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EVERY NORMAL LINEAR SYSTEM HAS A REGULAR TIME-OPTIMAL SYNTHESIS

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1. Introduction

In [2] (cf. also [3]) Bolťanskij introduced the concept of regular synthesis for the time-optimal control problem. This concept allowed him to formulate conditions under which Pontrjagin's maximum principle was a sufficient condition of optimality. Although all the known examples of time optimal synthesis for normal linear systems satisfy Bolťanskij's conditions, no proof has been given so far that this would be true in general.

The main aim of this paper is to prove that every normal linear system (cf. Section 5 for definition) admits a regular time- optimal synthesis (Section 6). For this purpose, however, the definition of regular synthesis has to be slightly modified. In Section 4 we prove that this modification is not essential: it is still possible to prove the optimality of the regular synthesis under this modified definition.

The proof of the existence of regular synthesis is largely based upon the theory of subanalytic sets. The necessary material is summarized in Sections 2, 3.

Although the existence theorem of Section 6 does not contribute directly to the sufficient conditions of optimality (the sufficiency of Pontrjagin's maximum principle for normal linear systems can be proved by other, simpler means — cf. [3, 19]), it gives an insight into the structure of the closed-loop optimal control. Moreover, it can be required that the synthesis has an additional transversality property which appears in [5] as a restrictive condition under which the coincidence of the open-loop optimal trajectories and the Filippov trajectories of the system under the action of the closed-loop optimal control is proved for systems with scalar control (Section 7).

2. *W*-stratification

Let M be a differential manifold and U, V be submanifolds of M such that $V \subset \bar{U} \setminus U$. We say that U, V have the *Whitney property* (a) if for every $x \in V$ and every sequence of points $\{x_k\}$ from U such that $x_k \to x$ and TU_{x_k} (the tangent space of U at x_k) converges, $TV_x \subset \lim_{k\to\infty} TU_{x_k}$ (the limit to be understood in the topology of the Grassmann manifolds of planes of dimension dim U in the tangent bundle TM of M — cf. [11]).

Let G be a subset of a differential manifold M. By a *W-stratification* of G we understand a locally finite (in M) partition $\mathcal{P}$ of G into submanifolds (called strata) of M such that if P, $Q \in \mathcal{P}$ and $\bar{P} \cap Q \neq \emptyset$, then $Q \subset \bar{P}$ and P, Q have the Whitney property (a). By the dimension of $\mathcal{P}$ we understand the maximum of the dimensions of its strata.

It is shown in [11] that if we denote $P > Q$ if $\bar{P} \supset Q$, $P \neq Q$, then $>$ is transitive and $P > Q$ implies $Q \not> P$.

Lemma 1. *Let $\mathcal{P}$ be a W-stratification of $G \subset M$. If P, $Q \in \mathcal{P}$ and $P > Q$, then dim $Q \leqslant$ dim P.*

Proof. Let $n = \dim M$, $K = \dim P$, $x \in Q$. Then there exists a sequence $x_i \to x$, $x_i \in P$. Since the set of k-dimensional planes through 0 is a compact subset of the Grassmann manifold of k-dimensional planes in R^n, we can choose the sequence $\{x_i\}$ in such a way that $\{T_{x_i}Q\}$ converges. Therefore, $T_xQ \subset \lim_{k\to\infty} T_{x_i}P$, which is possible only if $\dim Q \leqslant \dim P$.

3. Subanalytic and semianalytic sets.

Let A be a real analytic manifold (we shall drop the word "real" in the sequel) A subset $M \subset A$ is called *semianalytic* if for every $x \in A$ there exists a neighbourhood U of x such that $M \cap U$ is a finite union of sets of type $\{y \in U \mid g_i(y) = 0, f_j(y) > 0, i = 1, \ldots, p, j = 1, \ldots, q\}$, where g_i, f_j are analytic functions in U. A set $M \subset A$ is called *subanalytic,* if for every $x \in A$ there exists a neighbourhood U of x such that $M \cap U$ is a finite union of sets of type $f_1(Y_1) \setminus f_2(Y_2)$, where Y_1 and Y_2 are analytic manifolds and f_1, f_2 are proper analytic maps $Y_1 \to U$, $Y_2 \to U$, respectively. Recall that f is proper if $f^{-1}(K)$ is compact for every K compact.

For the following properties of semianalytic and subanalytic sets the reader is referred to [8]:

SA 1. The closure and interior of a semianalytic (subanalytic) set, the intersection, union and difference of two semianalytic (subanalytic) sets are semianalytic (subanalytic, respectively).

SA 2. Every semianalytic set is subanalytic.

SA 3. Let A, B be analytic manifolds, f: $A \to B$ analytic. The pre-image $f^{-1}(M)$ of any semianalytic set $M \subset B$ is semianalytic.

SA 4. Let A, B be as in SA 3, $f\colon A \to B$ analytic proper. The image $f(M)$ of any subanalytic set $M \subset A$ is subanalytic.

SA 5. Every semianalytic (subanalytic) subset of a second countable analytic manifold A admits a W-stratification (in A), the strata of which are connected analytic submanifolds of A, which are semianalytic (subanalytic, respectively).

Henceforth we shall call a CASA set any connected analytic submanifold of A which is subanalytic in A.

Lemma 2. *Let A be an analytic manifold and let M be a subanalytic set in $R^n \times A$, the natural projection of which on R^n is bounded. Then, the natural projection of M on A is subanalytic.*

Proof. Because of the boundedness of the projection of M on R^n we can consider M as a subanalytic subset of the product of the one-point compactification of R^n (which is the n-sphere S^n) and A, $S^n \times A$. The statement of the lemma follows from SA 4, because the natural projection map of $S^n \times A$ on A is proper.

Lemma 3. *Let M be a subanalytic subset of an analytic manifold A. Let X be an analytic vector field on A. Then, M admits a locally finite partition $\mathcal{P}$ into CASA sets such that for each $P \in \mathcal{P}$ the set P_X of those points $x \in P$ at which X is tangent to P is subanalytic.*

Proof. First assume that M is an analytic submanifold of A and that $M = f(Y)\backslash N$, where N is closed subanalytic, Y is an analytic manifold and f is proper analytic on Y with Df of constant rank on $f^{-1}(M)$. Then we have

$$T_{f(y)}M = Df(y)T_yY \tag{0}$$

for each $y \in f^{-1}(M)$. We prove that the set M_X of those points $x \in M$ at which X is tangent to M is subanalytic in A.

Take any point $x \in A$ and any subanalytic coordinate neighbourhood U of x with compact closure. We identify U with a subset of R^m, m being the dimension of A. Then, $f^{-1}(U)$ can be covered by a finite family $\mathcal{W}$ of subanalytic compact coordinate neighbourhoods. It suffices to prove that the set $M_X^W = \{x \mid x \in M_X \cap f(W)\}$ is subanalytic for every $W \in \mathcal{W}$.

We can consider W as a subanalytic subset of R^n, n being the dimension of Y. Let $e_1, \ldots, e_n$ be the coordinate basis of R^n. Consider the matrix $Q(y) = (Df(y)e_1, \ldots, Df(y)e_n, X(f(y)))$. It follows from (0) that the vectors $Df(y)e_1, \ldots, Df(y)e_n$ span $T_{f(y)}M$ for every $y \in f^{-1}(M \cap W)$. Therefore, M_X^W can be characterized as the intersection of M with the f-image of the set of those points $y \in W$ for which all the subdeterminants of $Q(y)$ of order $>r$ are zero. The values of these determinants are analytic functions of y. Consequently, S is a semianalytic subset of W from which it immediately follows that $f(S) \cap M = M_X^W$ is subanalytic.

Now we prove the lemma by induction. Assume that the statement of the lemma holds for all subanalytic subsets of A of dimension $\leq r$ (i.e. admitting a stratification of dimension $\leq r$). Let dim $M = r+1$. Due to SA 5 we may without loss of generality assume that M is a submanifold of A. By definition of subanalyticity we have $M = f(Y)\backslash g(Z)$, f, g analytic proper, Y, Z analytic manifolds. Denote $Y_i = \{y \in Y \mid \text{rank } Df(y) \leq i\}$. Locally at any $y \in Y$, Y_i can be characterized as the set of points at which all the subdeterminants of the Jacobian of f of order $>i$ are zero. Since these subdeterminants are analytic, Y_i is semianalytic in Y. Further, Y_i are obviously closed and $Y = \bigcup_i Y_i$, $M = f(Y_{r+1})\backslash g(Z)$ We have

$$M - f(Y)\backslash g(Z) = \{f(Y)\backslash[f(Y_r) \cup g(Z)]\} \cup [f(Y_r)\backslash g(Z)].$$

The set $f(Y_r)\backslash g(Z)$ is subanalytic and of dimension $\leq r$ Consequently, it admits the partition with required properties by the induction assumption. The set $f(Y)\backslash[f(Y_r) \cup g(Z)]$ is open in the submanifold M and therefore is a submanifold of A. The application of the first part of the proof to this set with $N = f(Y_r) \cup g(Z)$ concludes the proof.

Lemma 4. *Let M be a subanalytic subset of an analytic second countable manifold A and let $X^1, \ldots, X^r$ be analytic vector fields on A. Then there exists a locally finite partition $\mathcal{M}$ of M into CASA sets such that for every $N \in \mathcal{M}$ and $1 \leq i \leq r$, X^i is either everywhere or nowhere tangent to N.*

We shall call $\mathcal{M}$ as well as its components flow consistent (to the vector fields $X^1, \ldots, X^r$)

Proof. Due to SA 5 we can without loss of generality assume that M is a CASA set. By Lemma 3 there exists a partition $\mathcal{P}$ of M such that for every $P \in \mathcal{P}$ the set $P_{X^1} = \{x \in P,\ X^1(x) \in T_x P\}$ is subanalytic. Obviously, if $P_{X^1} \neq P$, $P'_{X^1} = P\backslash P_{X^1}$ is an open submanifold of P.

By SA 5, P_{X^1} admits a partition $\mathcal{N}_P$ into CASA sets. We prove that if $P_{X^1} \neq P$, then $\dim \mathcal{N}_P < \dim P \leq \dim M$.

Assume the contrary. Then there exists an $N \in \mathcal{N}_P$, $N \neq P$, $\dim N = \dim P$ and, consequently, N open in P. Let $x \in P$ be a boundary point of N. There exists a neighbourhood U of x such that $P \cap U$ is given by $P \cap U = \{x \in A \cap U \mid f_1(x) = 0, \ldots, f_s(x) = 0\}$, where $n-s$ is the dimension of P and f_i are properly chosen coordinate functions for P. Consider the functions $g_i\colon P \cap U \to R^1$ given by $g_i(x) = Df_i(x) \cdot X^1(x)$, where Df_i is the differential of f_i, $i = 1, \ldots, s$. Obviously, X^1 is tangent to P at x if and only if

$$g_i(x) = Df_i(x) \cdot X^1(x) = 0, \quad i = 1, \ldots, s. \tag{1}$$

Since x is a boundary point of N, the set of points $x \in P \cap U$ for which (1) is satisfied as well as the set of points $x \in P \cap U$ for which (1) is not satisfied are not empty, the former being open in P. This, however, is impossible, because g_i are analytic.

As a result of the partition we have obtained a partition $\mathcal{M}_1$ of M into CASA sets, consisting of the sets of $\mathcal{N}_P$ and the connected components of P'_X, $P \in \mathcal{P}$. As the next step we partition every component of $\mathcal{M}_1$ which is not flow consistent with respect to X^2 in the above way, X^1 replaced by X^2. Those components of $\mathcal{M}_1$ which are flow consistent are left unaltered. Then we take succesively $X^3, X^4, \ldots, X^r$ and repeat the partition of M consisting of the components obtained by the preceding partition and those unaltered by it. If the resulting partition is not flow consistent, we repeat the cycle of r partitionings again. We show that after a finite number of repetitions of the cycle the flow consistent partition will be obtained.

Indeed, if a component of some partition is flow consistent, it will not be affected by further partitionings. Furthermore, after every cycle the dimension of the components which are not flow consistent is lowered by at least one. This follows from the fact that if a vector field is everywhere transversal (parallel) to some submanifold of M, then it is so to every submanifold of M (open submanifold of M, respectively).

Lemma 5. *Let A be a second countable analytic manifold and let G be a locally finite union of CASA sets in A of dimension $\leqslant k$. Then $\bar{G}$ admits a W-stratification of dimension $\leqslant k$ with CASA strata.*

Proof. As a locally finite union of CASA sets, G is obviously subanalytic and therefore it admits a W-stratification $\mathcal{S}$ with CASA strata. Since no submanifold of dimension $>k$ can be contained in a locally finite union of submanifolds of dimension $\leqslant k$, the dimension of this stratification is $\leqslant k$. By SA 1, for each $S \in \mathcal{S}$, $\bar{S}$ is subanalytic and so is $\bar{G} = \bigcup_{S \in \mathcal{S}} \bar{S}$. Therefore, the statement of the lemma follows from the above dimension argument provided we prove that $\bar{S}$ admits a stratification of dimension $\leqslant k$.

Let $\mathcal{S}_1$ be a W-stratification of $\bar{S}$ into CASA strata. Assume that there exists an $S_1 \in \mathcal{S}_1$ such that $\dim S_1 > k$. If $S_1 \subset \bar{S} \setminus S$, then for $x \in S_1$ there exists a sequence of points $x_k \to x$, $x_k \in S$. Because of local finiteness of $\mathcal{S}_1$ we may assume that $x_k \in S_2$ for some $S_2 \in \mathcal{S}_1$. Then it follows from the definition of stratification that $S_1 \subset \bar{S}_2$ and, by Lemma 1, $\dim S_1 \leqslant \dim S_2$. Consequently, there exists a set $S_2 \in \mathcal{S}_1$ such that $S_2 \cap S \neq 0$ and $\dim S_2 > k$.

Let $x \in S_2 \cap S$. The set S is locally closed, i.e. for every $x \in S$ there exists a neighbourhood U of x such that $S \cap U$ is closed in U. This means that there exists a neighbourhood U of x such that $S_2 \cap U = S_2 \cap \bar{S} \cap U = S_2 \cap S \cap U \subset S \cap U$. This implies $\dim S \geqslant \dim S_1$, which contradicts $\dim S \leqslant k$.

4. Regular synthesis

Consider a control system

$$\dot{x} = f(x, u),$$

$x \in R^n$, $u \in U \subset R^m$, $f\colon R^n \times R^m \to R^n$ being C^1. Given an initial point x_0 and target point x_1, by an admissible control we understand a bounded measurable function u: $[0, T] \to U$ such that the solution $x(t, x_0, u)$ of the equation

$$\dot{x} = f(x, u(t)) \tag{2}$$

satisfying $x(0, x_0, u) = x_0$ exists on $[0, T]$ and satisfies $x(T, x_0, u) = x_1$. We say that the control $u(t)$ steers the system from x_0 to x_1. An admissible control will be called optimal if it steers the system from x_0 to x_1 in minimum time.

Given a target point x_1 and an open domain G containing x_1, by a *regular synthesis* we shall understand a pair $(\mathscr{S}, v)$, where $\mathscr{S}$ is a locally finite (in G) partition of G into C^1 connected submanifolds of R^n (called cells) and $v\cdot G \to U$ (the closed-loop optimal control) is a function satisfying the following properties:

A. The set $\bar{G}'$ admits a W-stratification of dimension $<n$, where $G' = \cup\{S \in \mathscr{S} \mid \dim S < n\}$ (if $\mathscr{S}$ is a family of sets, we shall use the notation $\cup\mathscr{S} = \cup\{S \mid S \in \mathscr{S}\} = \{x \in S \mid S \in \mathscr{S}\}$).

B. The set $\{x_1\}$ is a cell, $v(x)$ is C^1 on each cell S and can be extended into a C^1 function on some neighbourhood of S.

C. The cells of $\mathscr{S}$ are of two types, type I and type II. If S is a k-dimensional cell of type I, then $f(x, v(x))$ is everywhere tangent to S and through every point $x \in S$ there is a unique solution $\xi_x(t)$ of the differential equation

$$\dot{x} = f(x, v(x)), \tag{3}$$

which locally stays in S. There exists a $(k-1)$ — dimensional cell $\Pi(S)$ such that the vector field $x \mapsto f(x, v(x))$ is transversal to $\Pi(S)$ and every trajectory of (3) from any $x \in S$ enters $\Pi(S)$, the entering time being a continuous function of x. If S is of type II, there exists a unique cell $\Sigma(S)$ of dimension $k+1$ of type I such that from every point of S a unique trajectory of (3) starts and locally stays in $\Sigma(S)$; $v(x)$ is C^1 in $S \cup \Sigma(S)$.

D. Every trajectory $\xi_x(t)$ of (3) starting at $x \in G$ (which is uniquely defined by C until staying in G) reaches x_1 in finite time, passing only a finite number of cells, and satisfies Pontrjagin's maximum principle with the control $u_x(t) = v(\xi_x(t))$.

E. The time in which $\xi_x(t)$ reaches x_1 is a continuous function of x in G.

To avoid misunderstandings we now specify in which sense we shall understand transversality, since this concept is being used in two different contexts.

A vector field X on a manifold M is said to be *transversal* to a submanifold N of

M if $X(M) \cap TN = \emptyset$ (X to be understood as a section map X from M to TM, the tangent bundle of M) or, in other words, if for every $x \in N$, $X(x) \notin T_x N$.

A C^1 map f: $M \to N$ (M, N manifolds) is said to be *transversal* to a submanifold S of N if $Df(x)\,(T_x M) + T_{f(x)} S = T_{f(x)} N$ for every $x \in f^{-1}(S)$ (cf. [1]). In particular, if $\dim M < \operatorname{codim} S$, f transversal to S means $f^{-1}(S) = \emptyset$.

In most cases it will be clear from the context which meaning of transversality we have in mind, since the first of them is used for vector fields, the second for maps. The only case which needs some amplification is the case of a trajectory of a vector field. If we say that a trajectory $x(t)$, $t \in I$ of a vector field X on M is transversal to some manifold $N \subset M$ we shall always understand this as transversality of the map x: $I \to M$ to N (and not as transversality of X to N).

Our definition of regular synthesis differs from that of Bolťanskij ([2, 3]) in two ways. First it does not admit the exceptional set N from which more than one trajectory of (3) is allowed to start. Such a set can be included without complications and we omit it only because it does not occur in linear systems we deal with in this paper. Secondly instead of assuming that the sets $\cup\{S \in \mathscr{S} \mid \dim S < k\}$, $k = 1, \ldots, n$ are "piecewise smooth" we assume A. We show that the proof of optimality of the regular synthesis goes through under assumption A.

In order to do so we note that the only place where piecewise smoothness is needed is the following lemma (which appears as Lemma VI.6 in [3]):

Let M be a piecewise smooth set in G of dimension $< n$. Let $u(t)$, $t \in [0, T]$ be a piecewise continuous control which steers the system from x_0 to x_1, the trajectory $x(t, x_0, u)$, $t \in [0, T]$ of (2) *lying entirely in G. Then, in any neighbourhood of x_0 there exists a point y_0 such that the trajectory $x(t, y_0, u)$, $t \in [0, T]$ of* (2) *meets M for at most finitely many values of t.*

This lemma is applied to the set $M = G'$. We show now that from property A of the synthesis the statement of the lemma follows for $M = \bar{G}'$.

Let $x_0 \in G$ and let $u(t)$, $t \in [0, T]$ be a piecewise continuous control such that $x(t, x_0, u) \in G$ for $t \in [0, T]$. Let K be a subanalytic neighbourhood of the trajectory $\{x(t, x_0, u) \mid t \in [0, T]\}$ such that $\bar{K}$ is compact and $\bar{K} \subset G$. There exists a neighbourhood V_0 of x_0 such that $x(t, y, u) \in K$ for all $y \in V_0$ and $t \in [0, T]$.

As the set $\bar{K} \cap \bar{G}'$ is subanalytic and compact, it admits a finite W-stratification $\mathscr{M}$ of dimension $< n$. We associate with $\mathscr{M}$ an oriented graph as follows: we take the strata of $\mathscr{M}$ as vertices and the oriented pairs (M, N), $M, N \in \mathscr{M}$ as edges if $M < N$. Because of the property of the ordering mentioned before Lemma 1 this graph has no cycle. We define the *height* of $M \in \mathscr{M}$, $h(M)$, as the length of the longest path ending in M (i.e. the number of edges in the longest connected oriented sequence of edges ending in M). Since the graph has no cycle, $h(M)$ is defined for every $M \in \mathscr{M}$ and $h(N) \leqslant h(M)$ implies $N \not> M$.

First we show by induction that the following statement holds:

Denote $\mathscr{M}_i = \{M \in \mathscr{M} \mid h(M) \leqslant i\}$. Then the set of those points $y_0 \in V_0$ for which

$x(t, y_0, u)$, $t \in [0, T]$ does not meet $\cup \mathcal{M}_i$ at switching points t of $u(t)$ contains an open dense subset V_i of V. Henceforth $\cup \mathcal{M}_i$ etc. is understood in the sense specified in A, unless the index i appears below the union symbol.

Assume that the i-th induction statement holds. Let $M \subset \mathcal{M}_{i+1} \setminus \mathcal{M}_i$ and let τ be a switching point of $u(t)$. Since $\dim M < n$ and the map $y \mapsto x(\tau, y, u)$ is a diffeomorphism of V_i and $x(\tau, V_i, u)$, the set $W_{\tau,M}$ of points $y \in V_i$ such that $x(\tau, y, u) \notin M$ is dense in V_i. On the other hand, since $\bar{M} \subset M \cup \mathcal{M}_i$, M is closed in $x(\tau, V_i, u)$ and, consequently, $W_{\tau,M}$ is open in V_i. Thus, $W_{\tau,M}$ is open dense in V_i; since V_i is open dense in V_0, $W_{\tau,M}$ is open dense in V.

We take V_{i+1} as the intersection of the sets $W_{\tau,M}$ for τ, M running through the switching points of $u(t)$ and $\mathcal{M}_{i+1} \setminus \mathcal{M}_i$, respectively. Since the number of switching points of $u(t)$ as well as $\mathcal{M}_{i+1} \setminus \mathcal{M}_i$ are finite, V_{i+1} is open dense. The set $\tilde{V} = \bigcap_i V_i$ is open dense and for every $y \in \tilde{V}$, $x(t, y, u)$ does not meet $\cup \mathcal{M}$ if t is a switching point of $u(t)$.

Similarly, by induction in height we prove that the set of points $y \in \tilde{V}$ for which $x(t, y, u)$, $t \in [0, T]$ meets every stratum of $\mathcal{M}$ transversally is open dense in $\tilde{V}$ (note that since $x(t, y, u)$ does not meet $\cup \mathcal{M}$ if t is a switching point of $u(t)$, transversality makes sense). Assuming that the set $\tilde{V}_i$ of points of V such that $x(t, y, u)$ meets every stratum of $\mathcal{M}_i$ transversally is open dense in V, the density of the subset $\tilde{W}_M$ of $\tilde{V}_i$ of those points y for which $x(t, y, u)$, $t \in [0, T]$ meets $M \in \mathcal{M}_{i+1} \setminus \mathcal{M}_i$ transversally can be shown similarly as in [2, 3] or by the transversality theorem [1, 19.1]. To prove openness of $\tilde{W}_M$ in $\tilde{V}_i$ assume that there exists a sequence of points $y_k \in \tilde{V}_i \setminus \tilde{W}_M$, $y_k \to {}_0 \in \tilde{W}_M$. If $\dim M = n-1$ this means that there exists a sequence $\{t_k\}$, $t_k \in [0, T]$, $t_k \to t \in [0, T]$ such that $\dot{x}(t_k, y_k, u) \in T_{x(t_k, y_k, u)}M$. Since $\bar{M} \subset M \cup \bigcup \mathcal{M}_i$, we have $x(t_k, y_k, u) \to x(t_0, y_0, u) \in M \cup \bigcup \mathcal{M}_i$. Thus, t_0 is not a switching point and $\dot{x}(t_0, y_0, u) = \lim_{k\to\infty} \dot{x}(t_k, y_k, u)$. If $x(t_0, y_0, u) \in M$, we obtain $\dot{x}(t_0, y_0, u) \in T_{x(t_0, y_0, u)}M$, which violates $y_0 \in \tilde{W}_M$. If $x(t_O, y_0, u) \in N \in \mathcal{M}_i$, then necessarily $\dim N = n-1$, since by the assumed transversality $x(t, y_0, u)$ does not meet any stratum of $\mathcal{M}_i$ of dimension $< n-1$. Passing to a subsequence if necessary we may assume that $T_{x(t_k, y_k, u)}M$ converges and, consequently, from the Whitney property if follows that $T_{x(t_0, y_0, u)}N \subset \lim_{k\to\infty} T_{x(t_k, y_k, u)}$

Since $\dim N = n-1 \geqslant \dim M$, this is possible only if $T_{x(t_0\ y_0, u)} = \lim_{k\to\infty} T_{x(t_k\ y_k, u)}$. Consequently, $\dot{x}(t_0, y_0, u) \in T_{x(t_0, y_0, u)}N$, contrary to the induction hypothesis.

If $\dim M < n-1$, then there exists a sequence $\{t_k\}$, $t_k \in [0, T]$, $t_k \to [0, T]$ such that $x(t_k, y_k, u) \in M$. We have $x(t_0, y_0, u) \in \bar{M} \subset M \cup \bigcup \mathcal{M}_i$ and, since $y_0 \in \tilde{W}_M$, $x(t_0, y_0, u) \bigcup \mathcal{M}_i$. Since by Lemma 5 all the strata of $\mathcal{M}_i$ intersecting $\bar{M}$ have to be of dimension $< n-1$, this again violates the induction hypothesis. If $\dim M = n$, $\tilde{V}_i \setminus \tilde{W}_M = \emptyset$ trivially.

Now, the set $\tilde{V}=\bigcap_i \tilde{V}_i$ is open dense and every trajectory $x(t, y, u)$, $t \in [0, T]$ for $y \in \tilde{V}$ meets any $M \in \mathcal{M}$ transversally and, consequently, at isolated points. Since $\mathcal{M}$ is finite, this means that the number of intersection points of $x(t, y, u)$, $t \in [0, T]$ and $\bar{G}'$ (which coincide with the intersection points of $x(t, y, u)$, $t \in [0, T]$ and $\bigcup \mathcal{M}$) is finite.

5. Normal linear control systems

Consider a linear control system

$$(4) \qquad \dot{x} = Ax + u, \quad u \in U,$$

x, $u \in R^n$, A constant, with a polyhedral control domain $U = \text{co}\ \{w_1, \ldots, w_p\}$ (w_i being the extremal points of U and co standing for the convex hull), containing the origin in its relative interior. The system (4) is called *normal* ([7, 10]) or *in general position* ([3]) if for any $1 \leq i$, $j \leq p$, $i \neq j$ the vectors $w_i - w_j$, $A(w_i - w_j)$, ..., $A^{n-1}(w_i - w_j)$ are linearly independent, or, equivalently, if no vector $\psi \neq 0$ such that $\langle \psi, w_i - w_j \rangle = 0$ belongs to any proper invariant subspace of A^* (where $\langle \cdot, \cdot \rangle$ stand for scalar product, * for transpose). As a consequence of normality one obtains that for any $i \neq j$, any non-zero solution $\psi(t)$ of the adjoint equation

$$(5) \qquad \dot{\psi} = -A^* \psi$$

satisfies $\langle \psi(t), w_i - w_j \rangle = 0$ only at isolated points.

We recall some well known properties of normal systems, for which [3, 6, 10] can serve as a general reference, and we draw some simple corollaries from them:

N1. The set G of points from which the system can be steered to 0 is an open convex set containing 0 in its interior. The set $G(T)$ of points from which the system can be steered to 0 in time not exceeding T is a convex compact subset of G for every $T \geq 0$; $G(T_1) \subset G(T_2)$ for $0 \leq T_1 \leq T_2$.

N2. For every $x \in G$ there exists a unique optimal control $u_x(t)$, $t \in [0, T(x)]$, which steers the system from x to 0 in minimum time $T(x)$ (the unicity of the optimal control needs an agreement that as the value of a piecewise continuous function at any point its right-hand limit is understood). This optimal control is piecewise constant with vertices w_1, ..., w_p as values. The minimal steering time $T(x)$ is a continuous function of x.

N3. The (open-loop) optimal control $u_x(t)$ satisfies *Pontrjagin's maximum principle:* There exists a non- zero solution $\psi(t)$ of the adjoint equation (5) such that

$$(6) \qquad \langle \psi(t), u_x(t) \rangle = \max_{u \in U} \langle \psi(t), u \rangle$$

(we also say that $u_x(t)$ is *extremal* with respect to $\psi(t)$ for all $t \in [0, T(x)]$). The open- loop optimal controls can be synthesized into a *closed-loop control* $v(x)$, i.e. there exists a function v: $G \to U$ such that for every $x \in G$ and $t \in [0, T(x)]$, $u_x(t) = v(\xi_x(t))$, where $\xi_x(t) = x(t, x, u_x)$ is the optimal trajectory with the initial point x. The function v can be obtained as $v(x) = u_x(0)$. If we denote by W_j the normal cone of U at w_j, i.e. $W_j = \{\psi \in R^n | \langle \psi, w_j \rangle = \max_{u \in U} \langle \psi, u \rangle\}$, then (6) can be re-formulated as follows: $\psi(t) \in W_j$ as soon as $u_x(t) = w_j$.

Let us note that W_j are convex closed polyhedral cones and $W_i \cap W_j \subset \partial W_i \cap \partial W_j$ for all $i \neq j$ (cf. [4]).

N4 (cf. [4]). For $x \in G$, $x \neq 0$, denote $E(x)$ the set of $\psi \in R^n$ such that $u_x(t)$ is extremal with respect to the solution $\psi(t)$ of (5) with $\psi(0) = \psi$; for $x = 0$ denote $E(0) = R^n$. The set $E(x)$ is a closed convex cone for every $x \in G$ and $v(x) = w_j$ implies $E(x) \subset W_j$.

N5 (cf. [4]). Let $\psi(t)$ be any solution of the equation (5). Let $u^\psi(t)$ be the extremal control with respect to $\psi(t)$ for all $t \leqslant 0$. Let $x(t) = x(t, 0, u^\psi)$ for $t \leqslant 0$. Then, for every $\tau < 0$, $\psi(\tau) \in E(x(\tau))$, $u^\psi(t) = u_{x(\tau)}(t - \tau)$, $\xi_{x(\tau)}(t) = x(t - \tau)$, $T(x(\tau)) = \tau$, for $0 \leqslant t \leqslant -\tau$.

Corollary 1. Let $\psi(t)$ be a solution of (5) such that $\psi(0) \in E(x)$ for some $x \in G$. Let $x(t) = x(t, x, u^\psi)$, where $u^\psi(t)$ is the extremal control with respect to $\psi(t)$, $-\infty < t \leqslant T(x)$. Then, for every $\tau \in (-\infty, T(x))$, $\psi(\tau) \in E(x(\tau))$, $u^\psi(t) = u_{x(\tau)}(t - \tau)$, $\xi_{x(\tau)}(t) = x(t - \tau)$ for $0 \leqslant t \leqslant T(x) - \tau$, $T(x(\tau)) = T(x) - \tau$.

We denote by X^j the linear vector field $x \mapsto Ax + w_j$, $j = 1, \ldots, p$.

Lemma 6. *Let the problem* (4) *be normal. Then there exi ts no submanifold M of R^n of dimension $< n$ such that two vector fields X^i and X^j, $i \neq j$ are both everywhere tangent to M.*

Proof. Independence of $w_i - w_j$, $A(w_i - w_j)$, ..., $A^{n-1}(w_i - w_j)$ means that the dimension of the Lie algebra spanned by the vector fields X^i, X^j is n. Thus, by [9], in any neighbourhood V of any point x there exists an open ball, each point of which can be joined with x by a curve in V consisting of a finite number of trajectories of X^i and X^j. This however, is impossible if dim $M < n$ and X^i, X^j are both tangent to M, since by moving along trajectories of X^i and X^j we cannot leave M.

Corollary 2. If $\mathcal{N}$ is a flow consistent partition of $M \subset R^n$ (as defined in Lemma 4) with respect to $X^1, \ldots, X^p$, then for every $N \in \mathcal{N}$ with dim $N < n$ at most one of the vector fields is tangent to N.

Lemma 7. *Consider a linear differential equation in R^n*

$$\dot{x} = Ax, \tag{7}$$

A constant, and denote by φ the flow of (7). Further, denote by σ the radial projection of $R^n\setminus\{0\}$ on its unit sphere S^{n-1}, $\sigma(x)=x/|x|$, where $|x|$ is the Euclidean norm of x. Then, there exists a unique flow φ^0 on S^{n-1} such that $\sigma\circ\varphi_t(x)=\varphi_t^0\circ\sigma(x)$ for all t and $x\neq 0$ and this flow is analytic on S^{n-1}.

We shall call φ^0 the unit projection of φ.

Proof. Let $x(t)$ be a non- zero solution of (7), $y(t)=x(t)/|x(t)|$. Then,

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{x(t)}{|x(t)|}=A\frac{x(t)}{|x(t)|}-\frac{\langle x(t), Ax(t)\rangle x(t)}{|x(t)|^{3/2}}=$$

$$=Ay(t)-\langle y(t), Ay(t)\rangle y(t).$$

If we denote $X^0(y)=Ay-\langle y, Ay\rangle y$ for $y\in S^{n-1}$, then the flow φ^0 of X^0 is obviously the unit projection of φ.

Remark 1. Because the maximum principle (6) is homogeneous in ψ, the trajectories of the adjoint equation (5) in the definition of the extremal control and in the properties N3—N5 can be replaced in an obvious way by the trajectories of the unit projection of (5), W_i, $E(x)$ replaced by $W_i^0=W_i\cap S^{n-1}$, $E^0(x)=E(x)\cap S^{n-1}$, respectively. Since in the proof of Theorem 1 we shall mostly move backwards along the trajectories of (5) and (4), we denote by φ^* the unit projection of the backwards flow of (5), i.e. the projection of the flow of the equation $\dot{\psi}=A^*\psi$. Then, the maximum principle N4 can be reformulated as follows: there exists a $\psi\in S^{n-1}$ such that if $u_x(t)=w_i$, then $\varphi^{*t}(\psi)\in W_i^0$ and N5, N6 can be reformulated in a corresponding way.

For $x\in G$ denote $\vartheta(x)$ the number of switchings (discontinuities) of the optimal control $u_x(t)$. We have

Lemma 8. For every $T_0>0$ there exists a $\Theta>0$ such that $\vartheta(x)<\Theta$ as soon as $T(x)<T_0$.

Proof. From N3 it follows that the switching points are found among the zeros of the functions $\langle\psi(t), w_i-w_j\rangle$, where $\psi(t)$ is the solution of (5) with respect to which $u_x(t)$ is extremal. Therefore, the lemma will be proved if we show that for given T_0, i, j there exists an $N>0$ such that the number of zeros of $\langle\psi(t), w_i-w_j\rangle$ on any interval of length T_0 does not exceed N for any non-zero solution $\psi(t)$ of (5).

For given i, j denote $w=w_i-w_j$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi, w\rangle=\langle -A^*\psi, w\rangle=-\langle\psi, Aw\rangle,$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle\psi, w\rangle=\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi, Aw\rangle=-\langle -A^*\psi, Aw\rangle=\langle\psi, A^2w\rangle, \ldots,$$

$$\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\langle\psi, w\rangle=(-1)^{n-1}\langle\psi, A^{n-1}w\rangle, \frac{\mathrm{d}^n}{\mathrm{d}t^n}\langle\psi, w\rangle=(-1)^n\langle\psi, A^nw\rangle.$$

From the Cayley—Hamilton theorem it follows that

$$A^n = -a_{n-1}A^{n-1} - \dots - a_0 I,$$

where $\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0$ is the characteristic polynomial of A and I is the unity matrix. Thus, if we denote $(-1)^j\langle\psi, A^j w\rangle = y_{j+1}$, $j=0, \dots, n-1$, we have $\dot{y}_1 = y_2$, $\dot{y}_2 = y_3$, $\dots$, $\dot{y}_{n-1} = y_n$, $\dot{y}_n = -a_{n-1}y_{n-1} - \dots - a_0 y_1$, which means that $\langle\psi, w\rangle = y_1$ is a non-zero solution of the n-th order differential equation

(8) $$y_1^{(n)} + a_{n-1}y_1^{(n-1)} + \dots + a_0 y_1 = 0.$$

By de la Vallé—Poussin's theorem (cf. [12, IV, § 1.2]), there exists an $h > 0$ such that if a solution of (8) has n zeros on an interval of length h, then it is identically zero. As a conseqence we obtain that on an interval of length T_0, $y_1(t) - \langle\psi(t), w\rangle$ cannot have more than $n([T_0/h]+1)$ zeros ([ ] standing for the integer part).

From the continuity of the function $T(x)$ and Lemma 8 we obtain

Corollary 3. The function $T(x) + \vartheta(x)$ is bounded on any $K \subset G$ compact.

6. Main theorem

Theorem 1. *Every normal system* (4) *with target point* 0 *admits a regular time — optimal synthesis* (φ, v) *which has the following additional property*

F. For every cell $S \in \mathcal{S}$ of dimension $< n$, every vector field $X^j = Ax + w_j$ such that $w_j \neq v(x)$ for $x \in S$ is everywhere transversal to S.

Let us note that from N3 and B it follows that for a given $S \in \mathcal{S}$, $v(x)$ must be constant and equal to some vertex of U which we denote by $w_{\mu(S)}$.

Proof. Denote G the set of points from which the system can be steered to 0. By N1, G is an open neighbourhood of 0. We take v as in N3. Following the optimal trajectories backwards we shall construct inductively the cells of the partition $\mathcal{S}$ We show that they are CASA sets satisfying B – D and F and that each compact subset of G is covered by a finite number of those cells. Property A then follows from Lemma 5 and E holds by N2.

Since we shall mostly follow the trajectories of the vector fields X^j backwards, we denote by φ^j the backwards flow of X^j, $j=1, \dots, p$, i.e. the flo of the differential equation $\dot{x} = -Ax - w_j$.

A synthesis cell S will be called a *descendant* of a cell S' if the optimal trajectories of the points of S pass S'. Also, S' will be called an *ancestor* of S

We shall call the cells constructed at the k-th induction step cells of order k, the only cell of order 0 being $\{0\}$. We denote by $\mathcal{S}_k$ the set of cells of order $\leq k$ and by G_k their union. Among the cells of $\mathcal{S}_k$ we distinguish a certain class $\mathcal{S}_k$ of cells of type II which we shall call *border cells.*

Assume now that we have constructed $\mathcal{S}_k$ and that $\mathcal{S}_k$ satisfies the following induction hypotheses:

I1. G_k is closed and $\mathcal{S}_k$ is a finite partition of G_k into CASA sets.

I2. With every cell $S \in \mathcal{S}_k$ there is an associated integer $\mu(S) \in [1, p]$ such that $v(x) = w_{\mu(S)}$ for $x \in S$.

I3. $\mathcal{S}_k$ satisfies B – F.

I4. $T(x) \leq k$ for every $x \in G_k$.

I5. The set $E_S = \{(x, \psi) | x \in S, \psi \in E^0(x)\}$ is subanalytic.

I6. If S is a border cell of order k, then $\xi_x(t) \in G_k \backslash G_{k-1}$ for $t \in [0, 1)$, $\xi_x(1) \in G_{k-1}$ for every $x \in S$.

I7. If $u_x(t)$ is constant for $0 \leq t < t_1 \leq 1$ and $\xi_x(t_1) \in G_{k-1}$, then $x \in G_k$.

I8. If S is order $i \leq k$, its descendants are of order $\geq i$, its immediate descendant being of order $\leq i+1$. If S' is a descendant of S of order i, then $\mu(S') = \mu(S)$.

I9. If $S \in \mathcal{S}_k$ and not a border cell, $u_x(t) \in S$ for some $x \in G$ and $t > 0$ and u_x is constant in the neighbourhood of t, then $\xi_x(s) \in G_k$ for $s < t$ sufficiently close to t.

Each cell of order $k+1$ will be a descendant of some cell of order k. The descendants of a particular cell will be grouped into families associated with particular vertices w_j, $j = 1, \ldots, p$ (some of which may be empty). We specify how to obtain descendants of order $k+1$ of a given cell S of order k associated with a given vertex w_j. All the cells of order $k+1$ are then obtained if S runs through all cells of order k and j through $1, \ldots, p$, the pair (S, j) excluded if both $j = \mu(S)$ and S is not a border cell. Why we exclude such pairs we shall explain later.

The general idea behind the construction is the following one:

We denote by L the set of points y for which the optimal control satisfies $u_y(t) = w_j$ for $t \in [0, \tau)$ for $0 < \tau \leq 1$ and the optimal trajectory satisfies $\xi_y(\tau) \in S$, $\xi_y(t) \notin S$ for $0 < t < \tau$. By Corollary 1, L is precisely the set filled by the pieces of trajectories $x(t, x, u^{\psi})$, $t \in [\tau, 0)$ (in the notation of Corollary 1), where τ is such that $u^{\psi}(t) = w_j$ for $t \in [\tau, 0)$ or, equivalently, $\psi(t) \in W_j$, for $\psi(0)$ running over $E^0(x)$ and x over S (for the definition of $E^0(x)$, W_j cf. Remark 1 and N4 of §4, respectively). We prove that L is subanalytic and we obtain the descendants of S of order $k+1$ associated with w_j as the components of the partition of L into CASA sets. In order to satisfy C and F we have to split L in such a way that the partition of L into cells is flow consistent with respect to the flows $X^1, \ldots, X^p$ and that all cells have uniquely defined ancestors. Let us note that the restriction $\tau \leq 1$ is made for technical reasons to enable us to prove the subanalyticity of L. The bound 1 could be replaced by any other positive constant.

Throughout the proof we shall keep the following notation: If $A = A_1 \times A_2 \times \ldots \times A_r$ and $1 \leq i_1 < \ldots < i_s \leq r$, by $\pi_{i_1 \ldots i_s}$ we shall denote the natural projection of $A_1 \times \ldots \times A_r$ on $A_{i_1} \times \ldots \times A_{i_s}$. Note that if A_j are analytic manifolds for $j = 1, \ldots, r$ and compact for $j \in \{1, \ldots, r\} \backslash \{i_1, \ldots, i_s\}$, then $\pi_{i_1 \ldots i_s}$ is proper analytic.

For given S, j denote

$$H_1=\{(t, \varphi_t^j(x), \varphi_t^*(\varphi))|(x, \psi)\in E_S, 0<t<1\},$$

$$H_2=\{(s, t, \varphi_t^j(x), \varphi_t^*(\psi))|(x, \psi)\in E_S, 0<s\leqslant t<1,$$
$$\varphi_s^*(\psi)\in S^{n-1}\backslash W_j^0\},$$

$$B=\pi_{23}(H_1\backslash\pi_{234}(H_2)),$$

$$K=\pi_1(B)$$

$$H_1'=\{(\varphi_1^j(x), \varphi_1^*(\psi))|(x, \psi)\in E_S\},$$

$$H_2'=\{(s, \varphi_1^j(x)\ \varphi_1^*(\psi))|0<s<1, \varphi_s^*(\psi)\in S^{n\ \ 1}\ \ W_j^0\},$$

$$B'=H_1'\backslash\pi_{23}(H_2'),$$

$$K'=\pi_1(B'),$$

$$L=K\cup K',$$

$$\tilde{L}=\pi_{12}(H_1\backslash\pi_{234}(H_2))\cup(\{1\}\times K').$$

Analysing these expressions we see that $B(B')$ is the set of poınts $(\varphi_t^j(x), \varphi_t^*(\psi))$ for which $\psi\in E^0(x)$, $x\in S$, $0<t<1$ $(t=1)$ and $\varphi_s^*(\psi)\in W_j^0$ for $0<s\leqslant t$. From Corollary 1 it follows that its projection on the first compound, $K(K')$, is the set of points x for which $u_x(t)=w_j$ for $t\in[0, \tau(x))$, $\tau(x)>0$, where $\tau(x)$ is such that $\xi_x(\tau(x))\in S$ ($u_x(t)=w_j$ for $t\in[0, 1)$, $\xi_x(1)\in S$, respectively). Of course, K, K' may be empty. The sets B, B' can be considered as fibred sets over K, K', respectively, the fiber over x being $E^0(x)$. The set $\tilde{L}$ is a subset of $R^1\times L$ of the points $(\tau(x), x)$, $x\in L$.

We show that B, K are subanalytic; the subanalyticity of B', K' follows in a similar way. First, we note that $H_1=F((0, 1)\times E\ )$, where $F(t, x, \psi) = (t, \varphi_t^j(x), \varphi_t^*(x))$. Since F is an analytic diffeomorphism, it is proper and H_1 is subanalytic by SA4. To prove the subanalyticity of H we first note that $W_j^0=\{\psi|\langle\psi, \psi\rangle=1, \langle\psi, w_j-w_v\rangle\geqslant 0, v=1, \ldots, p\}$ ıs semianalytic and, by SA1, so is $S^{n-1}\backslash W_j^0$. By SA3, the set $\{(\tau, \psi)|\varphi_\tau^*(\psi)\in S^{n-1}\}$ is semianalytic and, by SA2, it is subanalytic. The subanalyticity of H_2 now follows in a similar way as that of H_1 The set B is subanalytic by Lemma 2 and K is subanalytic because S^{n-1} ıs compact and, consequently, the projection map π_1: $R^n\times S^{n-1}\to R^n$ i proper

The descendants of S of order $k+1$ associated with j will be obtained by a sequence of partitions of K and K', the subsets of K' becoming the border cells.

We first split K and K' (each of them separately) according to Lemma 4 to obtain a flow-consistent partition $\mathscr{R}_1$ of L into CASA sets with respect to the vector fields $X^1, \ldots, X^p$. We call the components of $\mathscr{R}_1$ to which X^j is tangent (transversal) components of type I (II) and denote by $\mathscr{R}_1'$ ($\mathscr{R}_1''$, respectively) the set of those components.

We could take the components of $\mathscr{R}_1$ as cells of order $k+1$ if they had uniquely defined ancestors. This, however, is not necessarily true and in order to achieve this, further partitioning may be necessary. This will be done as follows:

We project all the components of $\mathscr{R}_1$ on S along the trajectories of φ^j, split the projections into pairwise disjoint sets to obtain a partition $\mathscr{P}$ of S, the components of which are subsets of all the projections of components of $\mathscr{R}_1$ they intersect. Finally, we take as the components of the refinement $\mathscr{R}_2$ of $\mathscr{R}_1$ the intersections of the components of $\mathscr{R}_1$ with pre-images (under the projection map) of the components of $\mathscr{P}$. We show that this can be done in such a way that the components of $\mathscr{R}_2$ are CASA sets.

Denote by Φ the projection map of L on S along the trajectories of φ^j and note that $\Phi(x)=\varphi^j_{-\tau(x)}(x)$, where $\tau(x)$ is the least t such that $\xi_x(t)\in S$. The projection $\Phi(R)$ of a set $R\in\mathscr{R}_1$ can be written as $\Phi(R)=\pi_2\{(t,\ \varphi^j_{-t}(x)|(t,x)\in \tilde{L}\cap(R^1\times R)\}$. From this expression it follows that $\Phi(R)$ is subanalytic. Denote by $\mathscr{P}_1$ the partition of S into the sets $\Phi(R_1)\cap\ldots\ \Phi(R_k)\cap(S\setminus\Phi(R_{k+1}))\ \cap\ldots\cap (S\setminus\Phi(R_l))$, where $k\leq l$ run through all nonnegative integers and $R_1,\ldots,R_l$ through all finite sequences of components of $\mathscr{R}_1$. Obviously, $\mathscr{P}_1$ consists of subanalytic sets and has the property

$$\text{(9)}\qquad \text{If } P\in\mathscr{P}_1,\, P\cap\Phi(R)\neq\emptyset \text{ for some } R\in\mathscr{R}_1,\text{ then } P\subset\Phi(R).$$

We can further refine $\mathscr{P}_1$ to obtain a partition $\mathscr{P}$ of S into CASA sets having property (9) and we define the refinement $\mathscr{R}_2$ of $\mathscr{R}_1$ by taking the connected components of the sets $\Phi^{-1}(P)\cap R$, $R\in\mathscr{R}_1$, $P\in\mathscr{P}$ as its components. The subanalyticity of the components of $\mathscr{R}_2$ follows from the expression

$$\Phi^{-1}(P)\cap R=\pi_2(\{(t,\ \varphi^j_t(x))|0<t\leq 1,\, x\in P\}\cap\tilde{L}\cap(R^1\times R).$$

Furthermore, we show that $\Phi^{-1}(P)\cap R$ is also an analytic submanifold of R^n, which implies that $\mathscr{R}_2$ consists of CASA sets.

For this proof we need

Lemma 9. *The function τ is continuous.*

Proof. Assume the contrary. Then there exists a sequence of points $x_\nu\to x_0$, $x_\nu\in L$, $\nu=0,1,\ldots$, such that

$$\text{(10)}\qquad \tau(x_\nu)\not\to\tau(x_0).$$

Since $\{\tau(x_\nu)\}$ is a bounded sequence we may assume $\tau(x_\nu)\to\tau^*$.

Since G_k is closed, $\tau(x_0)\leq\tau^*$. To prove that $\tau(x_0)<\tau^*$, two cases have to be distinguished, namely $j=\mu(S)$ and $j\neq\mu(S)$.

Let first $j\neq\mu(S)$. Since u_{x_0} is right-hand continuous we have $u_{x_0}(t)=w_{(S)}$ for

$t \in [\tau(x_0), \tau(x_0)+\varepsilon]$ for some $\varepsilon > 0$ (here we understand $\Sigma(S) = \emptyset$ if S is of type I). By [5, Lemma 3.2] the open-loop controls u_x (if extended by 0 beyond $T(x_v)$) convergence weakly towards u_{x_0} for $x \to x_0$, which implies $u_{x_v}(t_v) = w_l$ for some $t_v \to \tau(x_0)$. This is impossible if $\tau^* > \tau(x_0)$.

If $j = \mu(S)$, then S is a border cell. It follows from I6 that if i is the largest possible integer such that the closest ancestor S' of S of order i is not a border cell and $\tau_1(x)$ is the time at which $\xi_x(t)$ enters S' for $x \in L$, then $\tau_1(x_v) - \tau(x_v) = \tau_1(x_0) - \tau(x_0) = k - i$. Therefore, if $\tau(x_v) \nrightarrow \tau(x_0)$, then also $\tau_1(x_v) \nrightarrow \tau_1(x_0)$. However, since S' is not a border cell, we have by I9 $u_{x_v}(t_1) \neq u_{x_v}(t_2)$ for $t_1 < \tau_1(x_v) \leq t_2$ and the contradiction to weak convergence follows similarly as in the case $j \neq \mu(S)$. This proves the lemma.

Since S is transversal to X^j, from Lemma 9 and the implicit function theorem it follows that τ is analytic and, consequently, that Φ is analytic.

Let now $R_1 \in \mathscr{R}_1''$. Since R_1 is transversal to φ^j, the projection map Φ is a local diffeomorphism $R_1 \to \Phi(R_1)$. Let $x \in R$. Then, $\Phi(x) \in P$ and, since P is a submanifold of R^n, it can locally at $\Phi(x)$ be represented as $P = \{y \mid f_1(y) = 0, \dots, f_r(y) = 0\}$, where $r = n - \dim P$ and the differentials Df_v of f_v at $\Phi(x)$ are independent, $v = 1, \dots, r$. Since Φ is a diffeomorphism, the differentials $D(f_v \circ \Phi) = Df_v \cdot D\Phi$ are independent. As R can locally at x be represented as $R = \{y \mid f_1 \circ \Phi(y) = 0, \dots, f_r \circ \Phi(y) = 0\}$, this proves that R is an analytic submanifold.

If $R_1 \in \mathscr{R}_1'$, then every $x \in R_1$ has a neighbourhood V such that $R_1 \cap V = \varphi_I^j(Z)$, where $I = (-\varepsilon, \varepsilon)$, $\varepsilon > 0$ and Z is an analytic submanifold transversal to X^j. Similarly as in the case $R_1 \in \mathscr{R}_1''$ (R_1 replaced by Z) it can be shown that $\Phi^{-1}(P) \cap V = \varphi_I^j(Z \cap \Phi^{-1}(P))$ is an analytic submanifold.

The fact that the components of $\mathscr{R}_2$ have uniquely defined ancestors (and descendants as well) follows from the property (9) of $\mathscr{P}$ and the fact that the partitioning of $\mathscr{R}_1$ has been done along the trajectories of φ^j. Moreover, it follows from the construction that every component of $\mathscr{R}_2$ is isomorphic with all its ancestors and descendants of the same type in $\mathscr{R}_2$. However, by the partition of $\mathscr{R}_1$ we may have destroyed flow consistency. More precisely, by partitioning a component R_1 of $\mathscr{R}_1''$ to which some X^v, $v \neq j$ was parallel, a component R_2 of $\mathscr{R}_2$ may be obtained with which X^v is not flow consistent. This cannot happen for $j = v$ (i.e. $R_1 \in \mathscr{R}_1'$), since splitting has been done along the trajectories of φ^j (because of the construction of $\mathscr{R}_1'$ and Lemma 6).

If this is the case, we repeat the cycle of flow consistent partitioning and subsequent partitioning into sets with uniquely defined ancestors. We show that every such cycle of two partitionings lowers the maximum of the dimensions of flow inconsistent components by at least one. This means that after a finite number of repetitions a flow consistent partition $\mathscr{R}$ with uniquely defined ancestors will be obtained.

Let R be a flow inconsistent component of $\mathscr{R}_2$. Then it is necessarily of type II. If we partition R into flow consistent subsets according to Lemma 4, from the subsequent partition into CASA sets with uniquely defined ancestors this will require at most a partition of the descendants and ancestors of R. Since those of them which are of type II are isomorphic to R their dimensions do not exceed $\dim R$. Consequently, since transversality is not destroyed by partitioning and parallelity can be lost only on the parts of lower dimension the flow inconsistent sets resulting from their partition will have dimension $< \dim R$. The assertion now follows if we let R run through all flow inconsistent components of $\mathscr{R}_2$ and if we admit for $\mathscr{R}_2$ the partition obtained after any repetition of the cycle.

As the cells of order $k+1$ we take the components of $\mathscr{R}$, the components of type I (II) becoming cells of order I (II, respectively), the subsets of K' becoming border cells. For $R \in \mathscr{R}$ we denote $E_R = (B \cup B') \cap (R \times S^{n-1})$. As mentioned at the beginning of the proof, to obtain all the cells of order $k+1$ we let S run through all the cells of order k and j through $1, \dots, p$, the pair (S, j) excluded if S is not a border cell and $j = \mu(S)$.

Next we show that $\mathscr{S}_{k+1}$ satisfies the induction hypothese I1—I9.

To prove that G_{k+1} is closed assume $x_j \in G_{k+1}$, $x_j \to x_0$. Since G_k is closed and $\mathscr{S}_{k+1}$ is finite we may assume $x_j \in S'$ for some $S' \in \mathscr{S}_{k+1}$. Let S be the closest ancestor of S' of order k and let τ be defined as in Lemma 9. By Lemma 9, τ is continuous and therefore $\tau(x_j) \to \tau^*$ for some $0 \leqslant \tau^* \leqslant 1$. Since u_{x_j} converge weakly to u_{x_0} and $u_{x_j}(t) = w_{\mu(S')}$ for $t \in [0, \tau_j)$, $u_{x_0}(t) = w_{\mu(S')}$ for $t \in [0, \tau^*]$ and $\xi_{x_j}(\tau_j) \to \xi_{x_0}(\tau^*)$. This implies $\xi_{x_0}(\tau^*) \in G_k$ and $x_0 \in L$ for the set L associated with the pair $(S'', \mu(S'))$, where S'' is the cell of order k containing $\xi_{x_0}(\tau^*)$.

The rest of I1, as well as I2, B and I8 are satisfied trivially. Due to Corollary 2, C and F are immediate consequences of the flow consistency of the partition of L into cells and the fact that each cell has uniquely defined ancestors. The property D (for $x \in G_{k+1}$) follows from the finiteness of $\mathscr{S}_k$ and $\mathscr{R}$, I4 follows from the restriction $t < 1$ in the definition of H_1. The property I5 follows directly from the definition of E_R, $R \in \mathscr{R}$ and I6 is a consequence of the fact that the border cells are subsets of K'.

Since G_k is closed, we may assume for the proof of I7 without loss of generality that $x' = \xi_x(t_1) \in S \subset G_k$, but $\xi_x(t) \notin G_k$ for $t < t_1$. Let $u_x(t) = w_j$ for $t \in [0, t_2)$. Then there exists a $\psi \in E^0(x)$ such that $u_x(t) = u^{\psi}(t - t_1)$ in the notation of Corollary 1. Since $t_1 < 1$ and $u_x(t) = w_j$ on $[0, t_1)$, this means $\xi_x(t) \in L$ for the pair (S, j). Because of the induction hypothesis I9 either $j \neq \mu(S)$ or S is a border cell, which means that the pair (S, j) is admissible.

To prove I9 observe that if $x' = \xi_x(t) \in S$, where S is of order $k+1$ and not a border cell, then $x' \in L$ for some admissible pair (S', j), where S' is of order k. This means $x = \varphi^j_{x'}(\tau)$ for some $x' \in S'$ and $0 < \tau < 1$. Since u_x is constant in the neighbourhood of t and $\tau<1$, $\varphi^{i}_{x'}(\sigma)\in L$ for $\sigma>\tau$ sufficiently close to τ or, equivalently, $\xi_x(s)\in L$ for $s<t$ sufficiently close to t.

Now, by induction it follows that $\mathcal{S}=\bigcup_{k\geq 0}\mathcal{S}_k$ is a regular synthesis provided we prove that every $C\subset G$ compact is covered by a finite number of cells of $\mathcal{S}$. This is an immediate consequence of Corollary 3 and the following lemma, which concludes the proof of the theorem.

Lemma 10. *Let $x\in G$. Then, $x\in G_{T(x)+\vartheta(x)}$, i.e. x is contained in some cell of order $\leq T(x)+\vartheta(x)$, where T, ϑ are defined in Section 5.*

Proof. We prove this lemma by induction in $T(x)+\vartheta(x)$. If $T(x)=0$, then $x=0$ and the statement is trivial. Assume that the statement of the lemma holds for all x with $T(x)+\vartheta(x)\leq i+1$.

Let $x\in G$ be such that $T(x)+\vartheta(x)\leq i+1$. Denote t_1 the first switching point of u_x, $\tau=\min\{1, t_1\}$ and $x'=\xi_x(\tau)$. Then, $T(x')+\vartheta(x')\leq i$ and by the induction hypothesis $x'\in G_i$. By I7, $x\in G_{i+1}$, which proves the lemma.

7. Filippov trajectories

As we have mentioned in Section 1, the existence of regular synthesis has an application to the problem of coincidence of the open-loop optimal trajectories of (4) and the Filippov trajectories of the equation

$$\dot{x}=Ax+v(x). \tag{11}$$

This problem has been studied in [4—6]. As shown in [6] it is not always true that the Filippov trajectories of (11) coincide with the optimal trajectories of (4) and in [4] the problem, for which systems the coincidence does take place, is completely solved for systems of dimension 2. The results demonstrate that the class of systems for which coincidence does not take place is not exceptional. While there is no general solution to the problem for systems of higher dimension, in [5] it is proved that for systems of arbitrary dimension with scalar control (i.e. $U=\{bu\mid |u|\leq 1\}$, $b\in R^n$) the optimal trajectories are Filippov trajectories of (11), provided the system admits a regular synthesis. The converse statement is proved under the additional hypothesis requiring that for each cell S of type I $Ax-bv(x)$ is everywhere transversal to S.

Because this hypothesis is a consequence of property F, using Theorem 1 we obtain from [5]

Theorem 2. *For any normal system with scalar control all optimal trajectories of* (4) *are Filippov trajectories of* (11) *and, conversely, all Filippov trajectories of* (11) *are optimal trajectories of* (4).

8. Concluding remarks.

If a system admits a regular synthesis, then it admits infinitely many regular syntheses. Indeed, if $(\mathscr{S}, v)$ is a regular synthesis and $\mathscr{S}'$ is a flow consistent locally finite refinement of $\mathscr{S}$ the components of which have uniquely defined ancestors, then $(\mathscr{S}', v)$ is again a regular synthesis. This suggests an ordering in the family of regular syntheses: $(\mathscr{S}, v) \geq (\mathscr{S}', v')$ if $v = v'$ and $\mathscr{S}'$ is a refinement of $\mathscr{S}$. It is quite obvious that every increasing sequence of syntheses has a maximal element, which by Zorn's lemma implies the existence of a maximal synthesis, i.e. a synthesis $(\hat{\mathscr{S}}, \hat{v})$ such that no other synthesis $(\mathscr{S}, v)$ satisfies $(\mathscr{S}, v) \geq (\hat{\mathscr{S}}, \hat{v})$.

In the constructions of Section 6 we have done many redundant splittings, which have not been dictated by the structure of the closed-loop optimal control but rather by the technique of proof. Simple examples for which the closed loop optimal control can be explicitely constructed indicate the existence of a unique "natural" synthesis $(\hat{\mathscr{S}}, \hat{v})$, which satisfies $(\hat{\mathscr{S}}, \hat{v}) \geq (\mathscr{S}, v)$ for all regular syntheses $(\mathscr{S}, v)$. Such a synthesis reflects the structure of the closed-loop optimal control in the most adequate way. Although all the regular syntheses which have been constructed for particular systems are of this natural type, we have not been able to prove that such a synthesis exists in general.

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ВСЯКАЯ ЛИНЕЙНАЯ СИСТЕМА В ОБЩЕМ ПОЛОЖЕНИИ ОБЛАДАЕТ РЕГУЛЯРНЫМ ОПТИМАЛЬНЫМ ПО ВЫСТРОДЕИСТВИЮ СИНТЕЗОМ

Павол Бруновски

Резюме

Доказывается, что если несущественным образом изменить понятие регулярного (по Болтянскому) синтеза оптимального управления, то всякая линейная система с многогранниковои областью управления обладает регулярным синтезом управления, оптимального по быстро ействию В доказательстве используется теорема Хиронаки о разбиении субаналитического множества на локально конечное семейство аналитических подмногообразий.

Одновременно доказывается, что можно регулярныи синтез построить так, что он имеет некоторые дальнеишие свойства, которые позволяют доказать, что в случае системы со скалярным управлением в общем положении, все оптимальные траектории являются траекториями системы с оптимальнои обратнои связью в смысле Филиппова и наоборот.