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PERMUTABILITY OF DISTRIBUTIVE CONGRUENCE RELATIONS IN A JOINT SEMILATTICE DIRECTED BELOW

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Introduction

The aim of this paper is to study some interesting properties of congruence relations obtained from distributive (standard) ideals in join semilattices (directed below). If L is a lattice and I is an ideal of L, it is of great interest to know how does the smallest congruence collapsing I look like (perhaps in a simpler form). However, a general description for such congruences was already given by P. Crawley and R. P. Dilworth, in a more general setting (see [1]), which, though, is not simple. If $(S, \vee)$ is a join semilattice, then it is known that for every ideal I of S, $\theta_I = \{(x, y) \in S \times S \mid x \vee i = y \vee i$ for some $i \in I\}$ is a congruence on S, and is the smallest congruence collapsing I if and only if S is directed below (see [6]). However, if $(L, \vee, \wedge)$ is a lattice and I is an ideal of L, then θ_I described above is a join congruence relation, not necessarily a lattice congruence relation. This led to the concept of a standard ideal in a lattice (due to Gratzer and Schmidt, [4]) in which they have shown that θ_T described above corresponding to a standard ideal T becomes a congruence relation and is the smallest congruence relation collapsing T. Also they have shown that any two standard congruence relations in a lattice are permutable. The concepts of distributive standard elements in lattices are due to Gratzer and Schmidt (see [4]). In this paper the concepts of distributive, standard elements are introduced in a join semilattice and it is shown that in a join semilattice S directed below an element t is distributive (standard) if and only if $(t]$ is a distributive (standard) element of the lattice of ideals of S. Also it has already been observed that if D is a distributive ideal of a lattice L, then θ_D is a lattice congruence relation on L (it follows from the dual of Corollary 1 of [5]). The concept of a nodal filter in a meet semilattice is due to Varlet [7] and the concept of a node of a poset is due to J. Duda (see [2]). A node of a poset is an element of P which is comparable with every element of P. In [2], Duda has proved that a node of a lattice is distributive. In fact in this paper we show that the node of a join semilattice directed below is neutral (see Theorem 6). Thus showing that the node of a lattice is neutral, which is much more stronger than saying that it is distributive.

In this paper it is shown that any two congruence relations induced by distributive ideals, i.e. distributive congruence relations, are permutable (see Theorem 7). Now it is suprising to note that even any two distributive congruence relations in a lattice are permutable, from which it follows as a corollary that any two standard congruence relations are permutable. A distributive (standard) ideal of a join semilattice directed below is characterized in terms of a congruence relation (see Theorems 8 and 9). A more interesting result (see Corollary 8) is obtained when the ideal is a standard ideal in a directed below join semilattice S, namely, in S; if T is a standard ideal of S, then for any ideal I of S, θ_T and θ_I are permutable, from which it follows that in a lattice L, for any standard ideal T of L, and for any ideal I of L the lattice congruence relation θ_T and the join congruence relation θ_I are permutable.

We now begin with the following definitions.

Let $(S; \vee)$ be a join semilattice. Then for any $a \in S$, θ_a stands for the congruence relation $\{(x, y) \in S \times S \mid x \vee a = y \vee a\}$. For the definition of an ideal in join semilattice see [3]. The set of ideals of S is denoted by $I(S)$. If S is a directed below join semilattice, then $I(S)$ is a lattice. If I is an ideal of S, then θ_I stands for $\{(x, y) \in S \times S \mid x \vee i = y \vee i$ for some $i \in I\}$ and is a congruence relation containing $I \times I$. Also if $a \in S$, $\theta_a = \theta_{(a]}$. If $H \subseteq S$, then $\theta(H)$ stands for the smallest congruence relation containing $H \times H$. For the standard definitions and results see [3] and [4]. Throughout this paper $(S; \vee)$ stands for a join semilattice, $(S_1; \vee)$ stands for a join semilattice directed below and $(L; \vee, \wedge)$ stands for a lattice unless otherwise stated.

Definition 1. *An element $d \in S$ is said to be distributive if $x \leqslant d \vee a$, $x \leqslant d \vee b$ $(x, a, b \in S)$ implies the existence of c in S, such that $c \leqslant a$, $c \leqslant b$ and $x \leqslant d \vee c$. An element $s \in S$ is said to be standard if $x \leqslant s \vee t (x, t \in S)$ implies the existence of $s_1 \leqslant s$, $t_1 \leqslant t (s_1, t_1 \in S)$ such that $x = s_1 \vee t_1$. An element $n \in S_1$ is said to be neutral if $(n]$ is neutral in $I(S_1)$.*

The proofs of Lemma 1, and the following Theorems 1 to 5, are straightforward and hence are omitted.

Lemma 1. *Every standard element of S is a distributive element of S.*

Remark 1. It is easy to observe from the standard nonmodular five element lattice that the converse of the above Lemma 1 is not true. However, this happens in the case of a modular semilattice as in the following

Theorem 1. *Let $(S; \vee)$ be a modular semilattice. Then for $t \in S$ the following are equivalent:*

1) t is distributive; 2) *t is standard;* 3) *t is neutral.*

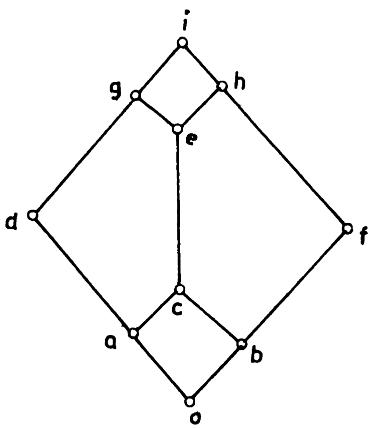
Theorem 2. *An element d of S_1 is distributive (standard) if and only if $(d]$ is a distributive (standard) ideal of the ideal lattice $I(S_1)$ of S_1.*

Theorem 3. *Let $s \in S$ be a standard element. If $a \wedge s$ exists for some $a \in S$, then $a \wedge s$ is standard in $(a]$.*

Theorem 4. *The sets of distributive, standard, neutral elements form subsemilattices of S.*

Remark 2. The meet of two distributive elements even in a lattice need not be distributive because of the following

Example 1.



It is easy to observe that d and e are distributive elements of L but a is not distributive.

Theorem 5. *If S contains a distributive element, then S is directed below.*

Corollary 1. *Let $d \in D$ be a distributive element. Then $\theta_d = \theta((d])$.*

Remark 3. The standard nonmodular five element lattice serves as an example to show that the converse of the above Corollary 1 is not true. Further from Corollary 1 it immediately follows that if t is a standard element of S, then $\theta_t = \theta((t])$. However, from the same example it can be observed that the converse of this statement is also not true even when t is distributive.

In [7] Varlet has introduced the concept of a nodal filter in a meet semilattice. The concept of a node of a poset is due to J. Duda [2]. He has proved that every node of a lattice is distributive. In fact in the following theorem we show that every node of a join semilattice directed below is neutral, which is much stronger than saying that it is distributive, thus showing that the node of a lattice in particular is neutral and every nodal filter of a lattice is a neutral filter (see Corollary 2 of page 4, [2]).

Theorem 6. *Every node of S_1 is neutral.*

Proof. Let s be a node of S_1. We first show that s is standard. Let $x \leqslant s \vee t$. If $s \leqslant x$ and $s \leqslant x < t$ so that $x = s \vee x$. If $s \leqslant x$ and $t \leqslant s$, then $x < s$ and hence $x = s$ so that $x = s \vee t$. If $x \leqslant s$ and $s \leqslant t$, then clearly $x = x \vee x$. If $x < s$ and $t \leqslant s$, then $x = x \vee a$, where a is any lower bound of t and x. Thus s is standard. Hence $(s]$ is a standard ideal. Now let $x, y \in S$ and $a \in (s] \cap ((x] \vee (y])$ so that $a \leqslant s$ and $a \leqslant x \vee y$. If either $s \leqslant x$ or $s \leqslant y$, then clearly a is in $(s] \cap (x]$ or $(s] \cap (y]$. So let $x \leqslant s$ and $y \leqslant s$. Then $a \leqslant x \vee y \in ((s] \cap (x]) \vee ((s] \cap (y])$. Since the reverse inequality holds trivially, it follows that $(s] \cap ((x] \vee (y]) - ((s] \cap (x]) \vee ((s] \cap (y])$. Thus by Theorem 3, page 26, and Theorem 1 page 28 of [4], $(s]$ is neutral.

Remark 4. Any distributive lattice possesing two noncomparable elements is an example to show that the converse of the above Theorem 6 is not true.

Corollary 2. *If N is a nodal ideal (filter) of S_1 (of a meet emilattice directed above), then N is a neutral ideal (filter).*

Lemma 2. *An ideal D of S_1 is distributive if and only if $t < x \vee a$, $t \leqslant y \vee b$ for some $x, y \in D$, $t, a, b \in S_1$ implies that there exist $z \in D$, $k \in S_1$ such that $k \leqslant a, b$ and $t \leqslant z \vee k$.*

Proof. Suppose that $t \leqslant x \vee a$ and $t \leqslant y \vee b$, where $x, y \in D$ and $a, b, t \in S_1$; then $t \in (D \vee (a]) \wedge (D \vee (b]) = D \vee ((a] \wedge (b])$ (since D is distributive) so that $t \leqslant z \vee k$, where $z \in D$ and $k \leqslant a, b$, $k \in S_1$. Conversely, assume the condition. Let $t \in (D \vee I) \wedge (D \vee J)$ so that $t \leqslant x \vee a$, $y \vee b$ for some $x, y \in D$ and $a \in I$, $b \in J$ and hence by assumption, there exist $z \in D$ and $k \in S_1$ such that $k \leqslant a$, b and $t \leqslant z \vee k$. Therefore $t \in D \vee (I \wedge J)$. Since the reverse inequality holds in general, D is distributive.

Theorem 7. *If I and J are distributive ideals of S_1, then θ_I and θ_J are permutable.*

Proof. Let $(x, y) \in \theta_J \supset \theta_I$ so that there exists $z \in S_1$ such that $(x, z) \in \theta_J$ and $(z, y) \in \theta_I$ and hence $x \vee j = z \vee j$ for some $j \in J$ and $z \vee i = y \vee i$ for some $i \in I$. Therefore $x \vee i \vee j = z \vee i \vee j = y \vee i \vee j$. Since $y \leqslant (x \vee i) \vee j$, $y \leqslant j$ and J is distributive, $y \leqslant t \vee k$, where $t \in J$ and $k \leqslant x \vee i$, y (from Lemma 2) so that $t \vee y \vee j = t \vee k \vee j$. Now since $x \leqslant (y \vee j) \vee i$, $x \leqslant x \vee i$, and I is distributive, there exist $p \in I$ and $l \in S_1$ such that $l \leqslant y \vee j$, $l \leqslant x$ and $p \vee x \vee i = p \vee l \vee i$. Now observe that $(x, k \vee l) \in \theta_I$ and $(k \vee l, y) \in \theta_J$ and hence $(x, y) \in \theta_J \supset \theta_I$. Thus θ_I and θ_J are permutable.

Remark 5. The standard nonmodular five element lattice is an example to observe that the result does not hold if just one of the two ideals is distributive.

Corollary 3. *Let $a, b \in S_1$ be two distributive elements. Then θ_a and θ_b are permutable.*

It is known that if L is a lattice, then an ideal I of L is distributive if and only if θ_I is a congruence relation (see [5]). This θ_I will be called as distributive congruence relation.

Corollary 4. *In L, any two distributive congruence relations are permutable.*

Corollary 5. *Any two standard congruence relations of L are permutable.*

Theorem 8. *An ideal D of S_1 is distributive if and only if $\theta = \{(x, y) \in S_1 \times S_1 \mid x \vee y \leqslant k \vee d$ for some $k \leqslant x, y$ and $d \in D\}$ is a congruence relation and $(x, y) \in \theta$, $z \leqslant y$ implies that there exist $w \leqslant x, z$ such that $(w, z) \in \theta$.*

Proof. Suppose that D is distributive. Clearly θ is reflexive and symmetric. Let $(x, y) \in \theta$ and $(y, z) \in \theta$ so that $x \vee y \leqslant k_1 \vee d_1$, $y \vee z \leqslant k_2 \vee d_2$ for some $k_1 \leqslant x, y$, $k_2 \leqslant y, z$ and $d_1, d_2 \in D$ and hence $y \in ((k_1] \vee D) \wedge ((k_2] \vee D)$. Therefore $y \leqslant k_3 \vee d_3$, where $k_3 \leqslant k_1$, $k_3 \leqslant k_2$, $d_3 \in D$. Also $x \in ((k_1] \vee D) \wedge ((x] \vee D)$ so that $x \leqslant k' \vee d_4$, where $k' \leqslant k_1$, $k' \leqslant x$ and $z \in ((k_2] \vee D) \wedge ((Z] \vee D)$ so that $z \leqslant k'' \vee d_5$, where $k'' \leqslant k_2$, z and hence

$$x \vee z \leqslant k' \vee k'' \vee d_4 \vee d_5 \leqslant k_1 \vee k_2 \vee d_4 \vee d_5 \leqslant k_3 \vee d_3 \vee d_4 \vee d_5,$$

where $d_3 \vee d_4 \vee d_5 \in D$. Thus $(x, z) \in \theta$, so that θ is transitive. Now it is easy to observe that θ is a congruence relation. Let $(x, y) \in \theta$ and $z \leqslant y$ so that $x \vee y \leqslant k \vee d$ for some $k \leqslant x, y$, $d \in D$. Hence $z \in ((k] \vee D) \wedge ((z] \vee D)$ so that $z \leqslant w \vee d_1$ for some $w \leqslant k, z$. Since $w \vee z = z \leqslant w \vee d_1$, it follows that $(w, z) \in \theta$. Thus the condition holds. Conversely, assume the condition. Let $x \in (D \vee A) \wedge (D \vee B)$ so that $x \leqslant d_1 \vee a$, $d_2 \vee b$ for some $d_1, d_2 \in D$, $a \in A$ and $b \in B$. Let $d_1 \vee d_2 = d$. Clearly $(a, \ \vee d) \in \theta$. $x \leqslant a \vee d$ so that there exist $w \leqslant a, x$ such that $(w, x) \in \theta$. Similarly since $(b, b \vee d) \in \theta$, there exists $w_1 \leqslant b, x$ such that $(w_1, x) \in \theta$. $(w, x) \in \theta$ implies that $x = w \vee x \leqslant k \vee d_3$, where $k \leqslant w, x$ and $d_3 \in D$ so that $k \vee d_3 = x \vee d_3$. Similarly since $(w_1, x) \in \theta$, there exists $k_1 \leqslant w_1, x$, $d_4 \in D$ such that $k_1 \vee d_4 = x \vee d_4$ and hence $x \vee d_3 \vee d_4 = k \vee k_1 \vee d_3 \vee d_4$. Thus $k \vee k_1 \leqslant k \vee d_3 \vee d_4$, $k_1 \vee d_3 \vee d_4$ so that $(k, k \vee k_1) \in \theta$ and $(k \vee k_1, k_1) \in \theta$ and hence $(k, k_1) \in \theta$. Therefore $k \vee k_1 \leqslant t \vee e$, where $t \leqslant k$, k_1 and $e \in D$ so that $x \leqslant x \vee d_3 \vee d_4 = k \vee k_1 \vee d_3 \vee d_4 \leqslant t \vee d_3 \vee d_4 \vee e$, where $t \leqslant k \leqslant w \leqslant a$, $t \leqslant k_1 \leqslant w_1 \leqslant b$ and $d_3 \vee d_4 \vee e \in D$. Thus $x \in D \vee (A \wedge B)$ and hence D is distributive.

Corollary 6. *An element d of S_1 is distributive if and only if $\theta = \{(x, y) \in S_1 \times S_1 \mid x \vee y \leqslant k \vee d$ for some $k \leqslant x, y\}$ is a congruence relation and $(x, y) \in \theta$, $z \leqslant y$ implies that there exist $w \leqslant x, z$ such that $(w, z) \in \theta$.*

Theorem 9. *An ideal T of S_1 is standard if and only if $\theta = \{(x, y) \in S_1 \times S_1 \mid x \vee t = y \vee t = x \vee y$ for some $t \in T\}$ is a congruence relation and $(x, y) \in \theta$, $z \leqslant y$ implies that there exists $w \leqslant x, z$ such that $(w, z) \in \theta$.*

Proof. Let T be standard ideal. It will be shown that $\theta = \{(x, y) \in S_1 \times S_1 \mid x \vee t = y \vee t$ for some $t \in T\}$ $(= \theta_T)$ so that θ is a congruence relation. Clearly $\theta \subseteq \theta_T$. So, let $(x, y) \in \theta_T$ so that $x \vee t = y \vee t$ and hence $x \in (x] \cap ((y] \vee T) = ((x] \cap (y]) \vee ((x] \cap T$. Thus there exist $y_1 \leqslant y, x$, $t_1 \in T$, $t_1 \leqslant x$ such that $x \leqslant y_1 \vee t_1$ and hence $x = y_1 \vee t_1$. By a similar argument there exist $x_1 \leqslant x$, $t_2 \in T$ such that

$y = x_1 \vee t_2$. Therefore $x \vee y = y_1 \vee t_1 \vee x_1 \vee t_2$ so that $x \vee y = x \vee t_1 \vee t_2 = y \vee t_1 t_2$ and $t_1 \vee t_2 \in T$. Thus $(x, y) \in \theta$. Let $(x, y) \in \theta$ and $z \leqslant y$ so that $x \vee t = y \vee t$ for some $t \in T$ and hence $z \leqslant z \vee t \leqslant y \vee t = x \vee t$. Thus $z \in (z] \cap ((x] \vee T) = ((z] \cap (x]) \vee ((z] \cap T)$ so that $z \leqslant x_1 \vee t_1$, where $x_1 \leqslant z$, x, $t_1 \leqslant z$, $t_1 \in T$ and hence $z = x_1 \vee t_1$. Thus $z \vee t_1 = x_1 \vee t_1$ so that $(z, x_1) \in \theta_T = \theta$. Conversely let $x \in A \cap (T \vee B)$ so that $x \in A$ and $x \leqslant t \vee b$ for some $t \in T$ and $b \in B$ and since $(t \vee b, b) \in \theta$ and hence there exist $w \leqslant x$, b such that $(w, x) \in \theta$. Hence $w \vee t_1 = x \vee t_1 = w \vee x = x$ for some $t_1 \in T$. Thus T is standard.

Corollary 7. *An element $t \in S_1$ is standard if and only if $\theta = \{(x, y) \in S_1 \times S_1 \mid x \vee t_1 = y \vee t_1 = x \vee y$ for some $t_1 \leqslant t\}$ is a congruence relation and $(x, y) \in \theta$, $z \leqslant y$ implies that there exists $w \leqslant x$, z such that $(w, z) \in \theta$.*

Definition 2. *Let $I, J \in I(S_1)$. We write IMJ if and only if for any $X \in I(S_1)$, $I \vee J \supseteq X \supseteq I$* implies that $X = I \vee (X \cap J)$.

In the following theorem a necessary and sufficient condition for θ_I (where I is an ideal of S_1) to be permutable with θ_J for every ideal J is obtained.

Theorem 10. *Let I be an ideal of S_1. Then θ_I is permutable with θ_J for all ideals J of S_1 if and only if IMJ and JMI hold for all J.*

Proof. Assume that IMJ and JMI hold for all J. It will be shown that $\theta_I \supset \theta_J = \theta_{I \vee J}$ (since $\theta_I \vee \theta_J = \theta_{I \vee J}$). Clearly $\theta_I \supset \theta_J \subseteq \theta_{I \vee J}$. Let $(s, t) \in \theta_{I \vee J}$ so that $s \vee a \vee x = t \vee a \vee x$ for some $a \in I$ and $x \in J$. Since $I \vee (x \vee s] \supseteq I \vee (t] \supseteq I$, $a \vee t \in I \vee (t] = I \vee ((I \vee (t]) \cap (x \vee s])$ (since IMJ holds) so that $a \vee t \leqslant a_1 \vee r$, where $a_1 \in I$ and $r \leqslant a_2 \vee t$, $r \leqslant x \vee s$, $a_2 \in I$ and hence $a \vee a_1 \vee a_2 \vee t = a \vee a_1 \vee a_2 \vee r$. Let $p = a \vee a_1 \vee a_2$ so that $p \in I$ and $p \vee r \vee x = p \vee t \vee x = p \vee s \vee x \geqslant s \vee x \geqslant r \vee x$ and hence $I \vee (r \vee x] \supseteq (s \vee x] \supseteq (r \vee x]$. Since $(r \vee x]\, MI$ holds, $x \vee s \in (s \vee x] = (r \vee x] \vee (I \cap (s \vee x])$ so that $x \vee s \leqslant a_3 \vee r_1$, where $a_3 \in I$, $a_3 \leqslant s \vee x$ and $r_1 \leqslant r \vee x$. Therefore $s \vee x = a_3 \vee r \vee x$ and hence $(s, a_3 \vee r) \in \theta_J$, $(a_3 \vee r, t) \in \theta_I$. Therefore $(s, t) \in \theta_I \supset \theta_J$. Hence θ_I and θ_J are permutable. Conversely, assume that θ_I and θ_J are permutable and $I \vee J \supseteq X \supseteq I$. Let $x \in X$ so that $x \leqslant i \vee k$ for some $i \in I$, $j \in J$ and hence $(x, j) \in \theta_{I \vee J} = \theta_J \supset \theta_I$. Therefore there exists $z \in S$ such that $(x, z) \in \theta_I$ and $(z, j) \in \theta_J$ so that $x \vee i_1 = z \vee i_1$ for some $i_1 \in I$ and $z \vee j_1 = j \vee j_1$ for some $j_1 \in J$. Now $i_1 \in I \subseteq X$ so that $z \in X$ and since $j \vee j_1 \in J$, $z \in J$ and hence $z \in X \cap J$. Thus $x \leqslant i_1 \vee z \in I \vee (X \cap J)$. Since the reverse inequality is trivial, $X = I \vee (X \cap J)$. Similarly it can be shown that JMI holds for all J.

Corollary 8. *Let T be a standard ideal of $(S_1, \vee)$. Then for any ideal J of S_1, θ_T and θ_J are permutable.*

Proof. Observe that if T is a standard ideal, then IMJ and JMT hold for all ideals J of S.

Corollary 9. *If T is a standard ideal of L, and I is an ideal of L, then the lattice congruence relation θ_T and the join congruence relation θ_I are permutable.*

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ЗАМЕНИТЕЛЬНОСТЬ ДИСТРИБУТИВНЫХ КОНГРУЭНЦИЙ НА $\vee$-ПОЛУРЕШЕТКЕ НАПРАВЛЕННОЙ ВНИЗ

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Резюме

Определяются дистрибутивные, стандартные и нейтральные элементы (идеалы) полурешетки. Показано, что для конгруэнций определеных этими элементами (идеаламы) имеют места теоремы подобные к раньше известным для решеток.