1,3)=1, H(2,1,1,3)=1$ and $H(1,1,1,1,4)= =3$. All the other values can be calculated by using the following theorem.

**Theorem 1.** *Let $n \geqslant 2$, $m_1 \geqslant 0$, $m_i \geqslant 1$ $(i=2, \dots, n)$ be integers, where $m_1 + m_2 + \dots + m_n \geqslant 4$. Then we have*

$$\text{(1)} \quad \begin{aligned} H(m_1+1, m_2, \dots, m_n, n) &= \Big(\sum_{i=1}^{n} m_i - 2m_1\Big) H(m_1, m_2, \dots, m_n, n) + \\ &+ \sum_{i=2}^{n} m_i(m_i-1) H(m_1, m_2, \dots, m_i - 1, \dots, m_n, n) . \end{aligned}$$

Remark. The formula (1) is efficient for practical use.

Proof. Let $G = K(m_1+1, m_2, \dots, m_n)$ be a complete $n$-partite graph, where $m_1+m_2+\dots+m_n \geqslant 4$. Let $v$ be a fixed vertex from the first part of $G$.

First we shall discuss the number of hamiltonian cycles of $G$ in which vertices adjacent to $v$ belong to different parts of $G$.

If $a, \dots, u, v, w, \dots, b, a$ is a hamiltonian cycle of $G$ of this kind, then $a, \dots, u, w, \dots, b, a$ is a hamiltonian cycle of the graph $G-\{v\}$. But from each of the $H(m_1, m_2, \dots, m_n, n)$ hamiltonian cycles of $G-\{v\}$ there arise in such a way exactly $\sum_{i=1}^{n} m_i - 2m_1$ distinct hamiltonian cycles of $G$ of the considered kind. Hence, the number of such hamiltonian cycles of $G$ is

$$\Big(\sum_{i=1}^{n} m_i - 2m_1\Big) H(m_1, m_2, \dots, m_n, n) .$$

To complete the proof we count those hamiltonian cycles of $G$ in which both vertices adjacent to $v$ belong to the same part of $G$.

If $a, \dots, t, u, v, w, x, \dots, b, a$ is a hamiltonian cycle of $G$ of this kind, then $a, \dots, t, w, x, \dots, b, a$ is a hamiltonian cycle of the graph $G-\{u, v\}$ and $a, \dots, t, u, x, \dots, b, a$ is a hamiltonian cycle of the graph $G-\{v, w\}$.

Each of the $H(m_1, \dots, m_j-1, \dots, m_n, n)$ hamiltonian cycles of the graph $G-\{u, v\}$ arises in such a way exactly from two hamiltonian cycles of $G$ (analogously to the graph $G-\{v, w\}$).

The number of hamiltonian cycles of $G$, in which $v$ is adjacent to the fixed vertices $u$, $w$ from the $j$-th part of $G$ is equal to $2H(m_1, \dots, m_j-1, \dots, m_n, n)$ and the number of hamiltonian cycles of $G$ in which both vertices adjacent to $v$ belong to the $j$-part of $G$ is $\binom{m_j}{2} 2H(m_1, \ldots, m_j - 1, \ldots, m_n, n)$. Thus we have $\sum_{j=2}^{n} m_j(m_j - 1)$ $H(m_1, \ldots, m_j - 1, \ldots, m_n, n)$ hamiltonian cycles of the second kind. Q.E.D.

It is easy to find a condition for $H(m_1, m_2, \ldots, m_n, n) \neq 0$.

**Theorem 2.** *The graph* $K(m_1, m_2, \ldots, m_n)$ *is hamiltonian if and only if* $\sum_{i=1, i\neq j}^{n} m_i \geqslant m_j$ $(j = 1, 1, \ldots, n)$.

Proof. The necessity of these conditions is straightforward.

Let $\sum_{i=1, i\neq j}^{n} m_i \geqslant m_j$ $(j = 1, 2, \ldots, n)$. Then for every vertex $v$ of the graph $K(m_1, m_2, \ldots, m_n)$ we have $\deg v \geqslant \frac{p}{2}$, where $p = m_1 + m_2 + \ldots + m_n$. Then from Pósa's theorem (see [3]) it follows that the graph is hamiltonian. Q.E.D.

**Theorem 3.** *For any integer* $n \geqslant 1$ *we have:*

$$H(2, 2, \ldots, 2, n+1) = n!\,2^{n-1} w_n\,, \tag{2}$$

*where*

$$w_n = D(n, 0) + \sum_{r=1}^{n} \sum_{0 \leqslant i_1 < \ldots < i_r \leqslant n-1} D(n, i_r + 1) D(i_r, i_{r-1} + 1) \ldots D(i_2, i_1 + 1) D(i_1, 0)$$

*and*

$$D(i, j) = i^2 - j^2 + 2i - 1.$$

Remark. This value has been obtained independently in other terms by A. D. Korshunov and J. Ninčák [2].

Proof. Put $a_n = H(2, 2, \ldots, 2, n)$, $b_n = H(2, 2, \ldots, 2, 1, n)$ and $c_n = H(2, 2, \ldots, 2, 1, 1, n)$.

From (1) it follows that

$$b_{n+1} = 2na_n + 2nb_n \qquad (n \geqslant 2)\,, \tag{3}$$

$$c_{n+1} = (2n-1)b_n + 2(n-1)c_n \qquad (n > 2)\,, \tag{4}$$

$$a_{n+1} = (2n-1)b_{n+1} + 2nc_{n+1} \qquad (n \geqslant 2)\,. \tag{5}$$

According to (3) and (4) we can write

$$b_{n+1} = n! \sum_{i=2}^{n} \frac{2^{n-i+1}}{(i-1)!} a_i \qquad (n \geqslant 2)\,, \tag{6}$$

$$c_{n+1} = (n-1)! \left( \sum_{i=3}^{n} \frac{2^{n-i}(2i-1)}{(i-1)!} b_i + 2^{n-2} \right) \qquad (n \geqslant 2)\,, \tag{7}$$

$$c_2 = 1\,.$$

If we substitute (6) and (7) into (5) and use

$$\sum_{i=3}^{n}(2i-1)\sum_{j=2}^{i-1}\frac{2^{n-j+1}}{(j-1)!}a_j=\sum_{i=2}^{n}(n^2-i^2)\frac{2^{n-i+1}}{(i-1)!}a_i \tag{8}$$

where ($n \geq 2$), we have

$$a_{n+1}=n!\left(\sum_{i=2}^{n}D(n,i)\frac{2^{n-i+1}}{(i-1)!}a_i+2^{n-1}\right). \tag{9}$$

We prove by induction the statement (2).

For $m=1$

$$a_2=1!\,2^0(D(1,0)+D(1,1)D(0,0))=1\,.$$

Supposing the statement is true for $p \leq n-1$ and prove it for $p=n$.

According to (9) we have

$$a_{n+1}=n!\left(\sum_{i=2}^{n}D(n,i)\frac{2^{n-i+1}}{(i-1)!}(i-1)!\,2^{i-2}w_{i-1}+2^{n-1}\right). \tag{10}$$

To complete the proof we must show that

$$w_n=\sum_{i=2}^{n}D(n,i)w_{i-1}+1\,.$$

Simplifying the same members from both sides of this equality gives

$$D(n,0)+D(n,1)D(0,0)=1\,. \tag{11}$$

However, the equation (11) holds for arbitrary $n \geq 1$.

Q.E.D.

The theorem yields for $n=2, 3, \ldots, 6$:

$$a_2=1,\ a_3=16,\ a_4=744,\ a_5=56\,256,\ a_6=6\,385\,920\,.$$

We shall apply the above results to estimate the number $H(G)$ of an arbitrary graph $G$.

A set of points in $G$ is independent if no two of them are adjacent. The largest number of points in such a set is called the point independence number of $G$ and is denoted by $\beta_0(G)$.

**Theorem 4.** *Let G be a graph with p vertices. Let $\beta_0(G) \geq m$. Then we have:*

$$H(G) \leq \frac{(p-m)!}{2}\prod_{i=2}^{m}(p-m+1-i)\,.$$

Proof. From the recursive expression (1) for the complete $n$-partite graph $K(m, 1, \ldots, 1)$ we have

$$H(m, 1, \ldots, 1, n) = (n-m)H(m-1, 1, \ldots, 1, n)\,. \tag{12}$$

It is clear that

$$H(1, 1, \ldots, 1, n) = \frac{(n-1)!}{2} \tag{13}$$

Hence from (12) and (13) it follows that

$$H(m, 1, \ldots, 1, n) = \frac{(n-1)!}{2}\prod_{i=2}^{m}(n-i)\,. \tag{14}$$

If $G_1$ is a spanning subgraph of a graph $G_2$ then one can easily verify that $H(G_1) \leq H(G_2)$. Let $G$ be a graph with $p$ vertices and let $\beta_0(G) \geq m$. Therefore $G$ is a spanning subgraph of the complete $(p-m+1)$-partite graph $K(m, 1, \ldots, 1)$ and from (14) it follows:

$$H(G) \leq H(m, 1, \ldots, 1, p-m+1) = \frac{(p-m)!}{2}\prod_{i=2}^{m}(p-m+1-i)\,.$$

The proof is complete.

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## О ГАМИЛЬТОНОВЫХ ЦИКЛАХ ПОЛНОГО $n$-ДОЛЬНОГО ГРАФА


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Резюме

Приводится рекуррентное соотношение для числа гамильтоновых циклов полного $n$-дольного графа $K(m_1, m_2, \ldots, m_n)$. В случае $m_1 = m_2 = \ldots = m_n$ 2 доказывается явная формула.