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HASHIMOTO TOPOLOGIES AND QUASI-CONTINUOUS MAPS

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ABSTRACT. In a topological space $(X,\mathcal{T})$ any ideal $\mathcal{P}$ of subsets of X induces a new topology $\mathcal{T}(\mathcal{P})$. If Y is a regular space, then under some assumptions on $\mathcal{P}$, for any upper $\mathcal{T}(\mathcal{P})$-quasi-continuous multivalued map $F\colon X \to Y$ the sets of all points at which F is lower quasi-continuous (lower semicontinuous) with respect to $\mathcal{T}(\mathcal{P})$ or $\mathcal{T}$ coincide. If F is compact-valued lower $\mathcal{T}(\mathcal{P})$-quasi-continuous, then the symmetrical result holds.

For a subset A of a topological space $(X,\mathcal{T})$ the symbols $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ denote the *closure* and the *interior* of A respectively.
A set A is said to be:

semi-open, if $A \subset \mathrm{cl}(\mathrm{int}(A))$, [9]
semi-closed, if $X \setminus A$ is semi-open, [2, 3].

The union of all semi-open sets contained in A is called the *semi-interior of* A and it is denoted as $\mathrm{sint}(A)$. The intersection of all semiclosed sets containing A is called the *semi-closure of* A and we denote it by $\mathrm{scl}(A)$, [2, 3].

Now, let $\mathcal{P}$ be an *ideal of subsets of* X and let

$$\mathcal{B}(\mathcal{P}) = \{U \setminus H\colon\ U \in \mathcal{T},\ H \in \mathcal{P}\}.$$

Then $\mathcal{B}(\mathcal{P})$ is a base of some topology $\mathcal{T}(\mathcal{P})$ in X and $\mathcal{T} \subset \mathcal{T}(\mathcal{P})$. For any set $A \subset X$ by $\mathrm{cl}_{\mathcal{P}}(A)$, $\mathrm{int}_{\mathcal{P}}(A)$, $\mathrm{scl}_{\mathcal{P}}(A)$ and $\mathrm{sint}_{\mathcal{P}}(A)$ are denoted the *closure, interior, semi-closure* and *semi-interior* of A in $(X,\mathcal{T}(\mathcal{P}))$. Let us put

$$D_{\mathcal{P}}(A) = \{x \in X : U \cap A \notin \mathcal{P} \text{ for each } \mathcal{T}\text{-neighbourhood } U \text{ of } x\}.$$

Then we have $A \cup D_{\mathcal{P}}(A) = \mathrm{cl}_{\mathcal{P}}(A)$ for each set $A \subset X$, [6].

Let us consider the following two properties:

$(*)$ $A \in \mathcal{P} \iff D_{\mathcal{P}}(A) = \emptyset \iff A \cap D_{\mathcal{P}}(A) = \emptyset$

$(**)$ If $\{A_j : j \in J\}$ is a family of sets belonging to $\mathcal{P}$ and each A_j is an open set in the subspace $\bigcup\{A_j : j \in J\}$, then $\bigcup\{A_j : j \in J\} \in \mathcal{P}$.

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1. PROPOSITION. *The conditions* $(*)$ *and* $(**)$ *are equivalent.*

Proof. Let $(*)$ be satisfied and let $\{A_j : j \in J\}$ be a family of ets belonging to $\mathcal{P}$ such that every A_j is open in $\bigcup\{A_j : j \in J\}$. For every point $x \in \bigcup\{A_j : j \in J\}$ there exist $j_x \in J$ and an open set U in $(X, \mathcal{T})$ such that $x \in U \cap \bigcup\{A_j : j \in J\} = A_{j_x} \in \mathcal{P}$. Thus

$$\Big(\bigcup\{A_j : j \in J\}\Big) \cap D_{\mathcal{P}}\Big(\bigcup\{A_j : j \in J\}\Big) = \emptyset,$$

which implies

$$\bigcup\{A_j : j \in J\} \in \mathcal{P}.$$

Conversely, let $(**)$ hold. Since the implications $A \in \mathcal{P} \implies D_{\mathcal{P}}(A) = \emptyset$ and $D_{\mathcal{P}}(A) = \emptyset \implies A \cap D_{\mathcal{P}}(A) = \emptyset$ are true, it suffices to prove $A \cap D_{\mathcal{P}}(A) = \emptyset \implies A \in \mathcal{P}$. If $A \cap D_{\mathcal{P}}(A) = \emptyset$ holds, then each point $x \in A$ has a $\mathcal{T}$-neighbourhood U_x such that $U_x \cap A \in \mathcal{P}$. So $\{U_x \cap A : x \in A\}$ is a family of sets belonging to $\mathcal{P}$ and every $U_x \cap A$ is open in the subspace $\bigcup\{U_x \cap A : x \in A\}$. From the assumption $A = \bigcup\{U_x \cap A : x \in A\} \in \mathcal{P}$ and the proof is completed.

If an ideal $\mathcal{P}$ satisfies $(*)$, then $\mathcal{B}(\mathcal{P}) = \mathcal{T}(\mathcal{P})$, [6]; in the other case this equality need not be satisfied. For instance, if $(\mathbb{R}, \mathcal{T})$ is the set of real numbers with the natural topology, $\mathbb{Q}$ is the set of all rational numbers and $\mathcal{P}$ is the ideal of all bounded subsets of $\mathbb{Q}$, then

$$\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^{\infty} (-n, n) \setminus [-n, n] \cap \mathbb{Q} \in \mathcal{T}(\mathcal{P}) \text{ and } \mathbb{R} \setminus \mathbb{Q} \notin \mathcal{B}(\mathcal{P}).$$

Let us denote

$$D^s_{\mathcal{P}}(A) = \{x \in X : U \cap A \notin \mathcal{P} \text{ for each } \mathcal{T}\text{-semi-open set } U \text{ containing } x\}.$$

If an ideal $\mathcal{P}$ has the property $(*)$, then $A \cup D^s_{\mathcal{P}}(A) = \mathrm{scl}_{\mathcal{P}}(A)$ for each set $A \subset X$, [4].

2. LEMMA. *If an ideal* $\mathcal{P}$ *in a space* $(X, \mathcal{T})$ *satisfies* $(*)$, *then for each set* $A \subset X$ *the sets* $\mathrm{cl}_{\mathcal{P}}(A) \setminus \mathrm{scl}_{\mathcal{P}}(A)$ *and* $\mathrm{sint}_{\mathcal{P}}(A) \setminus \mathrm{int}_{\mathcal{P}}(A)$ *are nowhere dense in* $(X, \mathcal{T})$.

Proof. Let $x \in \mathrm{int}(D_{\mathcal{P}}(A))$ and let W be a semi-open set containing x. Then the set $U = \mathrm{int}(D_{\mathcal{P}}(A)) \cap \mathrm{int}(W)$ is open non-empty, so $U \cap A \notin \mathcal{P}$, which implies $x \in D^s_{\mathcal{P}}(A)$. Thus we have $\mathrm{int}(D_{\mathcal{P}}(A)) \subset D^s_{\mathcal{P}}(A)$. Since $D_{\mathcal{P}}(A)$ is a closed set [6] and $D_{\mathcal{P}}(A) \setminus D^s_{\mathcal{P}}(A) \subset D_{\mathcal{P}}(A) \setminus \mathrm{int}(D_{\mathcal{P}}(A))$ it follows that $D_{\mathcal{P}}(A) \setminus D^s_{\mathcal{P}}(A)$ is nowhere dense.
Now we have $\mathrm{cl}_{\mathcal{P}}(A) \setminus \mathrm{scl}_{\mathcal{P}}(A) = (A \cup D_{\mathcal{P}}(A)) \setminus (A \cup D^s_{\mathcal{P}}(A)) \subset D_{\mathcal{P}}(A) \setminus D^s_{\mathcal{P}}(A)$;

so $\mathrm{cl}_{\mathcal{P}}(A) \setminus \mathrm{scl}_{\mathcal{P}}(A)$ is nowhere dense.
For each subset A of $(X, \mathcal{T})$ the following formula is true:
$\mathrm{sint}(A) = X \setminus \mathrm{scl}(X \setminus A)$; [2].
Thus we obtain $\mathrm{sint}_{\mathcal{P}}(A) \setminus \mathrm{int}_{\mathcal{P}}(A) = \big(X \setminus \mathrm{scl}_{\mathcal{P}}(X \setminus A)\big) \setminus \big(X \setminus \mathrm{cl}_{\mathcal{P}}(X \setminus A)\big) = \mathrm{cl}_{\mathcal{P}}(X \setminus A) \setminus \mathrm{scl}_{\mathcal{P}}(X \setminus A)$, so $\mathrm{sint}_{\mathcal{P}}(A) \setminus \mathrm{int}_{\mathcal{P}}(A)$ is a nowhere dense set in $(X, \mathcal{T})$.

Let $(X, \mathcal{T})$, (Y, τ) be *topological spaces* and $F\colon X \to Y$ a *multivalued map which assigns non-empty subsets of* Y. Following [1], for any set $V \subset Y$ we will denote $F^+(V) = \{x \in X : F(x) \subset V\}$ and $F^-(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$. A map $F\colon X \to Y$ is said to be *upper (lower) quasi-continuous at a point* $x_0 \in X$ if for each open set $V \subset Y$ with $F(x_0) \subset V$ (resp. $F(x_0) \cap V \neq \emptyset$) and each neighbourhood U of x_0 there exists an open non-empty set $U_1 \subset U$ such that $F(x) \subset V$ (resp. $F(x) \cap V \neq \emptyset$) for $x \in U_1$, [10].

A map F is upper (lower) quasi-continuous at x_0 if and only if for each open set $V \subset Y$ with $F(x_0) \subset V$ (resp. $F(x_0) \cap V \neq \emptyset$) we have $x_0 \in \mathrm{sint}\big(F^+(V)\big)$, (resp. $x_0 \in \mathrm{sint}\big(F^-(V)\big)$).

Any *single-valued map* $f\colon X \to Y$ can be considered as the multivalued map with values $\{f(x)\}$ for $x \in X$. In this case both upper and lower quasi-continuity mean quasi-continuity in the sense of K e m p i s t y [8].

For a multivalued map $F\colon (X, \mathcal{T}) \to (Y, \tau)$ we denote by $E^+(F, \mathcal{T}, \tau)$ and $E^-(F, \mathcal{T}, \tau)$ the *sets of all points at which* F *is upper or lower quasi-continuous.* Similarly $C^+(F, \mathcal{T}, \tau)$ and $C^-(F, \mathcal{T}, \tau)$ denote the *sets of points of upper or lower semicontinuity*, respectively. When there is no possibility of confusion, then the letter τ above will be omitted.
A multivalued map is called *upper (lower) quasi-continuous* if it is upper (lower) quasi-continuous at each point.

3. THEOREM. *Let $\mathcal{P}$ be an ideal of subsets of a topological space $(X, \mathcal{T})$ satisfying $(*)$. Assume that (Y, τ) is a second countable space and $F\colon X \to Y$ a multivalued map. Then:*

(a) *If F has compact values, then $E^+\big(F, \mathcal{T}(\mathcal{P}), \tau\big) \setminus C^+\big(F, \mathcal{T}(\mathcal{P}), \tau\big)$ is of the first category in $(X, \mathcal{T})$.*
(b) *The set $E^-\big(F, \mathcal{T}(\mathcal{P}), \tau\big) \setminus C^-\big(F, \mathcal{T}(\mathcal{P}), \tau\big)$ is of the first category in $(X, \mathcal{T})$.*

P r o o f. Let $\{V_n : n \geq 1\}$ be an open base of the topology τ in Y and let $\mathcal{A}$ be the set of all finite one-to-one sequences of natural numbers. Then $\mathcal{A} = \{\alpha_k : k \geq 1\}$, where $\alpha_k = \big(n_{k,1}, n_{k,2}, \dots, n_{k,j(k)}\big)$. Assume $W_k = \bigcup\{V_{n_{k,i}} : i = 1, 2, \dots, j(k)\}$. Since F has compact values we have

$$E^+\big(F, \mathcal{T}(\mathcal{P}), \tau\big) \setminus C^+\big(F, \mathcal{T}(\mathcal{P}), \tau\big) \subset \bigcup_{k=1}^{\infty} \mathrm{sint}_{\mathcal{P}}\big(F^+(W_k)\big) \setminus \mathrm{int}_{\mathcal{P}}\big(F^+(W_k)\big).$$

It follows from Lemma 2 that $E^+(F,\mathcal{T}(\mathcal{P}),\tau) \setminus C^+(F,\mathcal{T}(\mathcal{P}),\tau)$ is of the first category in $(X,\mathcal{T})$.

Similarly the second part follows from Lemma 2 and the inclusion

$$E^-(F,\mathcal{T}(\mathcal{P}),\tau) \setminus C^-(F,\mathcal{T}(\mathcal{P}),\tau) \subset \bigcup_{n=1}^{\infty} \operatorname{sint}_{\mathcal{P}}(F^-(V_n)) \setminus \operatorname{int}_{\mathcal{P}}((F^-(V_n)).$$

A set Y with two topologies is called a *bitopological space* [7]. In a bitopological space (Y,τ_1,τ_2) the topology τ_1 is said to be *regular with respect to* τ_2 if for each τ_1-open set U and each point $x \in U$ there exists a set $U_1 \in \tau_1$ such that $x \in U_1 \subset \mathrm{cl}_{(2)}(U_1) \subset U$, where $cl_{(i)}(A)$ denotes the *τ_i-closure of* A.
(Y,τ_1,τ_2) is said to be *pairwise regular* if τ_i is regular with respect to τ_j for $i,j \in \{1,2\}$, $i \neq j$.
A bitopological space (Y,τ_1,τ_2) is called *pairwise normal* if for each τ_1-closed set A and τ_2-closed set B with $A \cap B = \emptyset$ there exist disjoint sets $U \in \tau_2$, $V \in \tau_1$ such that $A \subset U$ and $B \subset V$, [7].

4. THEOREM. *Let $\mathcal{P}$ be an ideal of subsets of a topological space $(X,\mathcal{T})$ satisfying $\mathcal{T} \cap \mathcal{P} = \{\emptyset\}$ and let (Y,τ_1,τ_2) be a bitopological space in which τ_2 is regular with respect to τ_1. If $F\colon (X,\mathcal{T}(\mathcal{P})) \to (Y,\tau_1)$ is an upper quasi-continuous map, then*

$$E^-(F,\mathcal{T},\tau_2) = E^-(F,\mathcal{T}(\mathcal{P}),\tau_2),$$
$$C^-(F,\mathcal{T},\tau_2) = C^-(F,\mathcal{T}(\mathcal{P}),\tau_2).$$

Proof. The inclusion $E^-(F,\mathcal{T},\tau_2) \subset E^-(F,\mathcal{T}(\mathcal{P}),\tau_2)$, is evident. So let us assume that $x_0 \in E^-(F,\mathcal{T}(\mathcal{P}),\tau_2) \setminus E^-(F,\mathcal{T},\tau_2)$. Then there exists a τ_2-open set V_0 with $F(x_0) \cap V_0 \neq \emptyset$ and a $\mathcal{T}$-neighbourhood U of x_0 such that every non-empty $\mathcal{T}$-open set $U' \subset U$ contains a point x' for which $F(x') \cap V_0 = \emptyset$ holds. Let $y_0 \in F(x_0) \cap V_0$. By the regularity of τ_2 with respect to τ_1 we can choose a set $V \in \tau_2$ satisfying

$$y_0 \in V \subset \mathrm{cl}_{(1)}(V) \subset V_0.$$

Since $x_0 \in E^-(F,\mathcal{T}(\mathcal{P}),\tau_2)$, there exist $U_1 \in \mathcal{T}$, $H_1 \in \mathcal{P}$ such that $U_1 \setminus H_1 \subset U$ and

$$F(x) \cap V \neq \emptyset \qquad \text{for} \quad x \in U_1 \setminus H_1. \tag{1}$$

On the other hand there exists a point $x_1 \in U_1 \cap U$ such that $F(x_1) \subset Y \setminus \mathrm{cl}_{(1)}(V)$. The map $F\colon (X,\mathcal{T}(\mathcal{P})) \to (Y,\tau_1)$ is upper quasi-continuous at x_1, so for some $U_2 \in \mathcal{T}$, $H_2 \in \mathcal{P}$ we have $U_2 \setminus H_2 \subset U_1 \cap U$ and

$$F(x) \subset Y \setminus \mathrm{cl}_{(1)}(V) \qquad \text{for} \quad x \in U_2 \setminus H_2. \tag{2}$$

But $(U_1 \setminus H_1) \cap (U_2 \setminus H_2) \neq \emptyset$, hence it contradicts (1) and the proof of the first equality is completed.

The second part of the proof is analogous, so it is omitted.

Using similar arguments we can prove the following:

5. THEOREM. *Let $\mathcal{P}$ be an ideal of subsets of a topological space $(X,\mathcal{T})$ which satisfies $\mathcal{T} \cap \mathcal{P} = \{\emptyset\}$ and let (Y,τ_1,τ_2) be a bitopological space. Assume that $F\colon (X,\mathcal{T}(\mathcal{P})) \to (Y,\tau_1)$ is a lower quasi-continuous multivalued map. If one of the following conditions holds:*

(a) *τ_2 is regular with respect to τ_1 and the map F has τ_2-compact values,*
(b) *(Y,τ_1,τ_2) is pairwise normal and F has τ_1-closed values,*
then

$$E^+(F,\mathcal{T},\tau_2) = E^+(F,\mathcal{T}(\mathcal{P}),\tau_2),$$
$$C^+(F,\mathcal{T},\tau_2) = C^+(F,\mathcal{T}(\mathcal{P}),\tau_2).$$

6. COROLLARY. *Let $\mathcal{P}$ be an ideal of subsets of a topological space $(X,\mathcal{T})$ satisfying $\mathcal{T} \cap \mathcal{P} = \{\emptyset\}$ and let Y be a regular topological space. For a multivalued map $F\colon X \to Y$ with compact values the following properties are equivalent:*

(a) *$F\colon (X,\mathcal{T}) \to Y$ is upper and lower quasi-continuous;*
(b) *$F\colon (X,\mathcal{T}(\mathcal{P})) \to Y$ is upper and lower quasi-continuous.*

7. COROLLARY. *If $\mathcal{P}$ is an ideal of subsets of a topological space $(X,\mathcal{T})$ such that $\mathcal{T} \cap \mathcal{P} = \{\emptyset\}$ and (Y,τ_1,τ_2) is pairwise regular, then for any map $f\colon X \to Y$ the following are equivalent:*

(a) *$f\colon (X,\mathcal{T}) \to (Y,\tau_i)$ is quasi-continuous for $i \in \{1,2\}$;*
(b) *$f\colon (X,\mathcal{T}(\mathcal{P})) \to (Y,\tau_i)$ is quasi-continuous for $i \in \{1,2\}$.*

Denoting by $C(f,\mathcal{T})$ the set of all points at which $f\colon (X,\mathcal{T}) \to Y$ is continuous, from Theorem 4 we obtain

8. COROLLARY. *Let $(X,\mathcal{T})$ be a topological space, $\mathcal{P}$ an ideal of subsets of X such that $\mathcal{T}\cap\mathcal{P}=\{\emptyset\}$ and let Y be a regular space. If $f\colon X\to Y$ is a quasi-continuous map, then $C(f,\mathcal{T})=C\big(f,\mathcal{T}(\mathcal{P})\big)$.*

Let us remark that in Corollary 8 regularity of a space Y is not necessary.

9. Example. In the space (R,τ) of real numbers with the natural topology we denote by $\mathcal{P}_1$ the ideal of sets of the first category. The space $\big(R,\tau(\mathcal{P}_1)\big)$ is not regular. For instance, let $x_0=0$ and let $W=(-1,1)\setminus\left\{\frac{1}{n}\colon n\geq 1\right\}$. Evidently $x_0\in W\in\tau(\mathcal{P}_1)$. Every $\tau(\mathcal{P}_1)$-neighbourhood W_1 of x_0 is of the form $W_1=U\setminus H$, where $U\in\tau$, $H\in\mathcal{P}_1$ and $\mathrm{cl}_{\mathcal{P}_1}(W_1)=\mathrm{cl}(W_1)=\mathrm{cl}(U)$, [6], so $\mathrm{cl}_{\mathcal{P}_1}(W_1)\not\subset W$.

Now let $\mathcal{T}=\big\{(a,\infty)\colon a\in\mathbb{R}\big\}\cup\{\emptyset,\mathbb{R}\}$ and let $\mathcal{P}$ be an ideal satisfying $\mathcal{T}\cap\mathcal{P}=\{\emptyset\}$. A map $f\colon\big(\mathbb{R},\mathcal{T}(\mathcal{P})\big)\to\big(\mathbb{R},\tau(\mathcal{P}_1)\big)$ is quasi-continuous if and only if it is constant. Thus we have $C\big(f,\mathcal{T}(\mathcal{P})\big)=C(f,\mathcal{T})$ for each $\mathcal{T}(\mathcal{P})$-quasi-continuous map f.

Under some assumptions we can characterize regular spaces in terms of quasi-continuous maps.

10. THEOREM. *Let Y be a first countable T_1 Baire space. Then the following conditions are equivalent:*

(a) *Y is regular;*
(b) *for any topological space $(X,\mathcal{T})$, an ideal $\mathcal{P}$ satisfying $\mathcal{T}\cap\mathcal{P}=\{\emptyset\}$ and for each upper quasi-continuous multivalued map $F\colon\big(X,\mathcal{T}(\mathcal{P})\big)\to Y$ we have $C^-(F,\mathcal{T})=C^-\big(F,\mathcal{T}(\mathcal{P})\big)$;*
(c) *for any topological space $(X,\mathcal{T})$, an ideal $\mathcal{P}$ satisfying $\mathcal{T}\cap\mathcal{P}=\{\emptyset\}$ and for each lower quasi-continuous multivalued map $F\colon\big(X,\mathcal{T}(\mathcal{P})\big)\to Y$ with compact values we have $C^+(F,\mathcal{T})=C^+\big(F,\mathcal{T}(\mathcal{P})\big)$;*
(d) *for any topological space $(X,\mathcal{T})$, an ideal $\mathcal{P}$ satisfying $\mathcal{T}\cap\mathcal{P}=\{\emptyset\}$ and for each quasi-continuous map $f\colon\big(X,\mathcal{T}(\mathcal{P})\big)\to Y$ we have $C(f,\mathcal{T})=C(f,\mathcal{T}(\mathcal{P}))$.*

Proof. The implications $(a)\implies(b)$ and $(a)\implies(c)$ are consequences of Theorems 4 and 5; $(b)\implies(d)$ and $(c)\implies(d)$ are evident. Thus it suffices to prove $(d)\implies(a)$. Assume that Y is not regular. Then there exists an open set $W_0\subset Y$ and a point $y_0\in W_0$ such that

$$\mathrm{cl}(V)\not\subset W_0\qquad\text{for every neighbourhood } V \text{ of } y_0. \tag{1}$$

Let $\{W_n : n \geq 1\}$ be an open neighbourhoods base at y_0 such that $W_{n+1} \subset W_n \subset W_0$ for $n \geq 1$. Let us put

$$\mathcal{B}' = \Big\{A \subset Y : A \subset Y \setminus \Big(\{y_0\} \cup \bigcup_{n=0}^{\infty} \mathrm{Fr}(W_n)\Big)\Big\} \cup \big\{\mathrm{cl}(W_n) : n \geq 0\big\},$$

where $\mathrm{Fr}(W_n)$ denotes the boundary of the set W_n, and

$$\mathcal{P} = \Big\{A \subset Y : A \subset \bigcup_{n=0}^{\infty} \mathrm{Fr}(W_n)\Big\}.$$

Then $\mathcal{B}'$ is a base of some topology $\mathcal{T}$ on Y, $\mathcal{P}$ is an ideal and $\mathcal{T} \cap \mathcal{P} = \{\emptyset\}$. Let us consider the map $f : (Y, \mathcal{T}) \to Y$ given by $f(x) = x$ for $x \in Y$. Immediately we have

$$Y \setminus \Big(\{y_0\} \cup \bigcup_{n=0}^{\infty} \mathrm{Fr}(W_n)\Big) \subset C(f, \mathcal{T}). \tag{2}$$

Let V be a neighbourhood of $f(y_0)$. Then for some $m \geq 1$ we have $W_m \subset V$. Hence we obtain $y_0 \in U = \mathrm{cl}(W_m) \setminus \bigcup_{n=0}^{\infty} \mathrm{Fr}(W_n) \in \mathcal{T}(\mathcal{P})$ and $f(U) \subset V$. Thus, taking into account (1), we have shown

$$y_0 \in C\big(f, \mathcal{T}(\mathcal{P})\big) \setminus C(f, \mathcal{T}). \tag{3}$$

Now let $x \in \bigcup_{n=0}^{\infty} \mathrm{Fr}(W_n)$ and let V be a neighbourhood of $f(x)$. Then $\mathrm{cl}(W_m)$ is a $\mathcal{T}$-neighbourhood of x, $W_m \cap V \neq \emptyset$. Since Y is a T_1-space, the condition (1) implies that y_0 is not an isolated point. Thus $W_m \cap V \setminus \{y_0\}$ is an open non-empty set. From the assumption that Y is a Baire space it follows that $U = W_m \cap V \setminus \Big(\{y_0\} \cup \bigcup_{n=0}^{\infty} \mathrm{Fr}(W_n)\Big) \neq \emptyset$. Moreover we have $U \in \mathcal{T}(\mathcal{P})$, $U \subset \mathrm{cl}(W_m)$ and $f(U) \subset V$. It means that f is $\mathcal{T}(\mathcal{P})$-quasi-continuous at x and in consequence

$$\bigcup_{n=0}^{\infty} \mathrm{Fr}(W_n) \subset E\big(f, \mathcal{T}(\mathcal{P})\big), \tag{4}$$

where $E\big(f, \mathcal{T}(\mathcal{P})\big)$ is the set of all pints at which $f : \big(Y, \mathcal{T}(\mathcal{P})\big) \to Y$ is quasi-continuous. From (2), (3) and (4) we have that $f : \big(Y, \mathcal{T}(\mathcal{P})\big) \to Y$ is quasi-continuous but $C(f, \mathcal{T}) \neq C\big(f, \mathcal{T}(\mathcal{P})\big)$.

In a topological space (Y,τ) let $\tilde{\tau}$ be the *Vietoris topology* on the set $\mathcal{Z}(Y)$ of *all non-empty compact subsets* of Y. For *open sets* $W_1,\dots,W_m \subset Y$ we will denote

$$\mathcal{V}(W_1,\dots,W_m) = \Big\{B \in \mathcal{Z}(Y)\colon B \subset \bigcup_{i=1}^{m} W_i \text{ and } B \cap W_i \neq \emptyset \text{ for } i \leq m\Big\}.$$

If F is a multivalued map defined on a topological space $(X,\mathcal{T})$ with non-empty compact values in Y, then it can be considered also as the single valued map $F\colon (X,\mathcal{T}) \to (\mathcal{Z}(Y),\tilde{\tau})$. For the set $E(F,\mathcal{T},\tilde{\tau})$ of points at which this single valued map is quasi-continuous we have $E(F,\mathcal{T},\tilde{\tau}) \subset E^{+}(F,\mathcal{T},\tau) \cap E^{-}(F,\mathcal{T},\tau)$ and the inclusion cannot be replaced by the equality.

11. THEOREM. *Let $\mathcal{P}$ be an ideal of subsets of a topological space $(X,\mathcal{T})$ such that $\mathcal{T} \cap \mathcal{P} = \{\emptyset\}$ and let (Y,τ) be a regular space. If $F\colon X \to Y$ is a multivalued map with compact values which is upper and lower $\mathcal{T}(\mathcal{P})$-quasi-continuous, then $E(F,\mathcal{T},\tilde{\tau}) = E(F,\mathcal{T}(\mathcal{P}),\tilde{\tau})$.*

Proof. Assume that $x_0 \in E(F,\mathcal{T}(\mathcal{P}),\tilde{\tau}) \setminus E(F,\mathcal{T},\tilde{\tau})$. Then there exists a $\mathcal{T}$-neighbourhood U of x_0 and $\tilde{\tau}$-neighbourhood $\mathcal{V}(V_1,\dots,V_n)$ of $F(x_0)$ such that each non-empty $\mathcal{T}$-open set $U' \subset U$ contains a point x' for which $F(x') \notin \mathcal{V}(V_1,\dots,V_n)$. Since $F(x_0)$ is compact and $F(x_0) \subset \bigcup_{i=1}^{n} V_i$ we can choose open sets $W_1,\dots,W_n$ such that $\mathrm{cl}(W_i) \subset V_i$ for $i \leq n$ and $F(x_0) \in \mathcal{V}(W_1,\dots,W_n)$. Let us put $W = \bigcup_{i=1}^{n} W_i$. The condition $x_0 \in E\big(F,\mathcal{T}(\mathcal{P}),\tilde{\tau}\big)$ implies the existence of sets $U_1 \in \mathcal{T}$, $H_1 \in \mathcal{P}$ such that $\emptyset \neq U_1 \subset U$ and

$$F(x) \in \mathcal{V}(W_1,\dots,W_n) \qquad \text{for} \quad x \in U_1 \setminus H_1. \tag{1}$$

On the other hand for some point $x_1 \in U_1$ there holds $F(x_1) \notin \mathcal{V}\big(\mathrm{cl}(W_1),\dots,\mathrm{cl}(W_n)\big)$. Then

$$F(x_1) \cap \mathrm{cl}(W_i) = \emptyset \qquad \text{for some} \quad i \leq n, \tag{2}$$

or

$$F(x_1) \not\subset \mathrm{cl}(W). \tag{3}$$

If (2) holds, then using the upper $\mathcal{T}(\mathcal{P})$-quasi-continuity of F at x we can choose a non-empty set $U_2 \in \mathcal{T}$ and $H_2 \in \mathcal{P}$ such that

$$F(x) \subset Y \setminus \mathrm{cl}(W_i) \qquad \text{for} \quad x \in U_2 \setminus H_2. \tag{4}$$

But $U_1 \cap U_2 \neq \emptyset$, so $(U_2 \setminus H_2) \cap (U_1 \setminus H_1) \neq \emptyset$. Thus (4) is the contradiction to (1).

Now we assume that (3) is satisfied. Since F is lower $\mathcal{T}(\mathcal{P})$-quasi-continuous at x_1 there exist sets $U_3 \in \mathcal{T}$, $H_3 \in \mathcal{P}$ with $\emptyset \neq U_3 \subset U_1$ such that

$$F(x) \cap \big(Y \setminus \mathrm{cl}(W)\big) \neq \emptyset \qquad \text{for} \quad x \in U_3 \setminus H_3. \tag{5}$$

Because $(U_1 \setminus H_1) \cap (U_3 \setminus H_3) \neq \emptyset$ the condition (5) is in contradiction to (1), which finishes the proof.

The next results are consequences of Theorem 4 and 5.

A multivalued map $F\colon (X,\mathcal{T}) \to (Y,\tau)$ is called *upper (lower) c-quasi-continuous at* $x_0 \in X$ if for each open set $V \subset Y$ with $F(x_0) \subset V$ (resp. $F(x_0) \cap V \neq \emptyset$) and $Y \setminus V$ compact, and for each neighbourhood U of x_0 there exists a non-empty open set $U_1 \subset U$ such that $F(x) \subset V$ (resp. $F(x) \cap V \neq \emptyset$) for $x \in U_1$. A map F is called *upper (lower) c-quasi-continuous* if it is upper (lower) c-quasi-continuous at each point.

In a topological space (Y,τ) the family $\tau_c = \{V \in \tau : Y \setminus V \text{ is compact}\} \cup \{\emptyset\}$ is a topology and the upper (lower) c-quasi-continuity of a map $F\colon (X,\mathcal{T}) \to (Y,\tau)$ coincides with the upper (lower) quasi-continuity of $F\colon (X,\mathcal{T}) \to (Y,\tau_c)$. Moreover we have

12. Lemma. *If (Y,τ) is a locally compact T_2-space, then the bitopological space (Y,τ_c,τ) is pairwise regular.*

Proof. One can readily see that for any relatively compact set V there holds $\mathrm{cl}(V) = \mathrm{cl}_c(V)$, where cl_c denotes the *τ_c-closure.*

Let U be a τ-open set and $x \in U$. Then there exists a set $V \in \tau$ with $\mathrm{cl}(V)$ compact such that $x \in V \subset \mathrm{cl}(V) \subset U$. Since $\mathrm{cl}(V) = \mathrm{cl}_c(V)$, we have $\mathrm{cl}_c(V) \subset U$, so τ is regular with respect to τ_c.

Conversely, let us take $U \in \tau_c$ and a point $x \in U$. We can choose a τ-open set V such that $Y \setminus U \subset V$ and $x \notin \mathrm{cl}(V)$. Furthermore we can choose a τ-open relatively compact set W satisfying $Y \setminus U \subset W \subset \mathrm{cl}(W) \subset V$ and $x \in Y \setminus \mathrm{cl}(W) \subset U$. The set $Y \setminus \mathrm{cl}(W)$ is τ_c-open and $\mathrm{cl}\big(Y \setminus \mathrm{cl}(W)\big) \subset U$, thus τ_c is regular with respect to τ.

13. Theorem. *Let $\mathcal{P}$ be an ideal of subsets of a topological space $(X,\mathcal{T})$ with $\mathcal{P} \cap \mathcal{T} = \{\emptyset\}$ and let (Y,τ) be a locally compact T_2-space. If $F\colon X \to Y$ is an upper (lower) c-quasi-continuous multivalued map with compact values, then $E^-(F,\mathcal{T},\tau) = E^-\big(F,\mathcal{T}(\mathcal{P}),\tau\big)$ and $C^-(F,\mathcal{T},\tau) = C^-\big(F,\mathcal{T}(\mathcal{P}),\tau\big)$, (resp. $E^+(F,\mathcal{T},\tau) = E^+\big(F,\mathcal{T}(\mathcal{P}),\tau\big)$ and $C^+(F,\mathcal{T},\tau) = C^+\big(F,\mathcal{T}(\mathcal{P}),\tau\big)$).*

Proof. It is direct consequence of Lemma 12 and Theorems 4 and 5.

Finally we will consider *real functions*. A function $f\colon X \to \mathbb{R}$ is said to be *upper (lower) quasi-continuous at* $x_0 \in X$ if for each $\varepsilon > 0$ and each neighbourhood U of x_0 there exists a non-empty open set $U_1 \subset U$ such that $f(x) < f(x_0) + \varepsilon$ (resp. $f(x_0) - \varepsilon < f(x)$) for $x \in U_1$, [5]. By $E_u(f,\mathcal{T})$ and $E_l(f,\mathcal{T})$ will be denoted the sets of all points at which f is upper or lower quasi-continuous respectively; moreover $E(f,\mathcal{T})$ is the set of quasi-continuity points. Similarly $C_u(f,\mathcal{T})$ and $C_l(f,\mathcal{T})$ denote the *sets of points at which f is upper or lower semicontinuous.*

A function f is called *upper (lower) quasi-continuous* if $E_u(f,\mathcal{T}) = X$ ($E_l(f,\mathcal{T}) = X$).

14. THEOREM. *Let $\mathcal{P}$ be an ideal of subsets of a topological space $(X,\mathcal{T})$ such that $(*)$ is satisfied. For any function $f\colon X \to \mathbb{R}$ the sets $E_u(f,\mathcal{T}(\mathcal{P})) \setminus C_u(f,\mathcal{T}(\mathcal{P}))$, $E_l(f,\mathcal{T}(\mathcal{P})) \setminus C_l(f,\mathcal{T}(\mathcal{P}))$ and $E(f,\mathcal{T}(\mathcal{P})) \setminus C(f,\mathcal{T}(\mathcal{P}))$ are of the first category in $(X,\mathcal{T})$.*

Proof. Let us put $\tau_1 = \{(-\infty,a)\colon a \in \mathbb{R}\} \cup \{\emptyset,\mathbb{R}\}$, $\tau_2 = \{(a,\infty)\colon a \in \mathbb{R}\} \cup \{\emptyset,\mathbb{R}\}$ and let τ denotes the natural topology on $\mathbb{R}$. Then the conclusion is the consequence of Theorem 3 applied to the function $f\colon (X,\mathcal{T}) \to (\mathbb{R},\tau_1)$ or $f\colon (X,\mathcal{T}) \to (\mathbb{R},\tau_2)$ or $f\colon (X,\mathcal{T}) \to (\mathbb{R},\tau)$, respectively.

15. THEOREM. *Let $\mathcal{P}$ be an ideal of subsets of a topological space $(X,\mathcal{T})$ with $\mathcal{P} \cap \mathcal{T} = \{\emptyset\}$ and $f\colon X \to \mathbb{R}$ any function.*

(a) *If f is lower $\mathcal{T}(\mathcal{P})$-quasi-continuous, then $E_u(f,\mathcal{T}) = E_u(f,\mathcal{T}(\mathcal{P}))$ and $C_u(f,\mathcal{T}) = C_u(f,\mathcal{T}(\mathcal{P}))$.*

(b) *If f is upper $\mathcal{T}(\mathcal{P})$-quasi-continuous, then $E_l(f,\mathcal{T}) = E_l(f,\mathcal{T}(\mathcal{P}))$ and $C_l(f,\mathcal{T}) = C_l(f,\mathcal{T}(\mathcal{P}))$.*

Proof. Assume $\tau_1 = \{(-\infty,a)\colon a \in \mathbb{R}\} \cup \{\emptyset,\mathbb{R}\}$ and $\tau_2 = \{(a,\infty)\colon a \in \mathbb{R}\} \cup \{\emptyset,\mathbb{R}\}$. The upper (lower) quasi-continuity means the quasi-continuity with respect to τ_1 (or τ_2 resp.). Since the bitopological space $(\mathbb{R},\tau_1,\tau_2)$ is pairwise regular it suffices to use Theorem 4 to the single valued map f.

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