Osama Abdelkader; Sh. Khidr

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L^p-ESTIMATES FOR SOLUTIONS OF $\bar{\partial}$-EQUATION ON STRONGLY q-CONVEX DOMAINS

O. ABDELKADER* — SH. KHIDR**

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ABSTRACT. The goal of this article is to construct a solution with L^p-estimates, $1 \le p \le \infty$, of the equation $\bar{\partial}g = f$ on strongly q-convex domain D of a Kähler manifold M, $f \in L^p_{r,s}(D,E)$, $s \ge q$, where E is a holomorphic line bundle over M satisfying a certain positivity conditions.

Introduction

The existence of solutions of the equation $\bar{\partial}g = f$ when f is a form of type $(0,s)$, $\bar{\partial}f = 0$, $s \ge 1$, and satisfies L^p-estimates, $1 \le p \le \infty$, has been a central theme in complex analysis for many years. On strongly pseudo-convex domains with C^4-boundary in a Stien manifold, Kerzman [9] has obtained a solution with L^p-estimates, $1 \le p \le \infty$, for the equation $\bar{\partial}g = f$ when f is a complex valued form of type $(0,1)$, $\bar{\partial}f = 0$, and satisfies L^p-estimates. On strongly pseudo-convex domains with C^2-boundary in $\mathbb{C}^n$, Øvrelid [11] has generalized this results to $(0,s)$-forms. Using Kerzman's method, Abdelkader [1] has extended Øvrelid's results to strongly pseudo-convex domains in an n-dimensional complex manifold for forms of type $(0,s)$ with values in a holomorphic positive line bundle and to complex valued forms of type (n,s) when the complex manifold is a Stien manifold. Abdelkader and Khidr [3] have extended Abdelkader's results to forms of type $(0,s)$ with values in a holomorphic vector bundle which is Nakano positive and to complex valued forms of type (r,s), $0 \le r \le n$, when the complex manifold is a Stein manifold.

The local solvability of $\bar{\partial}g = f$ on strongly q-convex set when f is a complex valued form of type $(0,s)$, $\bar{\partial}f = 0$, $s \ge q$, is due to Andreotti and Grauert [5]. On strongly q-convex domains in $\mathbb{C}^n$, Ma [10] has obtained

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L^p-estimates, $1 \leq p \leq \infty$, for solution of $\bar{\partial} g = f$ when f is a complex valued form of type $(0,s)$, $\bar{\partial} f = 0$, $s \geq q$, and satisfies L^p-estimates.

In this paper, joining the results in [1] and [9] with those in [10], we extend the results of [1] to strongly q-convex domains and to forms of type (r,s), $s \geq q$, with values in a semi-positive (semi-negative) line bundle. The main aim of this paper is to establish the following existence theorem with L^p-estimates:

THEOREM. *Let M be a Kähler manifold of complex dimension n and E be a holomorphic line bundle over M. Let D be a strongly q-convex domain of M. If E is semi-positive (resp. semi-negative) of type k on $\bar{D}$, then for any $f \in L^1_{r,s}(D,E)$, $\bar{\partial} f = 0$, $s \geq q$ and $r + s \geq n + k$ (resp. $s \geq q$ and $r + s \leq n - k$), there is a form $g = T^s f \in L^1_{r,s-1}(D,E)$ satisfying $\bar{\partial} g = f$, where T^s is a bounded linear operator. Moreover, if $f \in L^p_{r,s}(D,E)$, $1 \leq p \leq \infty$, there is a constant C_s such that $\|g\|_{L^p_{r,s-1}(D,E)} \leq C_s \|f\|_{L^p_{r,s}(D,E)}$. The constant C_s is independent of f and p. If f is C^∞, then g is also C^∞.*

The plan of this paper is as follows: In Section 1, we fix the notation and recall some useful facts. In Section 2, we prove an existence theorem with L^2-estimates. In Section 3, we give local solution for the $\bar{\partial}$-equation with L^p-estimates for $1 \leq p \leq \infty$. In Section 4, we prove the existence theorem with L^p-estimates.

1. Notation and preliminaries

From now on, M denotes a Kähler manifold of complex dimension n and let $\pi\colon E \to M$ be a holomorphic line bundle over M. Let $\{u_j\}$, $j \in I$, be an open covering of M consisting of coordinates neighborhoods u_j with holomorphic coordinates $z_j = (z_j^1, z_j^2, \ldots, z_j^n)$ over which E is trivial, namely $\pi^{-1}(u_j) = u_j \times \mathbb{C}$. A Hermitian metric along the fibers of E is a system of positive C^∞-functions $h = \{h_j\}$, each defined on u_j, such that $h_j = |e_{ij}|^2 h_i$ on $u_i \cap u_j$, where $\{e_{ij}\}$ is the system of transition functions of E. The curvature form associated to the metric h is defined by $\Theta = \{\Theta_j\}$, $\Theta_j = \sqrt{-1}\bar{\partial}\partial \log h_j = \sqrt{-1} \sum\limits_{\alpha,\beta=1}^{n} \Theta_{j\alpha\bar{\beta}}\, \mathrm{d}z_j^\alpha \wedge \mathrm{d}\bar{z}_j^\beta$, where

$$\Theta_{j\alpha\bar{\beta}} = -\frac{\partial^2 \log h_j}{\partial z_j^\alpha \partial \bar{z}_j^\beta}.$$

Let $T(M)$ (resp. $T^\star(M)$) be the holomorphic tangent (resp.cotangent) bundle of M.

DEFINITION 1.1.

i) E is said to be *k-positive* (resp. *k-negative*) *at* $x \in u_j$, if the form

$$\sum_{\alpha,\beta=1}^{n} \Theta_{j\alpha\bar\beta} \tag{1.1}$$

is a Hermitian form on $T_x(M)$ having at least $n-k+1$ positive (resp. negative) eigenvalues.

ii) E is said to be *semi-positive* (resp. *semi-negative*) *at* $x \in u_j$, if the form (1.1) is positive (resp. negative) semi-definite Hermitian form on $T_x(M)$.

iii) E is said to be *semi-positive* (resp. *semi-negative*) *of type k at x*, if E is both semi-positive and k-positive (resp. semi-negative and k-negative) at x.

The notation $X \Subset M$ means that X is an open subset of M such that its closure is a compact subset of M.

DEFINITION 1.2. We say that $D \Subset M$ is *strongly q-convex*, $q \geq 1$, if there exist a neighborhood U of the boundary ∂D of D in M and a real-valued C^2-function ρ with $\mathrm{d}\rho(x) \neq 0$ on U such that $U \cap D = \{x \in U : \rho(x) < 0\}$, and its Levi form

$$L(\rho)(x) = \sum \frac{\partial^2 \rho(x)}{\partial z^\alpha \partial \bar z^\beta} \zeta^\alpha \bar\zeta^\beta , \qquad \zeta = (\zeta^1, \zeta^2, \dots, \zeta^n) \in \mathbb{C}^n , \tag{1.2}$$

has at least $n-q+1$ positive eigenvalues at each point $x \in U$. A function ρ satisfying (1.2) is called *strongly q-convex at x*.

Remark 1.3. By *shrinking* U we can assume that $U \Subset \tilde U$, where $\tilde U$ is open and the Levi form (1.2) has at least $n-q+1$ positive eigenvalues at each point $x \in \tilde U$. Thus ρ and its derivatives are bounded on U.

From now on, we assume that D is a strongly q-convex domain of M with ρ and U from Definition 1.2. We will use the standard notation of Hörmander [6] for differential forms. Let $\Lambda^{r,s}(M,E)$ (resp. $\mathcal{D}^{r,s}(M,E)$) be the space of E-valued differential forms (resp. with compact support) of type (r,s) and of class C^∞ on M. A differential form $\varphi = \{\varphi_j\} \in \Lambda^{r,s}(M,E)$ can be expressed, on u_j, as $\varphi_j(z) = \sum\limits_{A_r,B_s} \varphi_{jA_rB_s}(z)\,\mathrm{d}z_j^{A_r} \wedge \mathrm{d}\bar z_j^{B_s}$, where A_r and B_s are strictly increasing multi-indices with lengths r and s, respectively. We have the operator $\bar\partial\colon \Lambda^{r,s}(M,E) \to \Lambda^{r,s+1}(M,E)$, locally defined by:

$$\bar\partial\varphi_j(z) = \sum_{A_r,B_s} \sum_{k=1}^{n} \frac{\partial \varphi_{jA_rB_s}}{\partial \bar z_j^k}(z)\,\mathrm{d}\bar z_j^k \wedge \mathrm{d}z_j^{A_r} \wedge \mathrm{d}\bar z_j^{B_s} .$$

Let

$$\mathrm{d}s^2 = \sum_{\alpha,\beta=1}^{n} g_{j\alpha\bar{\beta}}\, \mathrm{d}z_j^{\alpha}\, \mathrm{d}\bar{z}_j^{\beta}\,,$$

be the Kähler metric defined on M.

For $\varphi, \psi \in \Lambda^{r,s}(M, E)$ we define a local inner product at $z \in u_j$ by

$$\frac{1}{h_j}\varphi_j(z) \wedge \star\overline{\psi_j(z)} = \big(\varphi(z), \psi(z)\big)\, \mathrm{d}v\,, \tag{1.3}$$

where the Hodge star operator $\star$ and the volume element $\mathrm{d}v$ are defined by $\mathrm{d}s^2$ and (φ, ψ) is a C^∞ function on M independent of j. Let $\Lambda^{r,s}(\bar{D}, E)$ be the subspace of $\Lambda^{r,s}(D, E)$ whose elements can be extended smoothly up to ∂D, i.e.,

$$\Lambda^{r,s}(\bar{D}, E) = \big\{\varphi|_{\bar{D}} :\ \varphi \in \Lambda^{r,s}(M, E)\big\}\,.$$

For $\varphi, \psi \in \Lambda^{r,s}(\bar{D}, E)$, we define the inner product $\langle\varphi, \psi\rangle$ and the norm $\|\varphi\|$ with respect to $\mathrm{d}s^2$ and h by:

$$\langle\varphi, \psi\rangle = \int_D (\varphi, \psi)\ \mathrm{d}v\,, \qquad \|\varphi\|^2 = \langle\varphi, \varphi\rangle\,. \tag{1.4}$$

Let $|f(z)| = \sqrt{\big(f(z), f(z)\big)}$ for $f \in \Lambda^{r,s}(D, E)$ and $L^p_{r,s}(D, E)$ be the Banach space of forms f in $\Lambda^{r,s}(D, E)$ for which $\int_D |f(z)|^p\ \mathrm{d}v < \infty$, $1 \le p < \infty$, and $\operatorname{ess\,sup}_{z\in D} |f(z)| < \infty$ for $p = \infty$. The norm on $L^p_{r,s}(D, E)$ is defined by $\|f\|_{L^p_{r,s}(D,E)} = \Big(\int_D |f(z)|^p\ \mathrm{d}v\Big)^{1/p}$ for $1 \le p < \infty$ and by $\|f\|_{L^\infty_{r,s}(D,E)} = \operatorname{ess\,sup}_{z\in D} |f(z)|$ for $p = \infty$.

2. Existence theorems with L^2-estimates

The L^2 completion of $\Lambda^{r,s}(D, E)$ with respect to the norm defined by (1.4) is denoted by $L^2_{r,s}(D, E)$. Let $\bar{\partial}$: $L^2_{r,s}(D, E) \to L^2_{r,s+1}(D, E)$ be the maximal closed extension of the original $\bar{\partial}$. A form $u \in L^2_{r,s}(D, E)$ is in the domain of $\bar{\partial}$ if $\bar{\partial}u$, defined in the sense of distributions, belongs to $L^2_{r,s+1}(D, E)$. Then $\bar{\partial}$ defines a linear closed, densely defined operator. From the general results of functional analysis, $\bar{\partial}$ admits a linear, closed and densely defined adjoint $\bar{\partial}^\star$: $L^2_{r,s+1}(D, E) \to L^2_{r,s}(D, E)$, called the *Hilbert space adjoint* of $\bar{\partial}$, such that

$$\langle\varphi, \bar{\partial}\psi\rangle = \langle\bar{\partial}^\star\varphi, \psi\rangle$$

for any $\psi \in \operatorname{Dom}(\bar{\partial})$ and $\varphi \in \operatorname{Dom}(\bar{\partial}^\star)$.

Let

$$\square_{r,s} = \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial\colon \operatorname{Dom}(\square_{r,s}) \to L^2_{r,s}(D,E),$$

be the Complex Laplacian operator of $\bar\partial\colon L^2_{r,s}(D,E) \to L^2_{r,s+1}(D,E)$, where

$$\begin{aligned}\operatorname{Dom}(\square_{r,s}) = \{\varphi \in L^2_{r,s}(D,E) :\ & \varphi \in \operatorname{Dom}(\bar\partial) \cap \operatorname{Dom}(\bar\partial^\star), \\ & \bar\partial\varphi \in \operatorname{Dom}(\bar\partial^\star) \text{ and } \bar\partial^\star\varphi \in \operatorname{Dom}(\bar\partial)\}.\end{aligned}$$

Let $\mathcal{H}^{r,s}(D,E)$ be the space of harmonic forms of type (r,s), i.e.,

$$\mathcal{H}^{r,s}(D,E) = \operatorname{Ker}(\square_{r,s}) = \{\alpha \in L^2_{r,s}(D,E) :\ \bar\partial\alpha = \bar\partial^\star\alpha = 0\}.$$

We define a linear operator

$$N_{r,s}\colon L^2_{r,s}(D,E) \to L^2_{r,s}(D,E)$$

as follows:

$$N_{r,s}\alpha = \begin{cases} 0 & \text{if } \alpha \in \operatorname{Ker}(\square_{r,s}), \\ \phi & \text{if } \alpha \in \mathcal{R}(\square_{r,s}), \end{cases}$$

where $\mathcal{R}(\square_{r,s})$ is the range of $\square_{r,s}$ and ϕ is the unique solution of $\square_{r,s}\phi = \alpha$ with $\phi \perp \operatorname{Ker}(\square_{r,s})$, and we extend $N_{r,s}$ by linearity. Let $H^{r,s}$ be the orthogonal projection of $L^2_{r,s}(D,E)$ onto $\mathcal{H}^{r,s}(D,E)$. For $s \geq q$, the operator $N_{r,s}$ satisfies the following theorem:

THEOREM 2.1. (cf. [8])

(1) *$N_{r,s}$ is a bounded operator;*
(2) *for any $\alpha \in L^2_{r,s}(D,E)$, $\alpha = \bar\partial\bar\partial^\star N_{r,s}\alpha + \bar\partial^\star\bar\partial N_{r,s}\alpha + H^{r,s}\alpha$, and the dimension of $\mathcal{H}^{r,s}(D,E)$ is finite;*
(3) *$N_{r,s}H^{r,s} = H^{r,s}N_{r,s} = 0$, $N_{r,s}\square_{r,s} = \square_{r,s}N_{r,s} = I - H^{r,s}$ on $\operatorname{Dom}(\square_{r,s})$, and if $N_{r,s+1}$ is defined on $L^2_{r,s+1}(D,E)$ (resp. $N_{r,s-1}$ is defined on $L^2_{r,s-1}(D,E)$), then $N_{r,s+1}\bar\partial = \bar\partial N_{r,s}$ on $\operatorname{Dom}(\bar\partial)$ (resp. $N_{r,s-1}\bar\partial^\star = \bar\partial^\star N_{r,s}$ on $\operatorname{Dom}(\bar\partial^\star)$);*
(4) *$N_{r,s}(\Lambda^{r,s}(\bar D,E)) \subset \Lambda^{r,s}(\bar D,E)$ and $H^{r,s}(\Lambda^{r,s}(\bar D,E)) \subset \Lambda^{r,s}(\bar D,E)$.*

Let $\omega = \frac{1}{2}\sqrt{-1}\sum\limits_{\alpha,\beta=1}^{n} g_{j\alpha\bar\beta}\,\mathrm{d}z_j^\alpha \wedge \mathrm{d}\bar z_j^\beta$ be the $(1,1)$ differential form associated to the Kähler metric $\mathrm{d}s^2$ on M. Let $e(\omega)\colon \Lambda^{r,s}(M,E) \to \Lambda^{r+1,s+1}(M,E)$ be the linear mapping locally defined by $(e(\omega)\varphi_j) = \omega \wedge \varphi_j$ and $\Gamma\colon \Lambda^{r,s}(M,E) \to \Lambda^{r-1,s-1}(M,E)$ be the linear mapping locally defined by $\Gamma = (-1)^{r+s} \star e(\omega) \star$. On $\Lambda^{r,s}(D,E)$, we write $\square$ and $h^{r,s}(D,E)$ for $\square_{r,s}$ and $\mathcal{H}^{r,s}(D,E)$ respectively. For $\varphi \in \Lambda^{r,s}(M,E)$, on Kähler manifolds, at any point, from classical differential geometry, we have the identity

$$((\square - \star^{-1}\square\star)\varphi, \varphi) = ((e(\Theta)\Gamma - \Gamma e(\Theta))\varphi, \varphi). \tag{2.1}$$

THEOREM 2.2. (cf. [4]) *Let M be a Kähler manifold of complex dimension n and E be a holomorphic line bundle over M. Let $D \Subset M$ be a strongly q-convex domain of M. If E is semi-positive (resp. semi-negative) of type k on $\bar{D}$, then we have $h^{r,s}(D,E) = 0$ for $r+s \geq n+k$ and $s \geq q$ (resp. $r+s \leq n-k$ and $s \geq q$).*

Proof. Firstly, we suppose that E is semi-positive of type k on $\bar{D}$. Then, from Definition 1.1, the matrix $(\Theta_{j\alpha\bar{\beta}})$ is positive semi-definite and of rank $\geq n-k+1$ at any point of $\bar{D}$. Let $x_0 \in \bar{D}$ be an arbitrary fixed point and v_i, $i = 1,2,\dots,n$, be the eigenvalues of $(\Theta_{\alpha\bar{\beta}})$ with respect to $(g_{\alpha\bar{\beta}})$ at x_0, with $v_1 \geq v_2 \geq \cdots \geq v_n \geq 0$. Then $m_1 = \inf\limits_{x\in\bar{D}} v_{n-k+1}(x) > 0$. We define another Kähler metric

$$\mathrm{d}s_1^2 = \sum_{\alpha,\beta=1}^{n} \left(g_{j\alpha\bar{\beta}} + \mu_1 \Theta_{j\alpha\bar{\beta}}\right) \mathrm{d}z_j^\alpha \, \mathrm{d}\bar{z}_j^\beta ,$$

on a neighborhood of $\bar{D}$, where $\mu_1 > 0$. Let ω_1 be the differential form associated to $\mathrm{d}s_1^2$ and $\Gamma_1 \colon \Lambda^{r,s}(M,E) \to \Lambda^{r-1,s-1}(M,E)$ be the linear mapping defined by $\Gamma_1 = (-1)^{r+s} \star e(\omega_1) \star$. Let b be the rank of $(\Theta_{\alpha\bar{\beta}})$ at x_0. Using (2.1), as [2; Lemma 1], we prove that at any point in $\bar{D}$, for $\varphi \in \Lambda^{r,s}(M,E)$, we have

$$\left(\left(\Gamma_1 e(\Theta) - e(\Theta)\Gamma_1\right)\varphi\right)_{j\alpha_1\alpha_2\dots\alpha_r\bar{\beta}_1\bar{\beta}_2\dots\bar{\beta}_s} = h_j \Omega \varphi_{j\alpha_1\dots\alpha_r\bar{\beta}_1\dots\bar{\beta}_s} , \tag{2.2}$$

where

$$\begin{aligned}\Omega &= \sum_{i=1}^{b} \frac{v_i}{1+\mu_1 v_i} - \sum_{i=1}^{r} \frac{v_{\alpha_i}}{1+\mu_1 v_{\alpha_i}} - \sum_{i=1}^{s} \frac{v_{\beta i}}{1+\mu_1 v_{\beta_i}} \\ &= \frac{1}{\mu_1}\left(\sum_{i=1}^{b}\left(1 - \frac{1}{1+\mu_1 v_i}\right) - \sum_{\alpha_i \leq b}\left(1 - \frac{1}{1+\mu_1 v_{\alpha_i}}\right) - \sum_{\beta_i \leq b}\left(1 - \frac{1}{1+\mu_1 \beta_i}\right)\right).\end{aligned}$$

Let $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_r \leq n$ (resp. $1 \leq \beta_1 < \beta_2 < \cdots < \beta_s \leq n$) be fixed. Suppose that in the set $\{\alpha_1,\dots,\alpha_r\}$ (resp. $\{\beta_1,\dots,\beta_s\}$) there exists s_1 (resp. s_2) among the first b of the set $\{1,2,3,\dots,n\}$. Then, we have

$$\Omega \leq \frac{1}{\mu_1}\left(2b - s_1 - s_2 + \sum_{i=1}^{n-k+1} \frac{1}{1+\mu_1 m_1} - (n-k+1)\right).$$

But $r+s \geq n+k$ implies that $r-(n-b)+s-(n-b) \geq 2b-n+k$. Hence, there exist at least $2b-n+k$ of α_i and β_i among the first b of the set $\{1,2,3,\dots,n\}$. Therefore, $s_1+s_2 \geq 2b-n+k$, i.e., $2b-s_1-s_2-(n-k+1) \leq -1$. Then, for any fixed $\mu_1 \geq \frac{2n-1}{m_1}$, we have $\Omega \leq -1/2\mu_1$. Then from (2.2), at any point of $\bar{D}$, for $r+s \geq n+k$ we have

$$\left(\left(\Gamma_1 e(\Theta) - e(\Theta)\Gamma_1\right)\varphi, \varphi\right) \leq -(1/2\mu_1)(\varphi,\varphi),$$

where the inner product $(\cdot,\cdot)$ is defined by the metrics ds_1^2 and h. Therefore, from (2.1), we obtain

$$\begin{aligned}0 &\le \langle \star^{-1}\Box_1 \star\varphi,\varphi\rangle = \big\langle\big(\Gamma_1 e(\Theta) - e(\Theta)\Gamma_1\big)\varphi,\varphi\big\rangle \\ &\le -(1/2\mu_1)\langle\varphi,\varphi\rangle \le 0\end{aligned}$$

for $\varphi \in h^{r,s}(D,E)$ with $r+s \ge n+k$ where $\Box_1 = \bar{\partial}\bar{\partial}_1^\star + \bar{\partial}_1^\star\bar{\partial}$ and $\bar{\partial}_1^\star$ is the formal adjoint of $\bar{\partial}$ when the inner product $\langle\cdot,\cdot\rangle$ is defined by ds_1^2 and h, i.e., $h^{r,s}(D,E) \cong 0$ for $r+s \ge n+k$.

Secondly, we suppose that E is semi-negative of type k on $\bar{D}$. Then, from Definition 1.1, the matrix $(\Theta_{j\alpha\bar{\beta}})$ is negative semi-definite and of rank $\ge n-k+1$ at any point of $\bar{D}$. We define another Kähler metric

$$ds_2^2 = \sum_{\alpha,\beta=1}^{n} \big(g_{j\alpha\bar{\beta}} - \mu_2\Theta_{j\alpha\bar{\beta}}\big)\, dz_j^\alpha\, d\bar{z}_j^\beta\,,$$

on a neighborhood of $\bar{D}$, where $\mu_2 > 0$. Let $x_0 \in D$ be an arbitrary fixed point and γ_i, $i = 1,2,\dots,n$, be the eigenvalues of $(\Theta_{j\alpha\bar{\beta}})$ at x_0 with respect to $(g_{j\alpha\bar{\beta}})$ with $\gamma_1 \le \gamma_2 \le \dots \le \gamma_n \le 0$. Then $m_2 = \inf_{x\in\bar{D}} \gamma_{n-k+1} < 0$. As in the first case, if $\Box_2$ is defined by ds_2^2 and h, we have

$$0 \le \langle \star^{-1}\Box_2 \star\varphi,\varphi\rangle \le -(1/2\mu_2)\langle\varphi,\varphi\rangle \le 0$$

for $\varphi \in h^{r,s}(D,E)$ with $r+s \le n-k$ and $\mu_2 \ge (1-2n)/m_2$, i.e., $h^{r,s}(D,E) \cong 0$ for $r+s \le n-k$. The proof is complete. □

The boundedness of the operator $H^{r,s}$, [7; Proposition 1.2.3, Proposition 1.2.4] (as they are applied to [7; Proposition 2.1.1]) and Theorem 2.1 imply that $H^{r,s}\big(\Lambda^{r,s}(\bar{D},E)\big)$ is dense in the finite dimensional vector space $\mathcal{H}^{r,s}(D,E)$, $s \ge q$, with respect to the graph norm $\big(\|\varphi\|^2 + \|\bar{\partial}\varphi\|^2\big)^{\frac{1}{2}}$. Then $H^{r,s}\big(\Lambda^{r,s}(\bar{D},E)\big) = \mathcal{H}^{r,s}(D,E)$, $s \ge q$. Therefore,

$$h^{r,s}(D,E) \cong \mathcal{H}^{r,s}(D,E) \qquad s \ge q\,.$$

Hence, the image of the orthogonal projection operator $H^{r,s}$ is identically equal to zero, i.e.,

$$H^{r,s}\alpha = 0\,. \tag{2.3}$$

Form the above results, we prove the following existence theorem with L^2-estimates:

THEOREM 2.3. *Let D be a strongly q-convex domain of an n-dimensional Kähler manifold M and $E \to M$ be a holomorphic line bundle over M. If E is semi-positive (resp. semi-negative) of type k on $\bar{D}$, then for any $\alpha \in L^2_{r,s}(D,E)$, $\bar{\partial}\alpha = 0$, $s \geq q$ and $r+s \geq n+k$ (resp. $s \geq q$ and $r+s \leq n-k$), there is a unique form $\phi \in L^2_{r,s-1}(D,E)$ satisfying $\bar{\partial}\phi = \alpha$ with $\phi \perp \operatorname{Ker}(\bar{\partial})$, and the estimate*

$$\|\phi\|_{L^2_{r,s-1}(E,D)} \leq C\|\alpha\|_{L^2_{r,s}(E,D)}$$

holds. If α is C^∞, then ϕ is also C^∞.

Proof. Using (3) of Theorem 2.1 and the condition on α, $\bar{\partial}\alpha = 0$, we have $\bar{\partial}N_{r,s}\alpha = N_{r,s+1}\bar{\partial}\alpha = 0$. Using (2.3) in (2) of Theorem 2.1 we obtain $\alpha = \bar{\partial}\bar{\partial}^\star N_{r,s}\alpha$. Therefore we may take $\phi = \bar{\partial}^\star N_{r,s}\alpha$, the condition $\phi \perp \operatorname{Ker}(\bar{\partial})$ clearly implies the uniqueness. Moreover, if α is C^∞, then so is $N_{r,s}\alpha$, hence ϕ is C^∞. Using (2.3) in (3) of Theorem 2.1 and the fact that $N_{r,s}$ is bounded we have the following L^2-estimate:

$$\begin{aligned}
\|\phi\|^2_{L^2_{r,s-1}(D,E)} &= \langle \bar{\partial}^\star N_{r,s}\alpha, \bar{\partial}^\star N_{r,s}\alpha \rangle = \langle \bar{\partial}\bar{\partial}^\star N_{r,s}\alpha, N_{r,s}\alpha \rangle \\
&= \langle (\bar{\partial}\bar{\partial}^\star + \bar{\partial}^\star\bar{\partial}) N_{r,s}\alpha, N_{r,s}\alpha \rangle \\
&= \langle \alpha, N_{r,s}\alpha \rangle \leq \|\alpha\|_{L^2_{r,s}(D,E)} \|N_{r,s}\alpha\|_{L^2_{r,s}(D,E)} \\
&\leq C\|\alpha\|^2_{L^2_{r,s}(D,E)}
\end{aligned}$$

This proves the theorem. □

Theorem 2.3 is needed in the course of the proof of Theorem 4.3.

3. Local solution for the $\bar{\partial}$-equation with L^p-estimates

We consider the following situation: Let D be a strongly q-convex domain of M and $W_\delta = W(\zeta_0, \delta)$ be the open ball in M of an arbitrary fixed center $\zeta_0 \in \partial D$ and radius δ such that $W_\delta \Subset u_j \subset V \Subset U$ for a certain $j \in I$, where δ is a positive constant which depends continuously on the distance $d(\zeta_0, CV)$ from ζ_0 to the complement of V. Then, according to [10] and the fact that every E-valued form is a $\mathbb{C}$-valued form on W_δ, there exist an open set D_{ζ_0} and a linear operator

$$T^s \colon L^1_{0,s}(D_{\zeta_0}, E) \to L^1_{0,s-1}(W_{\delta/2} \cap D, E), \qquad s \geq q,$$

with $W_{\delta/2} \cap D \subset D_{\zeta_0} \subset W_\delta \cap D$ and

$$f = T^{s+1}(\bar{\partial}f) + \bar{\partial}T^s f$$

for $f \in L^1_{0,s}(D_{\zeta_0}, E)$ with $\bar{\partial} f \in L^1_{0,s+1}(D_{\zeta_0}, E)$. Moreover, if $f \in L^p_{0,s}(D_{\zeta_0})$, $\bar{\partial} f = 0$, then there is a constant C such that the estimate $\|T^s f\|_{L^p_{0,s-1}(W_{\delta/2} \cap D)} \leq C \|f\|_{L^p_{0,s}(D_{\zeta_0})}$ holds. Then using Kerzman's techniques, [9; Theorem 1.3.1], as [1] and [3] we can prove the following theorem:

THEOREM 3.1 (LOCAL THEOREM). *Let T^s be the operator which is defined above and $f \in L^1_{0,s}(D_{\zeta_0}, E)$, $\bar{\partial} f = 0$, $s \geq q$. Then, there is $g = T^s f \in L^1_{0,s-1}(W_{\delta/2} \cap D, E)$ such that $\bar{\partial} g = f$. If f is C^∞, then so is g. If $f \in L^p_{0,s}(D_{\zeta_0}, E)$, then $g \in L^p_{0,s-1}(W_{\delta/2} \cap D, E)$ and satisfies*

$$\|g\|_{L^p_{0,s-1}(W_{\delta/2} \cap D, E)} \leq C \|f\|_{L^p_{0,s}(D_{\zeta_0}, E)}, \qquad 1 \leq p \leq \infty,$$

where $C = C(s)$ is a constant independent of f and p.

Using Theorem 3.1 (Local theorem), as [1] and [3], we can prove the following lemma:

LEMMA 3.2 (AN EXTENSION LEMMA). *Let $D \Subset M$ be a strongly q-convex domain. Then, there exists (slightly larger) open set $\hat{D} \Subset M$ with the following properties: $\bar{D} \Subset \hat{D}$; for any $f \in L^1_{0,s}(D, E)$ with $\bar{\partial} f = 0$ and $s \geq q$, there exist two bounded linear operators L_1, L_2, a form $\hat{f} = L_1 f \in L^1_{0,s}(\hat{D}, E)$ and a form $u = L_2 f \in L^1_{0,s-1}(D, E)$ such that:*

(i) *$\bar{\partial} \hat{f} = 0$ in $\hat{D}$.*

(ii) *$\hat{f} = f - \bar{\partial} u$ in D.*

(iii) *If $f \in L^p_{0,s}(D, E)$, then $\hat{f} \in L^p_{0,s}(\hat{D}, E)$ and $u \in L^p_{0,s-1}(D, E)$ with the estimates*

$$\|\hat{f}\|_{L^p_{0,s}(\hat{D}, E)} \leq C_1 \|f\|_{L^p_{0,s}(D, E)}$$

and

$$\|u\|_{L^p_{0,s-1}(D, E)} \leq C_2 \|f\|_{L^p_{0,s}(D, E)}, \qquad 1 \leq p \leq \infty,$$

where the constants C_1 and C_2 are independent of f and p. If f is C^∞ in D, then $\hat{f}$ is C^∞ in $\hat{D}$ and u is C^∞ in D.

4. Global solution for the $\bar{\partial}$-equation with L^p-estimates

Recall that ∂D is defined by a function $\rho\colon U \to \mathbb{R}$. Cover ∂D by finitely many balls $W_{i,\delta_i} = W(x_i, \delta_i)$, $x_i \in \partial D$, $i = 1, \dots m$, such that for each $x_i \in \partial D$ we have $W_{i,\delta_i} \Subset u_j \subset V \Subset U$. Put $\delta = \min_{1 \leq i \leq m} \delta_i$. Then as in [9; p. 321, Lemma 2.3.3, Claim] (see also [1; Proposition 3.2]), we can prove the following proposition:

PROPOSITION 4.1. *Let $\hat{D}$ be as in the extension lemma and let $W_{i,\delta}$ be as above such that $W_{i,\delta} \Subset u_j \subset \hat{D}$ for a certain $j \in I$. Then, for any $f \in L^1_{0,s}(W_{i,\delta}, E)$, $\bar{\partial} f = 0$, $s \geq q$, there exists $\alpha = Tf \in L^1_{0,s-1}(W_{i,\delta/2}, E)$ such that $\bar{\partial}\alpha = f$, where T is a bounded linear operator. If $f \in L^p_{0,s}(W_{i,\delta}, E)$, $1 \leq p \leq 2$, then we have $\alpha \in L^{p+1/4n}_{0,s-1}(W_{i,\delta/2}, E)$ and $\|\alpha\|_{L^{p+1/4n}_{0,s-1}(W_{i,\delta/2},E)} \leq c\|f\|_{L^p_{0,s}(W_{i,\delta},E)}$ and for any p, $1 \leq p \leq \infty$, we have*

$$\|\alpha\|_{L^p_{0,s-1}(W_{i,\delta/2},E)} \leq c\|f\|_{L^p_{0,s}(W_{i,\delta},E)},$$

where $c = c(n,a)$ is a constant independent of f and p.

Using Proposition 4.1 as [1; Proposition 3.2], we prove the following proposition:

PROPOSITION 4.2. *Let $\hat{D}$ be as in the extension lemma. Then, there exists a strongly q-convex domain $D_1 \Subset \hat{D}$ such that for every $\hat{f} \in L^1_{0,s}(\hat{D}, E)$, $\bar{\partial}\hat{f} = 0$, $s \geq q$, there are two bounded linear operators L_1 and L_2 and two forms $f_1 = L_1\hat{f} \in L^1_{0,s}(D_1, E)$ and $\eta_1 = L_2\hat{f} \in L^1_{0,s-1}(D_1, E)$ such that:*

(i) $\bar{\partial} f_1 = 0$ *on* D_1.
(ii) $\hat{f} = f_1 + \bar{\partial}\eta_1$ *on* D_1.
(iii) $\|f_1\|_{L^{p+1/4n}_{0,s}(D_1,E)} \leq c\|\hat{f}\|_{L^p_{0,s}(\hat{D},E)}$ *for* $\hat{f} \in L^p_{0,s}(\hat{D}, E)$, $1 \leq p \leq 2$.
(iv) *For every open set $W \Subset D_1$ and for every p, $1 \leq p \leq \infty$, we have*

$$\|f_1\|_{L^p_{0,s}(W,E)} \leq c\|\hat{f}\|_{L^p_{0,s}(\hat{D},E)},$$

and

$$\|\eta_1\|_{L^p_{0,s-1}(W,E)} \leq c\|\hat{f}\|_{L^p_{0,s}(\hat{D},E)},$$

where $c = c(\hat{D}, W, n)$ is a constant independent of $\hat{f}$ and p.

The solvability with L^2-estimates for $\bar{\partial} u = f$ on D_1 follows from Theorem 2.2 and Theorem 2.3.

Using Theorem 2.3, Proposition 4.2 and the interior regularity properties of the $\bar{\partial}$-operator, as [1; Theorem 3.1], we can prove the following theorem:

THEOREM 4.3. *Let $\hat{D}$ be the strongly q-convex domain of the extension lemma and $W \Subset \hat{D}$. Then, for any form $\hat{f} \in L^1_{0,s}(\hat{D}, E)$ with $\bar{\partial}\hat{f} = 0$, $s \geq q$, there exists a form $\eta \in L^1_{0,s-1}(W, E)$, $\eta = L\hat{f}$, such that $\bar{\partial}\eta = \hat{f}$, where L is a bounded linear operator. If $\hat{f} \in L^p_{0,s}(\hat{D}, E)$ with $1 \leq p \leq \infty$, then*

$$\eta \in L^p_{0,s-1}(W, E) \qquad \textit{and} \qquad \|\eta\|_{L^p_{0,s-1}(W,E)} \leq C\|\hat{f}\|_{L^p_{0,s}(\hat{D},E)},$$

where $C = C(\hat{D}, W)$ is a constant independent of $\hat{f}$ and p. If $\hat{f}$ is C^∞, then η is C^∞.

The idea of the proof of Theorem 4.3 is as follows:

If $p \geq 2$, Theorem 4.3 is an immediate consequence of the fact that $L^p_{0,s}(\hat{D}, E) \subseteq L^2_{0,s}(\hat{D}, E)$, that there exists a solution $u \in L^2_{0,s-1}(\hat{D}, E)$ of $\bar{\partial}u = f$ (if $f \in L^p_{0,s}(\hat{D}, E)$, $\bar{\partial}f = 0$) and the interior regularity properties for solutions of elliptic $\bar{\partial}$ operator. If $1 \leq p < 2$, then the problem of solving $\bar{\partial}u = f$ can be changed into one (in a smaller region) involving a form $f_1 \in L^\tau_{0,s}(D_1, E)$, $\tau > p$; this is the case in which Proposition 4.2 is used. The improvement of the exponent is small, but iterating $4n$ times, we finally obtain a form $f_{4n} \in L^2_{0,s}(D_{4n}, E)$ to which Theorem 2.3 can be applied.

Using Lemma 3.2, Theorem 4.3 and Theorem 2.3 we obtain our results.

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* *Mathematics Department*
Faculty of Science
Minia University
El-Minia
EGYPT
E-mail: usamakader882000@yahoo.com

** *Mathematics Department*
Faculty of Science
Branch of Cairo University
Beni Suef
EGYPT
E-mail: skhidr@yahoo.com