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ON STRONG CONVERGENCE OF MULTIVALUED MAPS

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ABSTRACT. The main results presented in this paper concern multivalued maps. We consider a convergence of nets of multivalued maps which is formulated in terms of open covers and we compare it with pointwise, uniform and continuous convergence. Furthermore we will show that the strong convergence keeps sets of points of lower and upper semicontinuity. The sufficient conditions are given under which the strong convergence implies the Kuratowski convergence of graphs.

In a topological space (Y,τ) we denote by $S(Y)$ the family of all nonempty subsets of Y. For an open set $U \subset Y$ we write

$$U^+ = \{B \in S(Y) :\ B \subset U\}, \qquad U^- = \{B \in S(Y) :\ B \cap U \neq \emptyset\}.$$

The families $\mathcal{B}^+ = \{U^+ :\ U \in \tau\}$ and $\mathcal{P}^- = \{U^- :\ U \in \tau\}$ form a base and a subbase of the upper and lower Vietoris topology, respectively. These topologies will be denoted by τ^+ and τ^-. We will write $A \in \tau^-\text{-}\lim A_j$ and $A \in \tau^+\text{-}\lim A_j$ if the net $\{A_j :\ j \in J\}$ converges to A in the space $(S(Y),\tau^-)$ or $(S(Y),\tau^+)$, respectively.

DEFINITION 1. The set $\mathrm{St}(A,\mathcal{A}) = \bigcup\{B \in \mathcal{A} :\ A \cap B \neq \emptyset\}$ is called a *star of a set A with respect to a cover $\mathcal{A}$* of a space Y.

The concept of strong convergence of functions and multifunctions was introduced by Kupka and Toma in [9]. They formulated the notions of strong convergence. In this paper we consider nets of multivalued maps lower and upper strongly convergent. The results presented here are in general stronger than the other ones.

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Let X be a nonempty set and let (Y,τ) be a topological space.

DEFINITION 2. A net $\{F_j : j \in J\}$ of multivalued maps $F_j\colon X \to Y$ is said to be *upper* (respectively: *lower*) *strongly convergent to a multivalued map* $F\colon X \to Y$ if for each open cover $\mathcal{A}$ of Y there exists $j_0 \in J$ such that

$$F_j(x) \subset \operatorname{St}\bigl(F(x),\mathcal{A}\bigr) \qquad (\text{respectively: } F(x) \subset \operatorname{St}\bigl(F_j(x),\mathcal{A}\bigr)\,)$$

for every $j \in J$, $j \geq j_0$, and $x \in X$.

DEFINITION 3. ([6]) Let (Y,τ) be a topological space. A subset A of Y is called *α-paracompact* if for every τ-open cover $\mathcal{A}$ of A there is a τ-open locally finite cover $\mathcal{B}$ such that $\mathcal{B}$ is a refinement of $\mathcal{A}$.

Evidently each compact set is α-paracompact.

LEMMA 1. ([6]) *Every α-paracompact subset of a Hausdorff space is closed.*

LEMMA 2. *Let (Y,τ) be a regular space. If A is α-paracompact subset of Y, U is open in Y and $A \subset U$, then there exists an open set V such that $A \subset V \subset \overline{V} \subset U$.*

Proof. Let A be an α-paracompact set and let U be an open set with $A \subset U$. Because Y is regular space, for every point x of A there exists an open set V_x satisfying $x \in V_x \subset \overline{V_x} \subset U$. Then the family $\mathcal{A} = \{V_x : x \in A\}$ forms an open cover of A. Thus we can choose a locally finite open cover $\mathcal{B} = \{W_s : s \in S\}$ of A which is a refinement of $\mathcal{A}$. For every $W_s \in \mathcal{B}$ there is $x_s \in A$ such that $W_s \subset V_{x_s}$. Hence we have $\overline{W_s} \subset \overline{V_{x_s}} \subset U$ and $\overline{\bigcup_{s\in S} W_s} = \bigcup_{s\in S} \overline{W_s} \subset U$. Let us denote $W = \bigcup_{s\in S} W_s$. The set W is open and $A \subset W \subset \overline{W} \subset U$, which completes the proof. □

THEOREM 1. *If Y is a regular space and a multivalued map $F\colon X \to Y$ has α-paracompact values, then the upper strong convergence of a net $\{F_j : j \in J\}$ to F implies the τ^+-pointwise convergence.*

Proof. Let us take $x_0 \in X$ and $U^+ \in \mathcal{B}^+$ such that $F(x_0) \in U^+$. This means that U is an open set and $F(x_0) \subset U$. Because the values of the map F are α-paracompact and Y is regular, there exists $W \in \tau$ with properties $F(x_0) \subset W \subset \overline{W} \subset U$. The family $\mathcal{A} = \{U, Y\setminus\overline{W}\}$ forms an open cover of Y. Now from the upper strong convergence of the net $\{F_j : j \in J\}$ to F there exists $j_0 \in J$ such that $F_j(x) \subset \operatorname{St}\bigl(F(x),\mathcal{A}\bigr)$ for each $j \geq j_0$ and $x \in X$.

In particular, $F_j(x_0) \subset \operatorname{St}\bigl(F(x_0),\mathcal{A}\bigr) = U$ and further $F_j(x_0) \in U^+$ for all $j \geq j_0$. This means $F \in \tau^+\text{-}\lim F_j$ and this finishes the proof. □

The next theorem could be proven in a similar way than Theorem 1 was. We omit the proof.

THEOREM 2. *Let Y be a regular space. Then the lower strong convergence of a net $\{F_j : j \in J\}$ of multivalued maps $F_j : X \to Y$ to a map F implies $F \in \tau^- \text{-}\lim F_j$.*

COROLLARY 1. ([9; Corollary 2]) *Let X be a topological space and Y be a T_1 or regular topological space. If a net of functions $\{f_j : j \in J\}$ converges strongly to a function $f : X \to Y$, then it converges pointwise to f.*

The symbols $C^+(F)$ and $C^-(F)$ are used to denote the sets of all points at which a map F is upper or lower semicontinuous, respectively.

In the next theorems we formulate the sufficient conditions under which the lower (upper) strong convergence preserves the upper (lower) semicontinuity. We use the general scheme of the proofs as the scheme in [8].

THEOREM 3. *Let X be a topological space, Y a regular space and let $F : X \to Y$ be a multivalued map with α-paracompact values. If a net $\{F_j : j \in J\}$ of multivalued maps $F_j : X \to Y$ is τ^+-pointwise and lower strongly convergent to F, then*

$$\bigcap_{i \in J} \bigcup_{j \geq i} C^+(F_j) \subset C^+(F).$$

Proof. Let

$$x_0 \in \bigcap_{i \in J} \bigcup_{j \geq i} C^+(F_j). \tag{1}$$

Assume that V is an open subset of Y with $F(x_0) \subset V$. Then an open set W can be chosen such that $F(x_0) \subset W \subset \overline{W} \subset V$.

Let us consider the open cover $\mathcal{A} = \{V, Y \backslash \overline{W}\}$ of the space Y. By the assumption $F \in \tau^+ \text{-}\lim F_j$, there exists $j_0 \in J$ such that $F_j(x_0) \subset W$ for $j \geq j_0$.

Moreover the lower strong convergence of the net $\{F_j : j \in J\}$ to F implies the existence of $j_1 \in J$ with the property $F(x) \subset \mathrm{St}(F_j(x), \mathcal{A})$ for every $j \geq j_1$ and $x \in X$. Let us fix $j_2 \geq j_i$ for $i \in \{0, 1\}$. We can choose $j \geq j_2$ such that $x_0 \in C^+(F_j)$. Then there exists a neighbourhood G of the point x_0 with $F_j(x) \subset W$ for every $x \in G$. Thus $F(x) \subset \mathrm{St}(F_j(x), \mathcal{A}) = V$ for each $x \in G$. We have proved that F is upper semicontinuous at the point x_0.

In consequence, we have shown the inclusion

$$\bigcap_{i \in J} \bigcup_{j \geq i} C^+(F_j) \subset C^+(F).$$

□

THEOREM 4. *Let X be a topological space, (Y,τ) a regular space. If a net $\{F_j : j \in J\}$ of multivalued maps $F_j\colon X \to Y$ is τ^--pointwise and upper strongly convergent to a multivalued map $F\colon X \to Y$, then*

$$\bigcap_{i\in J}\bigcup_{j\geq i} C^-(F_j) \subset C^-(F).$$

P r o o f. Assume that

$$x_0 \in \bigcap_{i\in J}\bigcup_{j\geq i} C^-(F_j)$$

and U is an open subset of Y with $U \cap F(x_0) \neq \emptyset$. Let $y_0 \in F(x_0) \cap U$. The space Y is regular, so we can choose an open set V such that $y_0 \in V \subset \overline{V} \subset U$.

Thus $F(x_0) \cap V \neq \emptyset$. Because $F \in \tau^-\text{-}\lim F_j$, there exists $j_0 \in J$ with $F_j(x_0) \cap V \neq \emptyset$ for each $j \geq j_0$.

Furthermore the net $\{F_j : j \in J\}$ is upper strongly convergent to F. We will consider the open cover $\mathcal{A} = \{U,\ Y\backslash\overline{V}\}$ of Y. Then we choose $j_1 \in J$ such that

$$F_j(x) \subset \operatorname{St}\big(F(x),\mathcal{A}\big) \qquad \text{for every} \quad j \geq j_1\,,\quad x \in X\,. \tag{2}$$

We can fix $j \geq j_i$ for $i \in \{0,1\}$ with the property $x_0 \in C^-(F_j)$. Hence there exists a neighbourhood G of the point x_0 with

$$F_j(x) \cap V \neq \emptyset \qquad \text{for every} \quad x \in G\,. \tag{3}$$

We are going to prove that $F(x) \cap U \neq \emptyset$ for each $x \in G$. Suppose that there is $x_1 \in G$ such that $F(x_1) \cap U = \emptyset$. It means $F(x_1) \subset Y \setminus U \subset Y \setminus \overline{V}$. Thus $\operatorname{St}\big(F(x_1),\mathcal{A}\big) = Y \setminus \overline{V}$. By virtue of (2) we obtain

$$F_j(x_1) \subset \operatorname{St}\big(F(x_1),\mathcal{A}\big) = Y \setminus \overline{V} \subset Y \setminus V \qquad \text{and} \qquad F_j(x_1) \cap V = \emptyset\,.$$

This contradiction means $x_0 \in C^-(F)$ and the inclusion

$$\bigcap_{i\in J}\bigcup_{j\geq i} C^-(F_j) \subset C^-(F)$$

is proved. □

In the sequel we will consider connections between the strong and the uniform convergence.

In a uniform space $(Y,\mathfrak{U})$ we will write

$$U[y] = \big\{z \in Y :\ (y,z) \in U\big\} \qquad \text{for all} \quad U \in \mathfrak{U},\quad y \in Y\,.$$

At first we will give necessary definitions.

Let X be a nonempty set and $(Y,\mathfrak{U})$ be a uniform space.

DEFINITION 4. A net $\{F_j : j \in J\}$ of multivalued maps $F_j \colon X \to Y$ is called *upper* (respectively: *lower*) *uniformly convergent* to a map $F \colon X \to Y$ if for every $U \in \mathfrak{U}$ there exists $j_0 \in J$ such that $F_j(x) \subset U[F(x)]$ (respectively: $F(x) \subset U[F_j(x)]$) for every $j \geq j_0$ and $x \in X$.

The upper (lower) uniform convergence with respect to a uniformity $\mathfrak{U}$ will be also denoted as $\mathfrak{U}^+$-uniform ($\mathfrak{U}^-$-uniform) convergence.

Under certain assumptions the strong convergence implies the uniform convergence, as it is shown in the following theorem:

THEOREM 5. *Let Y be a Tychonoff space and let $F_j, F \colon X \to Y$ be multivalued maps. If a net $\{F_j : j \in J\}$ is upper (respectively: lower) strongly convergent to F, then this net converges to F upper (lower) uniformly for every compatible uniformity $\mathfrak{U}$ on Y.*

P r o o f. Let $\mathfrak{U}$ be a compatible uniformity on Y and $U \in \mathfrak{U}$. We choose a set $W \in \mathfrak{U}$ satisfying the following properties: W is open, $W^{-1} = W$ and $W^2 \subset U$.

The family $\mathcal{A} = \{W[y] : y \in Y\}$ forms an open cover of Y. The net $\{F_j : j \in J\}$ is upper strongly convergent to the map F, so there exists $j_0 \in J$ such that

$$F_j(x) \subset \mathrm{St}\big(F(x), \mathcal{A}\big) \qquad \text{for each} \quad j \geq j_0 \,, \quad x \in X \,. \tag{4}$$

Let $j \geq j_0$ and $x \in X$ be fixed. We will prove the inclusion $F_j(x) \subset U[F(x)]$. From the condition (4), for a point $z \in F_j(x)$ there is $y \in Y$ such that $z \in W[y]$ and $F(x) \cap W[y] \neq \emptyset$.

Let $p \in F(x) \cap W[y]$. Because $(y,z) \in W$ and $(y,p) \in W$, $(p,z) \in W^2 \subset U$.

In the consequence

$$z \in U[p] \subset \bigcup_{p \in F(x)} U[p] = U[F(x)] \,.$$

Thus we have shown $F_j(x) \subset U[F(x)]$, which means the $\mathfrak{U}^+$-uniform convergence of the net $\{F_j : j \in J\}$ to F. For the lower strong convergence the proof is analogous. □

Remark. The above theorem is a generalization of [9; Proposition 4].

The next theorem is in some sense converse to the above, but at first we give a necessary definition.

DEFINITION 5. ([5]) A uniform space $(Y, \mathfrak{U})$ is said to *have the Lebesgue property* if for each open cover $\mathcal{A}$ of Y there is $U \in \mathfrak{U}$ such that $\{U[y] : y \in Y\}$ is a refinement of $\mathcal{A}$.

THEOREM 6. *Let* $(Y, \mathfrak{U})$ *be a uniform space with the Lebesgue property and let* $F_j, F\colon X \to Y$ *be multivalued maps. Then* $\mathfrak{U}^+$ *- (respectively* $\mathfrak{U}^-$ *-) uniform convergence of the net* $\{F_j : j \in J\}$ *to* F *implies the upper (lower) strong convergence to* F.

P r o o f . We will prove this theorem in the case of the upper uniform convergence of the net $\{F_j : j \in J\}$ to a multivalued map F. Let $\mathcal{A}$ be an open cover of Y. By the assumption Y has the Lebesgue property, there exists $U \in \mathfrak{U}$ such that $\mathcal{A}' = \{U[y] : y \in Y\}$ is a refinement of $\mathcal{A}$. For the set U we choose $W \in \mathfrak{U}$ with properties $W^{-1} = W$ and $W^2 \subset U$. Since the net $\{F_j : j \in J\}$ is $\mathfrak{U}^+$-uniformly convergent to F, there exists $j_0 \in J$ such that

$$F_j(x) \subset W[F(x)] \qquad \text{for each} \quad j \geq j_0, \quad x \in X. \tag{5}$$

We fix $j \geq j_0$ and $x \in X$. We will show the inclusion $F_j(x) \subset \mathrm{St}(F(x), \mathcal{A})$. Let us take $z \in F_j(x)$. From the condition (5) there exists $p \in F(x)$ with $z \in W[p]$. The cover $\mathcal{A}'' = \{W[y] : y \in Y\}$ is refinement of $\mathcal{A}'$, so we can find $y \in Y$ satisfying properties $p \in W[y] \subset U[y]$. Since $(y, p) \in W$ and $(p, z) \in W$, so $(y, z) \in W^2 \subset U$. In the consequence $z \in U[y] \subset G$ for certain $G \in \mathcal{A}$. Hence $F(x) \cap G \neq \emptyset$ which implies $z \in \mathrm{St}(F(x), \mathcal{A})$. Thus we have shown that $F_j(x) \subset \mathrm{St}(F(x), \mathcal{A})$, i.e. the net $\{F_j : j \in J\}$ upper strongly converges to F. □

In the proof of the next theorem we will use the following theorem:

THEOREM 7. ([5]) *Let* X *be a uniformisable Hausdorff topological space, then* X *possesses a uniform structure compatible with the topology of* X *for which* X *has the Lebesgue property if and only if* X *is paracompact. Under this condition, this uniform structure is the finest of all uniform structures compatible with the topology of* X, *and is unique.*

As a consequence of Theorems 5, 6 and 7 we have:

THEOREM 8. *Let* (Y, τ) *be a paracompact space and let* $\mathfrak{U}$ *be the finest compatible uniformity on* Y. *Assume that* $F_j, F\colon X \to Y$ *are multivalued maps. Then the upper (respectively: lower) strong convergence of the net* $\{F_j : j \in J\}$ *to the map* F *is equivalent to the* $\mathfrak{U}^+$ *- (*$\mathfrak{U}^-$ *-) uniform convergence.*

If we will assume that $(Y, \mathfrak{U})$ is compact uniform space, then we will obtain:

THEOREM 9. ([8; Theorem 3]) *Let* X *be a set, let* $(Y, \mathfrak{U})$ *be a compact uniform space. Let* $\{F_j : j \in J\}$ *be a net of multifunctions which converges uniformly to a multifunction* $F\colon X \to Y$. *Then this net is strongly convergent to* F.

For a net $\{A_j : j \in J\}$ of subsets of the space Y, $\mathrm{Li}\, A_j$ and $\mathrm{Ls}\, A_j$ are subsets of Y consisting of all points $y \in Y$ each neighbourhood of which meets

$\{A_j:\ j \in J\}$, eventually or frequently, respectively. If $\mathrm{Li}\, A_j = \mathrm{Ls}\, A_j$, then this set is denoted by $\mathrm{Lt}\, A_j$ ([10]).

It is known that:

LEMMA 3. ([4]) *Let (Y,τ) be a topological space. $A \in \tau^{-}$-$\lim A_j$ if and only if $A \subset \mathrm{Li}\, A_j$.*

The symbol $\mathrm{Gr}\, F$ is used to denote the graph of the multivalued map F, i.e. $\mathrm{Gr}\, F = \{(x,y):\ x \in X \text{ and } y \in F(x)\}$.

LEMMA 4. ([3]) *If $F \in \tau^{-}$-$\lim F_j$, then $\mathrm{Gr}\, F \subset \mathrm{Li}\, \mathrm{Gr}\, F_j$.*

LEMMA 5. *If (Y,τ) is a regular space and A is a subset of Y, then*

$$\overline{A} = \bigcap_{\mathcal{A}} \mathrm{St}(A,\mathcal{A}),$$

where the intersection runs through all open covers of Y.

Proof. At first assume that $x_0 \in \overline{A}$ and $\mathcal{A}$ is an open cover of Y. Then there exists $V \in \mathcal{A}$ such that $x_0 \in V$; hence $V \cap A \neq \emptyset$. Thus $x_0 \in \mathrm{St}(A,\mathcal{A})$ and in consequence

$$x_0 \in \bigcap_{\mathcal{A}} \mathrm{St}(A,\mathcal{A}).$$

We have proved the inclusion

$$\overline{A} \subset \bigcap_{\mathcal{A}} \mathrm{St}(A,\mathcal{A}). \tag{6}$$

Now suppose that $x_0 \notin \overline{A}$. Then there is a neighbourhood U of the point x_0 with the property $U \cap A = \emptyset$. From the regularity of the space Y there exists an open subset W of Y satisfying conditions:

$$x_0 \in W \subset \overline{W} \subset U.$$

A family $\mathcal{B} = \{Y\backslash\overline{W}, U\}$ forms an open cover of Y. Let us observe that $\mathrm{St}(A,\mathcal{B}) = Y \setminus \overline{W}$ and $x_0 \notin \mathrm{St}(A,\mathcal{B})$, so

$$x_0 \notin \bigcap_{\mathcal{A}} \mathrm{St}(A,\mathcal{A}).$$

Thus we obtain

$$\bigcap_{\mathcal{A}} \mathrm{St}(A,\mathcal{A}) \subset \overline{A} \tag{7}$$

and the proof is completed. □

THEOREM 10. *Let X be a topological space, Y a T_3 space. Assume that $F\colon X \to Y$ is an upper semicontinuous multivalued map with α-paracompact values. If a net $\{F_j : j \in J\}$ of multivalued maps $F_j\colon X \to Y$ is upper strongly convergent to F, then the inclusion $\operatorname{Ls}\operatorname{Gr} F_j \subset \operatorname{Gr} F$ holds.*

Proof. Let $(x_0, y_0) \in \operatorname{Ls}\operatorname{Gr} F_j$. For every neighbourhoods U, V of x_0, y_0, respectively, the set

$$J(U,V) = \{j \in J : (U \times V) \cap \operatorname{Gr} F_j \neq \emptyset\} \qquad \text{is cofinal in } J.$$

By Σ we denote the set of all triplets $\sigma = (U, V, j)$, where $U \times V$ is a neighbourhood of the point (x_0, y_0) and $j \in J(U,V)$. For $\sigma_k = (U_k, V_k, j_k) \in \Sigma$, $k = 1, 2$, we will write $\sigma_1 \le \sigma_2$ if and only if $U_2 \subset U_1$, $V_2 \subset V_1$ and $j_1 \le j_2$. Thus $(\Sigma, \le)$ is a directed set. For each $\sigma = (U, V, j) \in \Sigma$ we can select a point $(x_\sigma, y_\sigma) \in U \times V$ with $y_\sigma \in F_j(x_\sigma)$. Let $\mathcal{A}$ be an open cover of Y. Because Y is regular and $F(x_0)$ is α-paracompact set, there exist open sets W and V with properties

$$F(x_0) \subset W \subset \overline{W} \subset V \subset \overline{V} \subset \operatorname{St}\big(F(x_0), \mathcal{A}\big).$$

The map F is upper semicontinuous, therefore there is a neighbourhood U_0 of the point x_0 such that $F(U_0) \subset W$. By assumptions the net $\{F_j : j \in J\}$ is upper strongly convergent to F, for the open cover $\mathcal{A}_1 = \{V, Y \setminus \overline{W}\}$ we can choose $j_0 \in J$ such that $F_j(x) \subset \operatorname{St}\big(F(x), \mathcal{A}_1\big)$ for each $j \ge j_0$ and $x \in X$. Since $x_\sigma \to x_0$, some $\sigma_1 = (U_1, V_1, j_1) \in \Sigma$ can be chosen such that $j_1 \ge j_0$ and $x_\sigma \in U_0$ for $\sigma \ge \sigma_1$. Thus $y_\sigma \in F_j(x_\sigma) \subset \operatorname{St}\big(F(x_\sigma), \mathcal{A}_1\big)$ for $\sigma \ge \sigma_1$ and $F(x_\sigma) \subset W \subset \overline{W}$. This implies the equality $\operatorname{St}\big(F(x_\sigma), \mathcal{A}_1\big) = V$ for $\sigma \ge \sigma_1$. On the other hand $y_\sigma \to y_0$. It implies $y_0 \in \overline{V} \subset \operatorname{St}\big(F(x_0), \mathcal{A}\big)$. By virtue of Lemma 5 and since $F(x_0)$ is closed we obtain

$$y_0 \in \bigcap_{\mathcal{A}} \operatorname{St}\big(F(x_0), \mathcal{A}\big) = \overline{F(x_0)} = F(x_0).$$

Finally $(x_0, y_0) \in \operatorname{Gr} F$, which completes the proof. □

From Lemma 4, Theorem 10 and the inclusion $\operatorname{Li}\operatorname{Gr} F_j \subset \operatorname{Ls}\operatorname{Gr} F_j$ we have:

THEOREM 11. *Let X be a topological space and Y be a T_3 space. Assume that $F\colon X \to Y$ is an upper semicontinuous multivalued map with α-paracompact values. If a net $\{F_j : j \in J\}$ of multivalued maps $F_j\colon X \to Y$ is upper strongly convergent to F and $F \in \tau^-\text{-}\lim F_j$, then $\operatorname{Gr} F = \operatorname{Lt}\operatorname{Gr} F_j$.*

The last theorem implies the suitable result for continuous functions.

COROLLARY 2. *Assume that X is a topological space, Y — a T_3 space and $f\colon X \to Y$ is a continuous function. If a net $\{f_j : j \in J\}$ of functions $f_j\colon X \to Y$ is strongly convergent to f, then* $\operatorname{Gr} f = \operatorname{Lt} \operatorname{Gr} f_j$.

Convergence of graphs was investigated in several papers, see for example [2] and [7]. Now we will consider connections between the strong and continuous convergences.

DEFINITION 6. ([1]) A net $\{f_j : j \in J\}$ of functions $f_j\colon X \to Y$ is called *continuously convergent at* $x_0 \in X$ to a function $f\colon X \to Y$ if for each net $\{x_\sigma : \sigma \in \Sigma\}$ in X

$$x_0 \in \lim_{\Sigma} x_\sigma \implies f(x_0) \in \lim_{J\times\Sigma} f_j(x_\sigma).$$

If it holds at each point $x_0 \in X$, then the net is called *continuously convergent.*

Let $\Im$ denote a topology on a family $S(Y)$ of all subsets of Y. A net $\{F_j : j \in J\}$ of multivalued maps $F_j\colon X \to Y$ is said to be $\Im$*-continuously convergent to* $F\colon X \to Y$ if the net of functions $F_j\colon X \to \big(S(Y), \Im\big)$ is continuously convergent to the function $F\colon X \to \big(S(Y), \Im\big)$.

THEOREM 12. *Let X be a topological space and Y a regular space. If a net $\{F_j : j \in J\}$ of multivalued maps $F_j\colon X \to Y$ is upper strongly convergent to F and τ^--continuously convergent to a map $F\colon X \to Y$ at the point x_0, then $x_0 \in C^-(F)$.*

P r o o f. Assume that $x_0 \notin C^-(F)$. Then there exist a point $y_0 \in F(x_0)$, a neighbourhood V of y_0 and a net $\{x_\sigma : \sigma \in \Sigma\}$ such that

$$x_0 \in \lim_{\Sigma} x_\sigma \quad \text{and} \quad F(x_\sigma) \cap V = \emptyset \qquad \text{for every} \quad \sigma \in \Sigma.$$

The space Y is regular, so an open subset W of Y can be chosen such that $y_0 \in W \subset \overline{W} \subset V$. A family $\mathcal{A} = \{V, Y\backslash\overline{W}\}$ forms an open cover of Y. From the upper strong convergence of $\{F_j : j \in J\}$ to F there exists $j_0 \in J$ such that $F_j(x) \subset \operatorname{St}\big(F(x), \mathcal{A}\big)$ for each $j \geq j_0$ and $x \in X$.

In particular for every $\sigma \in \Sigma$ and $j \geq j_0$ we obtain $F_j(x_\sigma) \subset \operatorname{St}\big(F(x_\sigma), \mathcal{A}\big) = Y \setminus \overline{W}$. This means that $F_j(x_\sigma) \cap W = \emptyset$ for all $(j, \sigma) \in J \times \Sigma$, $j \geq j_0$.

Consequently

$$x_0 \in \lim_{\Sigma} x_\sigma \qquad \text{and} \qquad F(x_0) \notin \tau^-\text{-}\lim_{J\times\Sigma} F_j(x_\sigma).$$

But $\{F_j : j \in J\}$ is τ^--continuously convergent to F at x_0. Thus we have $x_0 \in C^-(F)$. □

We have also the "symmetrical" theorem, namely:

THEOREM 13. *Assume that X is a topological space, Y — a regular space. Let F be a multivalued map with α-paracompact values. If a net $\{F_j : j \in J\}$ is lower strongly and τ^+-continuously convergent to $F\colon X \to Y$ at x_0, then $x_0 \in C^+(F)$.*

Proof. Suppose that $x_0 \notin C^+(F)$. Thus there is an open set $V \subset Y$ such that $F(x_0) \subset V$ and for each neighbourhood U of x_0 there exist a point $x_u \in U$ with the property $F(x_u) \cap (Y \setminus V) \neq \emptyset$. The symbol Σ will denote a family of all neighbourhoods of x_0. We assume that $U_1 \leq U_2$ if and only if $U_2 \subset U_1$. Then $(\Sigma, \leq)$ is a directed set. From regularity of Y there exists an open set W satisfying inclusions:

$$F(x_0) \subset W \subset \overline{W} \subset V .$$

Let $\Sigma_1 = \{\sigma = (U, j) : U \in \Sigma \text{ and } j \in J\}$; we assume $\sigma_1 \leq \sigma_2$ if and only if $U_2 \subset U_1$ and $j_1 \leq j_2$. We will consider $\{F_j(x_u) : \sigma = (U, j) \in \Sigma_1\}$. The net $\{F_j : j \in J\}$ is τ^+-continuously convergent to F at the point x_0, hence

$$F(x_0) \in \tau^+\text{-}\lim_{\Sigma_1} F_j(x_u) .$$

We can choose $\sigma_0 = (U_0, j_0)$ such that $F_j(x_\sigma) \subset W$ for every $\sigma \geq \sigma_0$, $\sigma = (U, j)$. For the open cover $\mathcal{A} = \{V, Y \setminus \overline{W}\}$ and for every $\sigma \geq \sigma_0$ we obtain $\operatorname{St}(F_j(x_u), \mathcal{A}) = V$.

In the consequence, $F(x_u) \not\subset V = \operatorname{St}(F_j(x_u), \mathcal{A})$ and the net $\{F_j : j \in J\}$ is not lower strongly convergent to F. This is a contradiction. Thus we have $x_0 \in C^+(F)$. □

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