Mathematica Slovaca

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Mathematica Slovaca, Vol. 49 (1999), No. 2, 189--199

Persistent URL: http://dml.cz/dmlcz/131885

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Math. Slovaca, 49 (1999), No. 2, 189–199

Mathematica Slovaca
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CONVERGENCE PRESERVING PERMUTATIONS OF $\mathbb{N}$ AND FRÉCHET'S SPACE OF PERMUTATIONS OF $\mathbb{N}$

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(*Communicated by Stanislav Jakubec*)

ABSTRACT. Denote by P the metric space of all permutations of the set $\mathbb{N}$ of positive integers with Fréchet's metric. Let P_0 be the set of all $p = (p_n)_{n=1}^{\infty} \in P$ that preserve convergence of infinite series (i.e., if $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} a_{p_n}$ is also convergent). In the paper, topological and porosity properties of the set P_0 and related sets are investigated.

Introduction

Permutations of $\mathbb{N}$ preserving convergence of infinite series are studied in [2], [7], [9], [12], [13], [14]. These permutations form the set P_0. Denote by P the set of all permutations of the set $\mathbb{N} = \{1, 2, \dots, n, \dots\}$.

A permutation $q = \{q_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ is said to *preserve the convergence of infinite series* if for any convergent series $a = \sum_{k=1}^{\infty} a_k$ with real terms the series $\sum_{k=1}^{\infty} a_{q_k}$ converges and $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{q_k}$.

In [1], [3], [7], [9], [13], [14], a characterization of elements of P_0 as a subset of P is given and several interesting properties of P_0 are established. In this paper, we shall study some further properties of P_0 and the structure of the metric space P with Fréchet's metric. This space was introduced in [1] (see also [11]).

The paper is divided into two parts. In the first part we shall give some further results about P_0 and P not contained in [1], [3], [8] and [11]. These results

AMS Subject Classification (1991): Primary 40A05, 54A10.
Key words: convergence preserving permutation, residual set, uniformly distributed sequence, porous set, σ-1-porous set, σ-very porous set.

concern the structure of the space (P, d), d being the Fréchet's metric. In the second part we shall apply the concept of the porosity of sets for making more precise and completing results on (P, d) proved by other authors. Topological terminology in accordance with [5] will be used.

Notations and characterization of elements of P_0

The Fréchet's metric on P is defined as follows:

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} \frac{|x_k - y_k|}{1 + |x_k - y_k|},$$

$x = (x_k)_{k=1}^{\infty} \in P$, $y = (y_k)_{k=1}^{\infty} \in P$ (cf. [1], [8], [10]).

It is well known that (P, d) is not a complete metric space, but it is of the second category at every its point (cf. [1]).

In what follows we shall use the following well-known result (cf. [1], [8], [11]).

THEOREM A. *Let $a = \sum\limits_{k=1}^{\infty} a_k$ be a non-absolutely convergent series. Then the set of all $x = (x_k)_{k=1}^{\infty} \in P$ with*

$$\liminf_{n\to\infty} s_n(x,a) = -\infty, \qquad \limsup_{n\to\infty} s_n(x,a) = +\infty$$

is residual in P, where $s_n(x,a) = \sum\limits_{k=1}^{n} a_{x_k}$ $(n = 1, 2, \dots)$.

Similar statements are also obtained in [3] for P endowed by Baire's metric.

For $x \in P$ and $\delta > 0$, denote by $K(x, \delta)$ the ball in P with centre x and radius $\delta > 0$, i.e. $K(x,\delta) = \{y \in P;\ d(x,y) < \delta\}$.

If $\sum\limits_{k=1}^{\infty} a_k$ is a convergent series, denote by $P_0(a_1, a_2, \dots)$ the set of all $q = (q_k)_{k=1}^{\infty} \in P$ which preserve convergence of $\sum\limits_{k=1}^{\infty} a_k$, i.e. for which $\sum\limits_{k=1}^{\infty} a_{q_k}$ converges and has the same sum as $\sum\limits_{k=1}^{\infty} a_k$. A well-known result of elementary analysis says that if $\sum\limits_{k=1}^{\infty} a_k$ converges absolutely then $P_0(a_1, a_2, \dots) = P$.

By the definition of P_0 given in the Introduction we have

$$P_0 = \bigcap P_0(a_1, a_2, \dots), \tag{1}$$

where the intersection on the right-hand side of (1) is taken over all convergent series $\sum_{k=1}^{\infty} a_k$, or equivalently all non-absolutely convergent series (with real terms).

A finite set of positive integers of the form $\{b+1, b+2, \ldots, b+s\}$ ($b \in \{0\} \cup \mathbb{N}$, $s \in \mathbb{N}$) is called a block of positive integers. In papers [1], [3], [7], [9], [13], [14] the following characterization of elements of P_0 is given.

THEOREM B. *A permutation $q = (q_k)_{k=1}^{\infty} \in P$ belongs to P_0 if and only if it satisfies the following condition:*

(a) *There exists a positive integer M such that for each $n \in \mathbb{N}$ the set $\{q_1, q_2, \ldots, q_n\}$ can be expressed as a union of not more than M blocks of positive integers.*

In what follows $|A|$ stands for the cardinality of the set A.

1. Properties of P_0 and the structure of the space P

According to a result of [12], if $\sum_{k=1}^{\infty} a_k$ is a non-absolutely convergent series, then both sets $P_0(a_1, a_2, \ldots)$, $P \setminus P_0(a_1, a_2, \ldots)$ are dense in P. This result suggests the study of further topological properties of P_0.

THEOREM 1.1. *The sets P_0, $P \setminus P_0$ are dense in P, the set P_0 being an F_σ-set of the first Baire category in P.*

P r o o f. We prove the density of P_0 in P. It suffices to show that if $K(q, \delta)$ is an arbitrary ball in P, then $P_0 \cap K(q, \delta) \neq \emptyset$, ($q = (q_k)_{k=1}^{\infty} \in P$).

Choose $m \in \mathbb{N}$ such that $2^{-m} < \delta$ and put $b = \max\{q_1, q_2, \ldots, q_m\}$. Let $\nu_1, \nu_2, \ldots, \nu_j$ be positive integers not exceeding b which are not contained in $\{q_1, q_2, \ldots, q_m\}$. Construct the sequence $r = (r_k)_{k=1}^{\infty}$ as follows:

$$r = q_1, q_2, \ldots, q_m, \nu_1, \nu_2, \ldots, \nu_j, b+1, b+2, \ldots, b+n, \ldots .$$

Obviously $r \in P_0$ and

$$d(r, q) = \sum_{k=m+1}^{\infty} 2^{-k} \frac{|r_k - q_k|}{1 + |r_k - q_k|} \leq 2^{-m} < \delta .$$

The density of $P \setminus P_0$ follows from the density of the set $P \setminus P_0(a_1, a_2, \ldots)$, $\sum_{k=1}^{\infty} a_k$ being an arbitrary non-absolutely convergent series using the obvious inclusion $P_0 \subseteq P_0(a_1, a_2, \ldots)$ (see (1)).

We show that P_0 is an F_σ-set in P. For $v \in \mathbb{N}$ denote by $P_0(\nu)$ the set of all $q = (q_k)_{k=1}^{\infty} \in P_0$ satisfying condition (a) in Theorem B with constant $M = \nu$. Then we have

$$P_0 = \bigcup_{\nu=1}^{\infty} P_0(v) . \tag{2}$$

Hence it suffices to show that the sets $P_0(\nu)$ $(\nu \in \mathbb{N})$ are closed in P. But this is an easy consequence of the fact that convergence in P is equivalent to the coordinate-wise convergence.

By Theorem A the set $P_0(a_1, a_2, \dots)$ is of the first category if $\sum_{k=1}^{\infty} a_k$ is a non-absolutely convergent series. Hence, by (1), P_0 is also a set of the first category. □

Remark 1.1. If we consider the set P as an algebraic structure with operation $\oplus$, where $\oplus$ denotes composition of permutations, then $(P, \oplus)$ is obviously a group. It is easy to see that $(P_0, \oplus)$ is a semigroup, but it is not a group (cf. [7], [9], [14]). If $\sum_{k=1}^{\infty} a_k$ is a non-absolutely convergent series, then $P_0(a_1, a_2, \dots)$ is not a semigroup (cf. [7]). For each non-absolutely convergent series $\sum_{k=1}^{\infty} a_k$ we have

$$P_0 \subsetneq P_0(a_1, a_2, \dots) .$$

These sets are different since P_0 is a semigroup, but $P_0(a_1, a_2, \dots)$ is not a semigroup.

The definition of convergence preserving permutation can be stated in the following shorter form:

A permutation $q = (q_k)_{k=1}^{\infty} \in P$ is *convergence preserving* if for every convergent series $\sum_{k=1}^{\infty} a_k$ the series $\sum_{k=1}^{\infty} a_{q_k}$ also converges.

This observation is an easy consequence of the following proposition which has been proved in [7]. Here we give an another proof of it.

PROPOSITION 1.1. *Suppose that $q = (q_k)_{k=1}^{\infty} \in P$ satisfies the condition:*

(b) *If $\sum_{k=1}^{\infty} a_k$ is a convergent series, then $\sum_{k=1}^{\infty} a_{q_k}$ is also convergent.*

Then for every convergent series $\sum_{k=1}^{\infty} b_k$ we have

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} b_{q_k} . \tag{3}$$

P r o o f . In the proof of Theorem B in [13] (see the proof of Theorem 2 of [13]) the author has shown that if a permutation $x = (x_n)_{n=1}^{\infty} \in P$ does not satisfy condition (a) of Theorem B then there exists a convergent series $\sum_{k=1}^{\infty} a_k$ such that $\sum_{k=1}^{\infty} a_{x_k}$ does not converge. Hence (b) implies (a). But then, by Theorem B, we obtain the assertion of our Proposition (the equality (3)). □

At the end of this part of the paper we shall investigate the intersection of P_0 with the set U of all uniformly distributed sequences of positive integers.

Recall that a sequence $x = (x_n)_{n=1}^{\infty}$ of positive integers is said to be *uniformly distributed* mod m $(m \geq 2)$ provided that

$$\lim_{n\to\infty} \frac{A(j,m,n,x)}{n} = \frac{1}{m} \qquad (j = 1,2,\dots,m), \tag{4}$$

where $A(j,m,n,x)$ denotes the number of all x_k $(1 \leq k \leq n)$ with $x_k \equiv j \pmod{m}$. A sequence of positive integers is said to be *uniformly distributed* (in $\mathbb{Z}$) if it is uniformly distributed mod m for every $m = 2,3,\dots$ (cf. [4; p. 305]).

It is proved in [6] that the set U of all uniformly distributed sequences of positive integers is a dense set of the first category in the Baire space S of all sequences of positive integers ([6; Theorem 2]). The Baire metric d_1 in S is defined as follows: $d_1(x,y) = \frac{1}{m}$, where $m = \min\{k;\ x_k \neq y_k\}$ for $x \neq y$ and $d_1(x,x) = 0$ $(x = (x_k)_{k=1}^{\infty} \in S,\ y = (y_k)_{k=1}^{\infty} \in S)$.

The metrics d, d_1 are equivalent on P. In connection with the above the question arises naturally of how great is the cardinality of the set $U \cap P_0$, and what are its topological properties in the space (P,d). The answer is contained in the following theorem.

THEOREM 1.2. *The set $U \cap P_0$ is dense in (P,d), $|U \cap P_0| = c$ (c being the cardinality of the continuum), and $U \cap P_0$ is a set of the first category in (P,d).*

P r o o f . That $U \cap P_0$ is of the first category follows trivially from $U \cap P_0 \subseteq P_0$ by Theorem 1.1.

Denote by D_1 the set of all permutations of ℕ which arise from $1,2,\dots,n,\dots$ in such a way that we change only the numbers in pairs $(2n-1, 2n)$ $(n = 1,2,\dots)$. Clearly $|D_2| = c$ and $|D_2 \subseteq P_0$ by Theorem B.

We shall show that $D_2 \subseteq U$. Let $m \geq 2$ and $x = (x_k)_{k=1}^{\infty} \in D_2$. We give certain estimates for $A(j,m,n,x)$ (see (4)).

The following cases can occur for fixed n:

1) $x_n = 2l$,
2) $x_n = 2l - 1$ $(l \geq 1)$.

1) If x arises from $1,2,\dots,k,k+1,\dots$ in such a way that the numbers $2l-1$, $2l$ of the pair $(2l-1, 2l)$ are not changed, then $\{x_1, x_2, \dots, x_n\} = \{1,2,\dots,2l\}$

and then $A(j,m,n,x)$ equals the number of all numbers of the form $j+mt$ ($t \ge 0$) contained in the sequence $1,2,\dots,2l$. Hence

$$A(j,m,n,x) = \left[\frac{2l-j}{m}\right] + 1 \tag{5}$$

($[t]$ denotes the integer part of t).

If x arises from $1,2,\dots,k,k+1,\dots$ in such a way that the numbers $2l-1$, $2l$ in the pair $(2l-1,2l)$ are changed, then $\{x_1,x_2,\dots,x_n\} = \{1,2,\dots,2l-2,2l\}$ and by a simple calculation we get

$$\left[\frac{2l-2-j}{m}\right] + 1 \le A(j,m,n,x) \le \left[\frac{2l-2-j}{m}\right] + 2\,. \tag{6}$$

In both cases (see (5), (6)) we obtain

$$\liminf_{n\to\infty} \frac{A(j,m,n,x)}{n} = \frac{1}{m}\,. \tag{7}$$

Similarly we can show that (7) also holds in case 2). Hence $D_2 \subseteq U \cap P_0$, thus $c = |D_2| \le |U \cap P_0| \le |P| = c$. The Cantor-Bernstein theorem gives $|U \cap P_0| = c$.

The density of $U \cap P_0$ in P follows from the obvious fact that if $x = (x_k)_{k=1}^{\infty}$ belongs to $U \cap P_0$ then every $y = (y_k)_{k=1}^{\infty} \in P$ with finite $\{k :\ x_k \ne y_k\}$ also belongs to $U \cap P_0$. □

2. Structure of the space (P,d), and porosity of sets in P

Let $\sum\limits_{k=1}^{\infty} c_k$ be a non-absolutely convergent series. We introduce some sets of permutations (subsets of P) which are related to the convergence character of rearrangements of $\sum\limits_{k=1}^{\infty} c_k$. Put

$$\begin{aligned}
H^+(c_1,c_2,\dots) &= \Big\{x = (x_j)_{j=1}^{\infty} \in P;\ \limsup_{n\to\infty} s_n(x,c) = +\infty\Big\}\,,\\
H^-(c_1,c_2,\dots) &= \Big\{x = (x_j)_{j=1}^{\infty} \in P;\ \liminf_{n\to\infty} s_n(x,c) = -\infty\Big\}\,,\\
B^+(c_1,c_2,\dots) &= P \setminus H^+(c_1,c_2,\dots)\,,\\
B^-(c_1,c_2,\dots) &= P \setminus H^-(c_1,c_2,\dots)\,.
\end{aligned}$$

Hence

$$\begin{aligned}
B^+(c_1,c_2,\dots) &= \Big\{x = (x_j)_{j=1}^{\infty} \in P;\ \limsup_{n\to\infty} s_n(x,c) < +\infty\Big\}\,,\\
B^-(c_1,c_2,\dots) &= \Big\{x = (x_j)_{j=1}^{\infty} \in P;\ \liminf_{n\to\infty} s_n(x,c) > -\infty\Big\}\,.
\end{aligned}$$

Further put

$$\begin{aligned}B(c_1, c_2, \ldots) &= B^+(c_1, c_2, \ldots) \cap B^+(c_1, c_2, \ldots)\\ &= \Big\{x = (x_j)_{j=1}^{\infty} \in P;\ \limsup_{n\to\infty} |s_n(x, c)| < +\infty\Big\}\end{aligned}$$

$(s_n(x, c) = \sum_{i=1}^{n} c_{x_i}\,,\ x = (x_j)_{j=1}^{\infty})$. Define :

$$B^+ = \bigcap B^+(a_1, a_2, \ldots)\,, \tag{8}$$

$$B^- = \bigcap B^-(a_1, a_2, \ldots)\,, \tag{8'}$$

where the intersections on the right-hand sides are taken over all non-absolutely convergent series $\sum_{k=1}^{\infty} a_k$. Remark that $H^+(c_1, c_2, \ldots) \cap H^-(c_1, c_2, \ldots)$ is a residual set in P (see Theorem B).

Note that

$$P_0 \subseteq B^+ \cap B^-\,. \tag{9}$$

The following proposition is a simple observation about these sets.

PROPOSITION 2.1.

(i) *If* $\sum_{k=1}^{\infty} c_k$ *is a non-absolutely convergent series, then* $B^+(c_1, c_2, \ldots) \neq B^-(c_1, c_2, \ldots)$ *and neither of these sets is a subset of the other.*

(ii) *We have* $B^+ = B^-$.

Proof.

(i) According to a well-known theorem of Riemann on rearrangements of series there exists $q = (q_k)_{k=1}^{\infty} \in P$ such that $\sum_{k=1}^{\infty} c_{q_k} = -\infty$. This q belongs to $B^+(c_1, c_2, \ldots)$, but not to $B^-(c_1, c_2, \ldots)$. Similarly $B^-(c_1, c_2, \ldots)$ is not a subset of $B^+(c_1, c_2, \ldots)$.

(ii) We prove that $B^+ \subseteq B^-$. Let $q \in B^+$. Suppose that $\sum_{k=1}^{\infty} a_k$ is an arbitrary non-absolutely convergent series. Then $\sum_{k=1}^{\infty} (-a_k)$ is again a non-absolutely convergent series and, by assumption, q belongs to $B^+(-a_1, -a_2, \ldots)$. From this it follows immediately that q belongs to $B^-(-a_1, -a_2, \ldots)$. This holds for an arbitrary $\sum_{k=1}^{\infty} a_k$, hence $q \in B^-$ (see (8')). Similarly we obtain $B^- \subseteq B^+$. □

As a simple consequence of part (ii) we get the following form of (9):

$$P_0 \subseteq B^+ = B^-\,. \tag{9'}$$

In what follows we shall use the concept of porosity of sets (cf. [16], [17]).

Let (Y,ρ) be a metric space. Denote by $K(y,r)$ the ball with centre y and radius $r>0$, i.e. $K(y,r)=\{x\in Y;\ \rho(x,y)<r\}$.

Let $M\subseteq Y$, $y\in Y$, $r>0$. Put

$$\gamma(y,r,M)=\sup\Big\{t>0;\ \exists z{\in}Y\ \big[K(z,t)\subseteq K(y,r)\big]\wedge\big[K(z,t)\cap M=\emptyset\big]\Big\}.$$

Further, set

$$\overline{p}(y,M)=\limsup_{r\to 0+}\frac{\gamma(y,r,M)}{r},\qquad \underline{p}(y,M)=\liminf_{r\to 0+}\frac{\gamma(y,r,M)}{r}.$$

If $\overline{p}(y,M)=\underline{p}(y,M)$, then we set $p(y,M)=\overline{p}(y,M)=\underline{p}(y,M)=\lim\limits_{r\to 0+}\frac{\gamma(y,r,M)}{r}$.

Clearly, the numbers $\overline{p}(y,M)$, $\underline{p}(y,M)$, $p(y,M)$ belong to the interval $[0,1]$.

A set $M\subseteq Y$ is said to be *porous* (*c-porous*) *at* y provided that $\overline{p}(y,M)>0$ ($\overline{p}(y,M)\ge c>0$) and *σ-porous* (*σ-c-porous*) *at* y if $M=\bigcup\limits_{n=1}^{\infty}M_n$, where M_n is porous (c-porous) at y $(n=1,2,\dots)$.

Let $Y_0\subseteq Y$. A set $M\subseteq Y$ is said to be *porous, σ-porous,* or *σ-c-porous in* Y_0 if it is porous, σ-porous or σ-c-porous, respectively, at every $y\in Y_0$.

Every porous set in Y is a nowhere dense set in Y and hence every σ-porous set is a set of the first category in Y. The converse is not true even in $\mathbb{R}$ (cf. [15]).

From the definition of the numbers $\overline{p}(y,M)$, $\underline{p}(y,M)$ we get:

If $M_1\subseteq M_2$, then for each $y\in Y$ we have $\overline{p}(y,M_1)\ge\overline{p}(y,M_2)$, $\underline{p}(y,M_1)\ge\underline{p}(y,M_2)$.

A set $M\subset Y$ is said to be *very porous at* $y\in Y$ if $\underline{p}(y,M)>0$ (cf. [17; p. 327]). Obviously, if M is very porous at $y\in Y$, it is also porous at y.

A set $M\subseteq Y$ is said to be *very porous* (*σ-very porous*) *in* $Y_0\subseteq Y$ if it is very porous at every $y\in Y_0$ (if $M=\bigcup\limits_{n=1}^{\infty}M_n$ and each M_n $(n=1,2,\dots)$ is very porous at every $y\in Y_0$).

The application of the concept of porosity enables us to obtain a more precise and detailed view of the structure of sets in metric spaces.

We shall apply this concept to sets B^+, B^-, $B^+(c_1,c_2,\dots)$, $B^-(c_1,c_2,\dots)$ and P_0. In the first place, we shall give an application of porosity in connection with Theorem B for the set P_0.

LEMMA 2.1. *Let $v\in\mathbb{N}$. If $y\in P_0(v)$, then $\underline{p}(y,P_0(v))\ge 2^{-v-2}$. If $y\in P\setminus P_0(v)$, then $p(y,P_0(v))=1$.*

Proof. Let $y=(y_k)_{k=1}^{\infty}\in P_0(v)$ and $0<r<1$. Choose a natural number m such that $2^{-m}\le r<2^{-m+1}$. Put $a=\max\{y_1,\dots,y_m\}$. Denote by $v_1,\dots,v_t$

all positive integers not exceeding a, not contained in $\{y_1,\dots,y_m\}$. Construct the sequence

$$\begin{aligned} z &= (z_k)_{k=1}^{\infty} \\ &= y_1, y_2, \dots, y_m, a+2, a+4, \dots, a+2v, v_1, v_2, \dots, v_t, \\ &\quad a+1, a+3, \dots, a+2v-1, a+2v+1, a+2v+2, \dots, a+2v+j, \dots . \end{aligned}$$

Evidently the set $\{z_1, z_2, \dots, z_{m+v}\} = \{y_1, y_2, \dots, y_m, a+2, a+4, \dots, a+2v\}$ cannot be expressed as a union of not more than v blocks of positive integers.

Construct the ball $K(z, 2^{-m-v-1})$. It is easy to verify that

$$K(z, 2^{-m-v-1}) \subseteq K(y, 2^{-m}) \subseteq K(y, r). \tag{10}$$

Further if $x = (x_k)_{k=1}^{\infty} \in K(z, 2^{-m-v-1})$, then $x_k = z_k$ $(k = 1, 2, \dots, m+v)$ and so $\{x_1, x_2, \dots, x_{m+v}\} = \{z_1, z_2, \dots, z_{m+v}\}$. Thus

$$K(z, 2^{-m-v-1}) \cap P_0(v) = \emptyset. \tag{10'}$$

From (10), (10') we get

$$\gamma(y, r, P_0(v)) \geq 2^{-m-v-1}, \qquad \frac{\gamma(y, r, P_0(v))}{r} \geq \frac{2^{-m-v-1}}{2^{-m+1}} = 2^{-v-2}$$

and so

$$\underline{p}(y, P_0(v)) \geq 2^{-v-2} > 0.$$

The second part of the assertion is trivial because of the closedness of the set $P_0(v)$ (see the proof of Theorem 1.1). □

From the previous lemma, according to (2), we get:

THEOREM 2.1. *The set P_0 is σ-very porous set in the space (P, d).*

COROLLARY. *The set $U \cap P_0$ is σ-very porous in the space (P, d).*

We return to the sets B^+, B^-, $B^+(c_1, c_2, \dots)$, $B^-(c_1, c_2, \dots)$. The following results are similar to that proved in [10] for P endowed by Baire's metric.

LEMMA 2.2. *Let $\sum_{k=1}^{\infty} c_k$ be a non-absolutely convergent series. Then the set $B^+(c_1, c_2, \dots)$ is a σ-1-porous set in the set $H^+(c_1, c_2, \dots)$.*

P r o o f. We have

$$B^+(c_1, c_2, \dots) = \bigcup_{m=1}^{\infty} B_m^+(c_1, c_2, \dots), \tag{11}$$

where

$$B_m^+(c_1, c_2, \dots) = \{x = (x_j)_{j=1}^\infty \in P;\ s_n(x, c) \le m,\ n = 1, 2, \dots\}.$$

Let $y = (y_j)_{j=1}^\infty \in H^+(c_1, c_2, \dots)$. By definition of $H^+(c_1, c_2, \dots)$ there exists a sequence $v_1 < v_2 < \dots < v_k < \dots$ of positive integers such that

$$\lim_{k \to \infty} s_{v_k}(y, c) = +\infty. \tag{12}$$

Construct the ball $K(y, 2^{-v_k-1}) \subseteq P$. If $z = (z_j)_{j=1}^\infty \in K(y, 2^{-v_k-1})$ then $z_j = y_j$ $(j = 1, 2, \dots, v_k)$ and so $s_{v_k}(y, c) = s_{v_k}(z, c)$. By (12) there exists a k_0 such that $s_{v_k}(y, c) > m$ for $k > k_0$. Then (for $k > k_0$)

$$K(y, 2^{-v_k-1}) \cap B_m^+(c_1, c_2, \dots) = \emptyset.$$

Hence

$$\gamma(y, 2^{-v_k-1}, B_m^+(c_1, c_2, \dots)) = 2^{-v_k-1} \qquad (k > k_0).$$

From this we get

$$\overline{p}(y, B_m^+(c_1, c_2, \dots)) = 1 \qquad (m = 1, 2, \dots).$$

The lemma follows from (11). □

Similarly we can prove:

LEMMA 2.3. *Let* $\sum_{k=1}^{\infty} c_k$ *be a non-absolutely convergent series. Then the set* $B^-(c_1, c_2, \dots)$ *is* σ*-1-porous in the set* $H^-(c_1, c_2, \dots)$.

From the previous Lemmas 2.2 and 2.3 and Theorem A we get:

THEOREM 2.2. *Let* $\sum_{k=1}^{\infty} c_k$ *be a non-absolutely convergent series. Then each of the sets* $B(c_1, c_2, \dots)$, $P_0(c_1, c_2, \dots)$ *is a* σ*-1-porous set in the residual set* $H^+(c_1, c_2, \dots) \cap H^-(c_1, c_2, \dots)$.

Remark 2.1. In connection with inclusions (9) and (9') the question arises whether the set P_0 is equal to the set B^+ $(= B^-)$ or not. We are not able to give any answer to this question.

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Received November 14, 1996

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