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ON THE OSCILLATION OF SOLUTIONS OF PARABOLIC PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Oscillation properties of the solutions of parabolic partial functional differential equations are investigated via the method of differential inequalities. The oscillatory criteria of solutions with three kinds of boundary conditions are obtained.

1. Introduction

Recently there has been an increasing interest in studying parabolic differential equations with functional arguments. In the oscillatory behavior of solutions for such equations some results have been received, we can refer to [1]–[4] and references therein. Those works, however, considered only the discrete deviating arguments. It seems that very little is known about the case with continuous deviating arguments. The corresponding theory is not well-developed yet. In this paper, we consider the parabolic differential equation with continuous deviating arguments

$$\frac{\partial}{\partial t}\big[u+\lambda(t)u(x,\tau(t))\big] = a(t)\Delta u - c(x,t,u) - \int_a^b q(x,t,\xi)u\big[x,g(t,\xi)\big]\ \mathrm{d}\sigma(\xi) + f(x,t)\,, \tag{1.1}$$
$$(x,t)\in\Omega\times\mathbb{R}_+ = G\,,\quad \mathbb{R}_+ = [0,+\infty)$$

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and consider boundary value conditions of the type

$$u = \varphi(x,t) \qquad \text{on} \quad (x,t) \in \partial\Omega \times \mathbb{R}_+ , \tag{1.2a}$$

$$\frac{\partial u}{\partial n} = \psi(x,t) \qquad \text{on} \quad (x,t) \in \partial\Omega \times \mathbb{R}_+ , \tag{1.2b}$$

$$\frac{\partial u}{\partial n} + \nu(x,t)u = 0 \qquad \text{on} \quad (x,t) \in \partial\Omega \times \mathbb{R}_+ , \tag{1.2c}$$

where $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i{}^2}$, $u = u(x,t), \varphi(x,t), \psi(x,t) \in C(\partial\Omega \times \mathbb{R}_+, \mathbb{R})$, $\nu(x,t) \in C(\partial\Omega \times \mathbb{R}_+, \mathbb{R}_+)$, Ω is a bounded domain in $\mathbb{R}^n$ with a piecewise smooth boundary $\partial\Omega$, n denotes the unit exterior normal vector to $\partial\Omega$.

The aim of this paper is to establish oscillatory criteria for equation (1.1) satisfying three kinds of boundary conditions. Our approach is to reduce the multi-dimensional problem under study to a one-dimensional oscillation problem for ordinary differential equations or inequalities.

Suppose that the following conditions (H) hold.

(H_1) $a(t)$, $\lambda(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$; $q(x,t,\xi) \in C(\overline{\Omega} \times \mathbb{R}_+ \times [a,b], \mathbb{R}_+)$; $f(x,t) \in C(\overline{\Omega} \times \mathbb{R}_+, \mathbb{R})$;

(H_2) $\tau(t) \in C(\mathbb{R}_+, \mathbb{R})$ and $\lim_{t\to+\infty} \tau(t) = +\infty$;

(H_3) $g(t,\xi) \in C(\mathbb{R}_+ \times [a,b], \mathbb{R})$; $g(t,\xi) \le t$, $\xi \in [a,b]$; $g(t,\xi)$ are nondecreasing with respect to t and ξ, and $\lim_{t\to+\infty} \min_{\xi\in[a,b]} \{g(t,\xi)\} = +\infty$;

(H_4) $c(x,t,u) \in C(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$; $c(x,t,u) \ge p(t)r(u)$, in which $p(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$, $r(u) \in C([a,b], \mathbb{R})$; $r(u)$ is a positive and convex function in $(0,+\infty)$, and $c(x,t,-u) = -c(x,t,u)$;

(H_5) $\sigma(\xi) \in ([a,b], \mathbb{R})$ is nondecreasing, integral of equation (1.1) is Stieltjes integral.

Remark. Since integral of equation (1.1) is Stieltjes integral, equation (1.1) includes the following equation

$$\frac{\partial}{\partial t}\left[u + \lambda(t)u(x,\tau(t))\right] = a(t)\Delta u - c(x,t,u) - \sum_{j=1}^{n} q_j(x,t)u\left[x, g_j(t)\right] + f(x,t) .$$

Definition. A solution $u(x,t)$ of equation (1.1) is called *oscillatory* in the domain G if for each positive number μ there exists a point $(x_0, t_0) \in \Omega \times [\mu, +\infty)$ such that the condition $u(x_0,t_0) = 0$ holds.

2. Oscillation criteria

The smallest eigenvalue α_1 of the following Dirichlet problem in the domain Ω

$$\Delta u + \alpha u = 0 \qquad \text{in} \quad (x,t) \in \Omega \times \mathbb{R}_+ \,, \tag{2.1}$$

$$u = 0 \qquad \text{on} \quad (x,t) \in \partial\Omega \times \mathbb{R}_+ \tag{2.2}$$

is positive and the corresponding eigenfunction $\Phi(x)$ is positive in $x \in \Omega$.

LEMMA 1. *Suppose that u is a positive solution of problem* (1.1) *and* (1.2a) *on $\Omega \times [\mu, +\infty)$, $\mu \geq 0$, then the function $U(t)$ satisfying the following differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[U(t) + \lambda(t)U(\tau(t))\big] + \alpha_1 a(t)U(t) + p(t)r(U(t)) + \int_a^b Q(t,\xi)U\big[g(t,\xi)\big]\,\mathrm{d}\sigma(\xi) \leq H(t) \tag{2.3}$$

has eventually positive solution

$$U(t) = \frac{\int_\Omega u(x,t)\Phi(x)\,\mathrm{d}x}{\int_\Omega \Phi(x)\,\mathrm{d}x}\,, \tag{2.4}$$

in which

$$Q(t,\xi) = \min_{x\in\Omega}\{q(x,t,\xi)\}\,;$$

$$H(t) = \left[\int_\Omega \Phi(x)\,\mathrm{d}x\right]^{-1}\left[-a(t)\int_{\partial\Omega}\varphi(x,t)\frac{\partial\Phi(x)}{\partial n}\,\mathrm{d}\omega + \int_\Omega f(x,t)\Phi(x)\,\mathrm{d}x\right].$$

Proof. Let $u(x,t)$ be a positive solution of problem (1.1) and (1.2a) in $\Omega \times [\mu,+\infty)$, $\mu \geq 0$. By condition (H_3), there exists a $t_1 > \mu$ such that $g(t,\xi) \geq \mu$, $(t,\xi) \in [t_1,+\infty) \times [a,b]$ and $\tau(t) \geq \mu$, $t \geq \mu$, $t \geq t_1$, then $u(x,g(t,\xi)) > 0$, $(x,t,\xi) \in \Omega \times [t_1,+\infty) \times [a,b]$, $u(x,\tau(t)) > 0$, $(x,t) \in \Omega \times [t_1,+\infty)$. Multiplying both sides of equation (1.1) by $\Phi(x)$, and integrating with respect to x over the domain Ω, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega u\Phi(x)\,\mathrm{d}x + \lambda(t)\int_\Omega u\bigl(x,\tau(t)\bigr)\Phi(x)\,\mathrm{d}x\right]$$
$$= a(t)\int_\Omega \Delta u\Phi(x)\,\mathrm{d}x - \int_\Omega c(x,t,u)\Phi(x)\,\mathrm{d}x$$
$$- \int_\Omega\int_a^b q(x,t,\xi)u\bigl[x,g(t,\xi)\bigr]\Phi(x)\,\mathrm{d}\sigma(\xi)\,\mathrm{d}x + \int_\Omega f(x,t)\Phi(x)\,\mathrm{d}x\,, \qquad t \ge t_1\,. \tag{2.5}$$

Using the Green formula, we have

$$\int_\Omega \Delta u\Phi(x)\,\mathrm{d}x = \int_{\partial\Omega}\left(\Phi(x)\frac{\partial u}{\partial n} - u\frac{\partial\Phi(x)}{\partial n}\right)\mathrm{d}\omega + \int_\Omega u\Delta\Phi(x)\,\mathrm{d}x$$
$$= -\int_{\partial\Omega}\left(\varphi(x,t)\frac{\partial\Phi(x)}{\partial n}\right)\mathrm{d}\omega - \alpha_1\int_\Omega u\Delta\Phi(x)\,\mathrm{d}x\,, \qquad t \ge t_1\,. \tag{2.6}$$

Using condition (H_4) and Jensen's inequality, we have

$$\int_\Omega c(x,t,u)\Phi(x)\,\mathrm{d}x \ge p(t)\int_\Omega r(u)\Phi(x)\,\mathrm{d}x \ge p(t)r\left(\frac{\int_\Omega u\Phi(x)\,\mathrm{d}x}{\int_\Omega \Phi(x)\,\mathrm{d}x}\right)\int_\Omega \Phi(x)\,\mathrm{d}x\,,$$
$$t \ge t_1\,. \tag{2.7}$$

Let us notice that

$$\int_\Omega\int_a^b q(x,t,\xi)u\bigl[x,g(t,\xi)\bigr]\Phi(x)\,\mathrm{d}\sigma(\xi)\,\mathrm{d}x$$
$$= \int_a^b\int_\Omega q(x,t,\xi)u\bigl[x,g(t,\xi)\bigr]\Phi(x)\,\mathrm{d}x\,\mathrm{d}\sigma(\xi)\,. \tag{2.8}$$

From (2.5) – (2.8), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega u\Phi(x)\,\mathrm{d}x + \lambda(t)\int_\Omega u(x,\tau(t))\Phi(x)\,\mathrm{d}x\right] + \alpha_1 a(t)\int_\Omega u\Phi(x)\,\mathrm{d}x$$

$$+ p(t)r(U(t))\int_\Omega \Phi(x)\,\mathrm{d}x + \int_a^b Q(t,\xi)\left(\int_\Omega u[x,g(t,\xi)]\Phi(x)\,\mathrm{d}x\right)\mathrm{d}\sigma(\xi)$$

$$\le -a(t)\int_{\partial\Omega}\varphi\frac{\partial\Phi(x)}{\partial n}\,\mathrm{d}\omega + \int_\Omega f(x,t)\Phi(x)\,\mathrm{d}x\,, \qquad t \ge t_1\,.$$

From the above inequality, we know that $U(t)$ is an eventually positive solution of inequality (2.3). □

THEOREM 1. *Suppose that the function $U(t)$ satisfying the following differential inequalities*

$$\frac{\mathrm{d}}{\mathrm{d}t}[U(t) + \lambda(t)U(\tau(t))] + \alpha_1 a(t)U(t) + p(t)r(U(t)) + \int_a^b Q(t,\xi)U[g(t,\xi)]\,\mathrm{d}\sigma(\xi) \le H(t)\,, \tag{2.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[U(t) + \lambda(t)U(\tau(t))] + \alpha_1 a(t)U(t) + p(t)r(U(t)) + \int_a^b Q(t,\xi)U[g(t,\xi)]\,\mathrm{d}\sigma(\xi) \le -H(t) \tag{2.10}$$

have no eventually positive solution, then every solution of (1.1) *and* (1.2a) *is oscillatory in G.*

Proof. Assume to the contrary that there exists a nonoscillatory solution $u(x,t)$ of problem (1.1) and (1.2a). If $u(x,t) > 0$, $(x,t) \in \Omega \times [\mu,+\infty)$, $\mu \ge 0$, then from Lemma 1 it follows that $U(t)$ is an eventually positive solution of inequality (2.9), which contradicts the condition of Theorem 1.

If $u(x,t) < 0$, $(x,t) \in \Omega \times [\mu,+\infty)$, $\mu \ge 0$, let $v(x,t) = -u(x,t)$, then $v(x,t)$ is a positive solution of the problem

$$\frac{\partial}{\partial t}[u + \lambda(t)u(x,\tau(t))]$$

$$= a(t)\Delta u - c(x,t,u) - \int_a^b q(x,t,\xi)u[x,g(t,\xi)]\,\mathrm{d}\sigma(\xi) + f(x,t)\,,$$

$$u = -\varphi(x,t) \qquad \text{on} \quad (x,t) \in \partial\Omega \times \mathbb{R}_+$$

and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega u\Phi(x)\ \mathrm{d}x + \lambda(t)\int_\Omega u\big(x,\tau(t)\big)\Phi(x)\ \mathrm{d}x\right] + \alpha_1 a(t)\int_\Omega u\Phi(x)\ \mathrm{d}x$$

$$+ p(t)r\big(U(t)\big)\int_\Omega \Phi(x)\ \mathrm{d}x + \int_a^b Q(t,\xi)\left(\int_\Omega u\big[x,g(t,\xi)\big]\Phi(x)\ \mathrm{d}x\right)\ \mathrm{d}\sigma(\xi)$$

$$\leq -a(t)\int_{\partial\Omega}(-\varphi)\frac{\partial\Phi(x)}{\partial n}\ \mathrm{d}\omega + \int_\Omega f\Phi(x)\ \mathrm{d}x\,, \qquad t\geq t_1\,,$$

therefore

$$V(t) = \frac{\int_\Omega v(x,t)\Phi(x)\ \mathrm{d}x}{\int_\Omega \Phi(x)\ \mathrm{d}x} \tag{2.11}$$

would be an eventually positive solution of inequality (2.10), which contradicts the conditions of the theorem as well. This completes the proof of Theorem 1. □

COROLLARY 1. *Suppose that the function $U(t)$ satisfying the following differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[U(t)+\lambda(t)U\big(\tau(t)\big)\big] + \alpha_1 a(t)U(t) + p(t)r\big(U(t)\big) + \int_a^b Q(t,\xi)U\big[g(t,\xi)\big]\ \mathrm{d}\sigma(\xi)$$

$$\leq \int_\Omega f(x,t)\Phi(x)\ \mathrm{d}x$$

has no eventually positive solution, then every solution $u(x,t)$ of the below problem

$$\frac{\partial}{\partial t}\big[u+\lambda(t)u\big(x,\tau(t)\big)\big]$$

$$= a(t)\Delta u - c(x,t,u) - \int_a^b q(x,t,\xi)u\big[x,g(t,\xi)\big]\ \mathrm{d}\sigma(\xi) + f(x,t)\,,$$

$$u=0 \qquad on \quad (x,t)\in\partial\Omega\times\mathbb{R}_+$$

is oscillatory in G.

THEOREM 2. *Suppose that*

$$\liminf_{t\to+\infty}\int_{t_1}^{t}\left[-a(s)\int_{\partial\Omega}\varphi(x,s)\frac{\partial\Phi(x)}{\partial n}\,\mathrm{d}\omega+\int_{\Omega}f(x,s)\Phi(x)\,\mathrm{d}x\right]\mathrm{d}s=-\infty\,, \tag{2.12}$$

$$\limsup_{t\to+\infty}\int_{t_1}^{t}\left[-a(s)\int_{\partial\Omega}\varphi(x,s)\frac{\partial\Phi(x)}{\partial n}\,\mathrm{d}\omega+\int_{\Omega}f(x,s)\Phi(x)\,\mathrm{d}x\right]\mathrm{d}s=+\infty\,, \tag{2.13}$$

for sufficiently large t_1, *then every solution of* (1.1) *and* (1.2a) *is oscillatory in* G.

P r o o f. Assume that there exist nonoscillatory solution $u(x,t)$. If $u(x,t) > 0$, $(x,t)\in\Omega\times[\mu,+\infty)$, $\mu\ge 0$, then from Lemma 1 it follows that the function $U(t)$ defined in (2.4) is an eventually positive solution of the inequality (2.9). Then

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big[U(t)+\lambda(t)U\big(\tau(t)\big)\big]\\&\le\left[\int_{\Omega}\Phi(x)\,\mathrm{d}x\right]^{-1}\left[-a(t)\int_{\partial\Omega}\varphi(x,t)\frac{\partial\Phi(x)}{\partial n}\,\mathrm{d}\omega+\int_{\Omega}f(x,t)\Phi(x)\,\mathrm{d}x\right],\qquad t\ge t_1\,.\end{aligned}$$

Integrating the above inequality in the segment $[t_1,t]$, we get

$$\begin{aligned}&U(t)+\lambda(t)U\big(\tau(t)\big)\\&\le\left[\int_{\Omega}\Phi(x)\,\mathrm{d}x\right]^{-1}\int_{t_1}^{t}\left[-a(s)\int_{\partial\Omega}\varphi(x,s)\frac{\partial\Phi(x)}{\partial n}\,\mathrm{d}\omega+\int_{\Omega}f(x,s)\Phi(x)\,\mathrm{d}x\right]\mathrm{d}s\,.\end{aligned}$$

From (2.12), we have $\liminf\limits_{t\to+\infty}\frac{1}{t}\big[U(t)+\lambda(t)U\big(\tau(t)\big)\big]=-\infty$, which contradicts the assumption that $U(t)>0$.

If $u(x,t)<0$, $(x,t)\in\Omega\times[\mu,+\infty)$, $\mu\ge 0$, let $v(x,t)=-u(x,t)$, then $V(t)$ is an eventually positive solution of the inequality (2.12). From (2.13), we get

$$\begin{aligned}&\liminf_{t\to+\infty}\int_{t_1}^{t}\left[-a(s)\int_{\partial\Omega}\varphi(x,s)\frac{\partial\Phi(x)}{\partial n}\,\mathrm{d}\omega+\int_{\Omega}f(x,s)\Phi(x)\,\mathrm{d}x\right]\mathrm{d}s\\&=-\limsup_{t\to+\infty}\int_{t_1}^{t}\left[-a(s)\int_{\partial\Omega}\varphi(x,s)\frac{\partial\Phi(x)}{\partial n}\,\mathrm{d}\omega+\int_{\Omega}f(x,s)\Phi(x)\,\mathrm{d}x\right]\mathrm{d}s=-\infty\end{aligned} \tag{2.14}$$

thus, it can be proved by using a method similar to the above-mentioned one. This proves Theorem 2. □

COROLLARY 2. *Suppose that*

$$\liminf_{t\to+\infty} \int_{t_1}^{t} \int_{\Omega} f(x,s)\Phi(x)\,\mathrm{d}x\,\mathrm{d}s = -\infty, \tag{2.15}$$

$$\limsup_{t\to+\infty} \int_{t_1}^{t} \int_{\Omega} f(x,s)\Phi(x)\,\mathrm{d}x\,\mathrm{d}s = +\infty, \tag{2.16}$$

for sufficiently large t_1, *then every solution* $u(x,t)$ *of the below problem*

$$\frac{\partial}{\partial t}\left[u + \lambda(t)u(x,\tau(t))\right]$$
$$= a(t)\Delta u - c(x,t,u) - \int_a^b q(x,t,\xi)u\left[x, g(t,\xi)\right]\,\mathrm{d}\sigma(\xi) + f(x,t),$$
$$u = 0 \qquad on \quad (x,t) \in \partial\Omega \times \mathbb{R}_+$$

is oscillatory in G.

Analogously to Theorem 1 and Theorem 2, we have following Theorems.

THEOREM 3. *Suppose that the function* $V(t)$ *satisfying the following differential inequalities*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[V(t) + \lambda(t)V(\tau(t))\right] + p(t)r(V(t)) + \int_a^b Q(t,\xi)V\left[g(t,\xi)\right]\,\mathrm{d}\sigma(\xi) \le G(t), \tag{2.17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[V(t) + \lambda(t)V(\tau(t))\right] + p(t)r(V(t)) + \int_a^b Q(t,\xi)V\left[g(t,\xi)\right]\,\mathrm{d}\sigma(\xi) \le -G(t), \tag{2.18}$$

have no eventually positive solution, then every solution of (1.1) *and* (1.2b) *is oscillatory in* G, *where*

$$G(t) = \left[\int_{\Omega} \mathrm{d}x\right]^{-1} \left[a(t)\int_{\partial\Omega} \psi(x,t)\,\mathrm{d}\omega + \int_{\Omega} f(x,t)\,\mathrm{d}x\right].$$

THEOREM 4. *Suppose that*

$$\liminf_{t\to+\infty} \int_{t_1}^{t} \left[a(s)\int_{\partial\Omega} \psi(x,s)\,\mathrm{d}\omega + \int_{\Omega} f(x,t)\,\mathrm{d}x\right]\mathrm{d}s = -\infty, \tag{2.19}$$

$$\limsup_{t\to+\infty} \int_{t_1}^{t} \left[a(s)\int_{\partial\Omega} \psi(x,s)\,\mathrm{d}\omega + \int_{\Omega} f(x,t)\,\mathrm{d}x\right]\mathrm{d}s = +\infty, \tag{2.20}$$

for sufficiently large t_1, *then every solution of* (1.1) *and* (1.2b) *is oscillatory in* G.

THEOREM 5. *Suppose that the function* $V(t)$ *satisfying the following differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[V(t)+\lambda(t)V\bigl(\tau(t)\bigr)\right]+p(t)r\bigl(V(t)\bigr)+\int_a^b Q(t,\xi)V\left[g(t,\xi)\right]\,\mathrm{d}\sigma(\xi)\le 0\,, \quad (2.21)$$

has no eventually positive solution, then every solution of (1.1) *and* (1.2c) *is oscillatory in* G.

THEOREM 6. *Suppose that*

$$\liminf_{t\to+\infty}\int_{t_1}^{t}\int_{\Omega} f(x,s)\,\mathrm{d}x\,\mathrm{d}s=-\infty\,, \quad (2.22)$$

$$\limsup_{t\to+\infty}\int_{t_1}^{t}\int_{\Omega} f(x,s)\,\mathrm{d}x\,\mathrm{d}s=+\infty \quad (2.23)$$

for sufficiently large t_1, *then every solution of* (1.1) *and* (1.2c) *is oscillatory in* G.

To conclude this paper, we give an example.

EXAMPLE. Consider the following equation

$$\frac{\partial}{\partial t}\left[u+u(x,t-\pi)\right]=\frac{\partial^2 u}{\partial x^2}-u-2\int_{-\pi}^{0}u(x,t+\xi)\,\mathrm{d}\xi+\mathrm{e}^t\cos x(2\sin t-\mathrm{e}^{-\pi}\cos t)\,, \quad (2.24)$$

$$\frac{\partial u}{\partial n}(0,t)=0\,,\qquad \frac{\partial u}{\partial n}\left(\tfrac{\pi}{2},t\right)=-\,\mathrm{e}^t\sin t\,. \quad (2.25)$$

It is easy to see that conditions (H) holds, and

$$\int_{t_1}^{t}\left[a(s)\int_{\partial\Omega}\psi(x,s)\,\mathrm{d}\omega+\int_{\Omega}f(x,s)\,\mathrm{d}x\right]\mathrm{d}s$$

$$=\int_0^t\left[\mathrm{e}^s\sin s-\mathrm{e}^{-\pi}\cos s\right]\mathrm{d}s$$

$$=\mathrm{e}^t\sin t\left[\frac{1}{2}-\mathrm{e}^{-\pi}\right]-\mathrm{e}^t\cos t\left[\frac{1}{2}+\mathrm{e}^{-\pi}\right]+\frac{1}{2}+\mathrm{e}^{-\pi}\,.$$

Hence all the conditions of Theorem 4 hold. It follows from Theorem 4 that every solution of problem (2.24), (2.25) is oscillatory in $\left(0,\frac{\pi}{2}\right)\times[0,+\infty)$. For example $u(x,t)=\mathrm{e}^t\cos x\sin t$ is one oscillatory solution.

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