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ON THE KURZWEIL INTEGRAL FOR FUNCTIONS WITH VALUES IN ORDERED SPACES II

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ABSTRACT. A limit theorem is formulated and proved for uniform convergent sequences of Kurzweil-Henstock integrable functions from a compact interval to a Riesz space.

The paper is a continuation of the article [5]. In the article there were presented the definitions and some elementary properties of the Kurzweil integral. This paper contains a limit theorem.

We recall that a function $f\colon I \to X$ ($I = \langle a, b\rangle \subset \mathbb{R}$, X being a boundedly σ-complete, σ-distributive linear lattice, i.e. for every bounded double sequence $(a_{ij})_{i,j}$ such that $a_{ij} \searrow 0$ ($j \to \infty$, $i = 1, 2, \dots$) it is $\bigwedge_{\varphi \in \mathbb{N}} \bigvee_i a_{i\varphi(i)} = 0$), is called integrable (in the Kurzweil sense) if there exist $x \in X$ and a bounded double sequence $(a_{ij})_{i,j}$ such that $a_{ij} \searrow 0$ ($j \to \infty$, $i = 1, 2, \dots$) and for every $\varphi\colon \mathbb{N} \to \mathbb{N}$ there exists $\sigma\colon I \to (0, \infty)$ such that for every $D \in A(\sigma)$

$$|x - S(f, D)| < \bigvee_i a_{i\varphi(i)} .$$

Here $A(\sigma)$ consists of all decompositions D of I such that $D = \{(J_1, t_1), (J_2, t_2), \dots, (J_n, t_n)\}$, where $J_i \subset (t_i - \sigma(t_i),\ t_i + \sigma(t_i))$, and $S(f, D) = \sum_{i-1}^{n} f(t_i)m(J_i)$, where $m(J_i)$ is the measure of the interval J_i, is the integral sum.

If $x_n, x \in X$, then $x_n \to x$ (with respect to the ordering) if and only if there exist $a_n \in X$, $a_n \searrow 0$ and $|x_n - x| \leqq a_n$ for all n.

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It is possible to prove that a sequence $(x_n)_n \subset X$ converges to $x \in X$ if and only if $(x_n)_n$ is bounded and

$$x = \bigwedge_{n=1}^{\infty} \bigvee_{i=n}^{\infty} x_i = \bigvee_{n=1}^{\infty} \bigwedge_{i=n}^{\infty} x_i .$$

We say that $f_n \to f$ uniformly ($f_n, f\colon I \to X$) if and only if there exist $a_n \in X$, $a_n \searrow 0$ such that

$$|f_n(t) - f(t)| \leqq a_n$$

for every $t \in I$ and every n.

LEMMA 1. *If $f_n\colon I \to X$ is integrable for $n = 1, 2, \dots$, $f_n \to f$ uniformly and f is bounded, then $\lim\limits_{n\to\infty} \int f_n \,\mathrm{d}m$ exists.*

P r o o f. It is sufficient to show that the sequence $\left(\int f_n \,\mathrm{d}m\right)_n$ is bounded and

$$\bigwedge_{n=1}^{\infty} \bigvee_{i=n}^{\infty} \int f_i \,\mathrm{d}m \leqq \bigvee_{n=1}^{\infty} \bigwedge_{j=n}^{\infty} \int f_j \,\mathrm{d}m .$$

The function f is bounded, then there exists $h \in X$, $h > 0$ such that $|f(t)| \leqq h$ for all $t \in I$.

If $f_n \to f$ uniformly, then there exists a sequence $(a_n)_n \subset X$, $a_n \to 0$ $(n \to \infty)$ and for any $t \in I$

$$|f_n(t) - f(t)| \leqq a_n$$

for all n. Hence

$$-h - a_1 \leqq f_n(t) \leqq h + a_1 \quad \text{and} \quad -2a_n \leqq f_i(t) - f_j(t) \leqq 2a_n$$

for any $t \in I$ and $i, j \geqq n$. It is evident that if for $f\colon I \to X$, $f(t) = a$ for all $t \in I$, then $\int f \,\mathrm{d}m = am(I)$. By Theorems 5 and 6 in [5] for any n we have

$$(-h - a_1)m(I) \leqq \int f_n \,\mathrm{d}m \leqq (h + a_1)m(I)$$

and

$$-2a_n m(I) \leqq \int (f_i - f_j) \,\mathrm{d}m = \int f_i \,\mathrm{d}m - \int f_j \,\mathrm{d}m \leqq 2a_n m(I)$$

for $i, j \geqq n$.

Then the sequence $\left(\int f_n \, dm\right)_n$ is bounded and

$$\bigvee_{i=n}^{\infty} \int f_i \, dm \leqq \bigwedge_{j=n}^{\infty} \int f_j \, dm + 2a_n m(I)$$

for all n and hence

$$\bigwedge_{n=1}^{\infty} \bigvee_{i=n}^{\infty} \int f_i \, dm \leq \bigvee_{n=1}^{\infty} \bigwedge_{j=n}^{\infty} f_j \, dm .$$

THEOREM 2. *Let $f_n \colon I \to X$ be integrable for $n = 1, 2, \dots$, $f_n \to f$ uniformly and f be bounded. Then f is integrable and $\int f \, dm = \lim_{n\to\infty} \int f_n \, dm$.*

Proof. By Lemma 1 $\lim_{n\to\infty} \int f_n \, dm = c$ exists and hence there exists a sequence $(c_n)_n \subset X$, $c_n \searrow 0$ $(n \to \infty)$ and

$$\left| \int f_n \, dm - c \right| \leqq c_n .$$

for any n.

The function f_n is integrable and then there exists a bounded double sequence $(a_{nij})_{i,j} \subset X$ such that $a_{nij} \searrow 0$ $(j \to \infty,\ i, n = 1, 2, \dots)$ and for every $\varphi \colon \mathbb{N} \to \mathbb{N}$ there exists $\sigma_n \colon I \to (0, \infty)$ such that for every $D \in A(\sigma_n)$

$$\left| \int f_n \, dm - S(f_n, D) \right| < \bigvee_i a_{ni\varphi(i+n+1)} .$$

When $f_n \to f$ uniformly, then there exists a sequence $(b_n)_n \subset X$, $b_n \searrow 0$ and $|f_n(t) - f(t)| \leqq b_n$ for any $t \in I$ and all n.

Let $\varphi \in \mathbb{N}^{\mathbb{N}}$. Put $k = \min_j \varphi(j+1)$ and take $D \in A(\sigma_k)$, $D = \{(J_1, t_1), (J_2, t_2), \dots, (J_r, t_r)\}$.

Then

$$\begin{aligned} |S(f, D) - c| &\leqq |S(f, D) - S(f_k, D)| + \left| S(f_k, D) - \int f_k \, dm \right| + \left| \int f_k \, dm - c \right| \\ &< \sum_{i=1}^{r} |f(t_i) - f_k(t_i)| m(J_i) + \bigvee_i a_{ki\varphi(i+k+1)} + c_k \\ &\leqq b_k \sum_{i=1}^{r} m(J_i) + \bigvee_i a_{ki\varphi(i+k+1)} + c_k \\ &= b_k m(I) + c_k + \bigvee_i a_{ki\varphi(i+k+1)} = d_k + \bigvee_i a_{ki\varphi(i+k+1)} , \end{aligned}$$

where $d_j = b_j\, m(I) + c_j$ for $j = 1,2,\dots$, $d_j \searrow 0$ $(j \to \infty)$, $d_k = d_{\min\limits_j \varphi(j+1)}$ $= \bigvee\limits_i d_{\varphi(i+1)}$. Put $b_{1ij} = d_j$ for $i,j = 1,2,\dots$ and $b_{n+1ij} = a_{nij}$ for $n,i,j = 1,2,\dots$.

Now

$$\begin{aligned}|S(f,D) - c| &< \bigvee_i d_{\varphi(i+1)} + \bigvee_i a_{ki\varphi(i+k+1)} \\ &= \bigvee_i b_{1i\varphi(i+1)} + \bigvee_i b_{k+1i\varphi(i+k+1)} \\ &\leqq \sum_{n=1}^{\infty} \bigvee_i b_{ni\varphi(i+n)} .\end{aligned}$$

There exists $h \in X$, $h > 0$ such that $|f(t)| \leqq h$ for any $t \in I$, since f is bounded. Then

$$|S(f,D) - c| \leqq h \cdot m(I) + |c| = a ,$$

when $a \in X$, $a > 0$ and

$$|S(f,D) - c| \leqq a \wedge \left(\sum_{n=1}^{\infty} \bigvee_i b_{ni\varphi(i+n)} \right).$$

By Lemma 2 in [7] there exists a bounded double sequence $(a_{ij})_{i,j} \subset X$, $a_{ij} \searrow 0$ $(j \to \infty,\ i = 1,2,\dots)$ and

$$a \wedge \left(\sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} b_{ni\varphi(i+n)} \right) \leqq \bigvee_{i=1}^{\infty} a_{i\varphi(i)} .$$

Therefore there exists $c \in X$, $c = \lim\limits_{n\to\infty} \int f_n \,\mathrm{d}m$ and the sequence $(a_{ij})_{i,j} \subset X$, $a_{ij} \searrow 0$ $(j \to \infty,\ i = 1,2,\dots)$ and for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $\sigma \colon I \to (0,\infty)$ $(\sigma = \sigma_{\min\limits_j \varphi(j+1)})$ such that

$$|S(f,D) - c| \leqq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}$$

for any $D \in A(\sigma)$. Hence f is integrable and

$$\int f \,\mathrm{d}m = \lim_{n\to\infty} \int f_n \,\mathrm{d}m .$$

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