Pham Ngoc Ánh; László Márki

A general theory of Fountain-Gould quotient rings

Mathematica Slovaca, Vol. 44 (1994), No. 2, 225--235

Persistent URL: http://dml.cz/dmlcz/131195

Math. Slovaca, **44** (1994), No. 2, 225–235

Mathematica Slovaca
©1994
Mathematical Institute
Slovak Academy of Sciences

Dedicated to Academician Štefan Schwarz on the occasion of his 80th birthday

A GENERAL THEORY OF FOUNTAIN–GOULD QUOTIENT RINGS

PHAM NGOC ÁNH — LÁSZLÓ MÁRKI [1]

(Communicated by Tibor Katriňák)

ABSTRACT. Fountain and Gould [1] introduced a new generalization of classical quotient rings. These new quotient rings have been described for some special classes of rings in subsequent papers. In the present paper we develop their general theory and compare Fountain-Gould quotient rings with the classical ones. It is seen, in particular, that for rings with identity the notion of Fountain-Gould quotient rings is more restrictive.

1. Definitions

In what follows we are going to look at four kinds of quotient rings. Since it is important not to confuse them, we recall the definitions of three of them here, and take only the notion of classical left quotient rings for known.

DEFINITION 1. (Utumi [6]) *Let R be a subring of a ring S. We say that S is a left quotient ring of R if for all $x, y \in S$ with $x \neq 0$ there is an $a \in R$ such that $ax \neq 0$ and $ay \in R$.*

Notice that a ring is a left quotient ring of itself if and only if it is right faithful.

AMS Subject Classification (1991): Primary 16A08.
Key words: Ring, Quotient.
[1] This research was supported partly by Hungarian National Foundation for Scientific Research, the Grant no. T4265 .

DEFINITION 2. *If R is a right faithful ring, then by* Utumi [6] *it has a unique maximal left quotient ring, which we shall call the Utumi left quotient ring of R.*

Utumi's construction [6] of the Utumi left quotient ring of a right faithful ring R goes back to R. E. Johnson [4] in the nonsingular case. Using homological language, this construction is just $S = \varinjlim \mathrm{Hom}_R(L, R)$, where L runs through those left ideals of R for which R is a left quotient ring. This S is an abelian group, on which multiplication is defined as composition of partial mappings of R.

DEFINITION 3. An element b of a ring R is a *group inverse* of the element a of R if $aba = a$, $bab = b$ and $ab = ba$. A group inverse, if it exists, is unique. The group inverse of a will be denoted by $a^{\#}$; in fact, $a^{\#}$ is the inverse of a, in the usual sense, in a subring of the form eRe of R, where e is an idempotent. Thus (group) inverses are taken with respect to any idempotent, not only an identity, element. If R has an identity, and an element a has an inverse in the usual sense, then $a^{\#} = a^{-1}$.

Remark. The unicity of the group inverse is a well-known fact in semigroups. It is easily seen e.g. as follows. Suppose $a_1^{\#}$, $a_2^{\#}$ are group inverses of a, then $a_2^{\#} = (a_2^{\#})^2 a = (a_2^{\#})^2 \left[aa^2(a_1^{\#})^2\right] = \left[(a_2^{\#})^2 a^2 a\right](a_1^{\#})^2 = a(a_1^{\#})^2 = a_1^{\#}$.

DEFINITION 4. An element a of R is said to be *left square-cancellable* if, for any $x, y \in R \cup \{1\}$, $a^2x = a^2y$ implies $ax = ay$. *Right square-cancellable* elements are defined dually; *square-cancellable* means both left and right square-cancellable.

The set of square-cancellable elements of R will be denoted by $\mathcal{S}(R)$. Clearly, any element of R which has a group inverse in an overring of R, must be square-cancellable in R.

DEFINITION 5. (Fountain and Gould [1]) *Let R be a subring of a ring Q. We say that R is a left order in Q and Q is a Fountain-Gould left quotient ring of R if*

(i) *every $a \in \mathcal{S}(R)$ has a group inverse in Q,*
(ii) *every $q \in Q$ can be written as $q = a^{\#}b$ for some $a \in \mathcal{S}(R)$ and $b \in R$.*

By [1; Lemma 2.1], we know that every $q \in Q$ can be written as $q = a^{\#}b$ with $aa^{\#}b = b$, $b \in R$: one just replaces $a^{\#}b$ by $(a^2)^{\#}ab$. This will be assumed throughout.

NOTATION. We denote by $\ell_R(a) = \{x \in R \mid xa = 0\}$ the *set of left annihilators of* $a \in R$. If there is no danger of confusion, then we shall omit the subscript R.

If R is a left order in Q and $a \in \mathcal{S}(R)$, then $aa^{\#} = a^{\#}a \in Q$ is idempotent, $\ell_Q(a) = \ell_Q(aa^{\#}) = Q\ell_R(a)$, $Qa = Qaa^{\#}$, and, for any idempotent $e \in Q$, $\ell_Q(e) + Qe = Q$. Hence

$$Q\big(\ell_R(a) + Ra\big) = Q\ell_R(a) + QRa = \ell_Q(a) + Qa = \ell_Q(aa^{\#}) + Qaa^{\#} = Q\,.$$

Furthermore, if $b \in R$, $a \in \mathcal{S}(R)$ and $\ell(a) \subseteq \ell(b)$, then $b = aa^{\#}b$ in Q because

$$Q(b - a^{\#}ab) = \big(\ell_Q(a) + Qa\big)(b - a^{\#}ab) = 0,$$

and writing $b - a^{\#}ab = c^{\#}d$ with $d = cc^{\#}d \in Qc^{\#}d$ we get $d = 0$, hence $b = a^{\#}ab$.

2. Results

THEOREM 1. *Let R be a left order in Q (in the sense of Fountain and Gould). Then Q is a left quotient ring of R.*

Proof. Take any $x \neq 0$ and y from Q. We may write y in the form $y = c^{\#}d$ with $cc^{\#}d = d$, hence $\big(\ell_R(c) + Rc\big)c^{\#}d \subseteq R$. Now $\ell_R(x) \supseteq \ell_R(c) + Rc$ would imply $\ell_Q(x) \supseteq Q\ell_R(x) \supseteq Q\big(\ell_R(c) + Rc\big) = Q$, hence $x = 0$, contrary to our assumption. Thus $\ell_R(x) \not\supseteq \ell_R(c) + Rc$, and for any $r \in \big(\ell_R(c) + Rc\big) \setminus \ell_R(x)$ we have $rx \neq 0$ and $rc^{\#}d \in R$, which proves our assertion.

COROLLARY 1. *If R is a left order in Q, then Q is contained in the Utumi left quotient ring S of R.*

COROLLARY 2. *If a commutative ring R is a left order in a ring Q, then Q is also commutative.*

Indeed, this is proven for the Utumi left quotient ring of R in Lambek [5] in case R has an identity, but the existence of the identity is not needed in the proof.

Remark. If a ring has a Fountain-Gould left quotient ring, then the latter is, in general, smaller than the Utumi left quotient ring. Namely, if the ring consists of zero divisors only, then the same is true for its Fountain-Gould left quotient ring, while the Utumi left quotient ring always has an identity.

Since every square-cancellable element of R belongs to a subgroup of Q, and thus to a subgroup of S, its inverse is uniquely determined in S. As Q is generated by R and the inverses of the square-cancellable elements of R, this means that Q is a uniquely determined subring of S.

Thus we obtain:

COROLLARY 3. (G o u l d [3; Theorem 5.9]) *If a ring R has a Fountain-Gould left quotient ring Q, then Q is unique up to isomorphisms.*

Next we prove that the Common Left Denominator Theorem holds in Fountain-Gould left quotient rings.

PROPOSITION 2. (cf. G o u l d [3; Proposition 5.5]) *Let R be a left order in Q. Then for any $a, b \in \mathcal{S}(R)$ there is an $r \in \mathcal{S}(R)$ such that $\ell_Q(a) \cap \ell_Q(b) \supseteq \ell_Q(r)$.*

P r o o f. Put $e = a^{\#}a$, $f = b^{\#}b$. Write $f - ef$ in the form $x^{\#}y$, where $xx^{\#}y = y$, and put $g = xx^{\#}$, $w = e + g - ge$. By straightforward computation we verify that $we = e$ and $wf = f$. Write w in the form $r^{\#}s$ with $rr^{\#}s = s$, then we have $r^{\#}rw = w$, hence $r^{\#}re = r^{\#}rwe = we = e$, and similarly $r^{\#}rf = f$. Thus $\ell_Q(r) \subseteq \ell_Q(e) \cap \ell_Q(f) = \ell_Q(a) \cap \ell_Q(b)$.

PROPOSITION 3. *Let R be a left order in Q and $p, q \in Q$ be arbitrary elements. Then there exist $x \in \mathcal{S}(R)$ and $y \in R$ such that $p = x^{\#}y$ and $x^{\#}xq = q$.*

P r o o f. Write p and q in the form $p = a^{\#}b$, $q = c^{\#}d$. By Proposition 2, there is an $r \in \mathcal{S}(R)$ such that $\ell(r) \subseteq \ell(a) \cap \ell(c)$. Put $e = r^{\#}r$, $f = a^{\#}a$, $g = c^{\#}c$, then we have $ef = f$ and $eg = g$. Put $h = e + f - fe$, then straightforward verification yields $hf = fh = f$ and $hg = g$. Put $u = h - f + a$, $v = h - f + a^{\#}$, then it is easy to check that $uv = vu = h$ and $vf = fv = a^{\#}$. Write v in the form $v = x^{\#}z$ with $xx^{\#}z = z$, then we get

$$p = a^{\#}b = vfb = vaa^{\#}b = vb = x^{\#}(zb),$$

and

$$\begin{aligned} x^{\#}xq &= x^{\#}xc^{\#}d = x^{\#}x(gc^{\#})d = x^{\#}x(hg)c^{\#}d = x^{\#}xhc^{\#}d \\ &= x^{\#}xvuc^{\#}d = vuc^{\#}d = hc^{\#}d = c^{\#}d = q\ . \end{aligned}$$

PROPOSITION 4. *Let R be a left order in Q. Then for any $a, b \in \mathcal{S}(R)$ there are $u \in \mathcal{S}(R)$, $v, w \in R$ such that $a^{\#} = u^{\#}v$ and $b^{\#} = u^{\#}w$.*

Proof. By Proposition 3, for $(a^2)^{\#}$, $b^{\#} \in Q$ there exist $x \in \mathcal{S}(R)$ and $y \in R$ such that $(a^2)^{\#} = x^{\#}y$ and $x^{\#}xb^{\#} = b^{\#}$. Again by Proposition 3, there exist $r \in \mathcal{S}(R)$, $s \in R$ such that $xb^{\#} = r^{\#}s$, $r^{\#}rya = ya$. Write $x^{\#}r^{\#}$ in the form $u^{\#}z$. Now we have

$$b^{\#} = x^{\#}xb^{\#} = x^{\#}r^{\#}s = u^{\#}zs,$$

and

$$a^{\#} = (a^2)^{\#}a = x^{\#}ya = x^{\#}r^{\#}rya = u^{\#}zrya.$$

As an immediate corollary to Proposition 4, we obtain:

THEOREM 5. (**Common Left Denominator Theorem**) *Let R be a left order in Q, then for any $p, q \in Q$ there exist $u \in \mathcal{S}(R)$, $v, w \in R$ such that $p = u^{\#}v$, $q = u^{\#}w$.*

This result has been known only in the case when Q is regular (see Fountain and Gould [1; Theorem 4.3]).

By induction, the Common Left Denominator Theorem holds for arbitrary finite subsets of Q.

THEOREM 6. *A ring R has a Fountain-Gould left quotient ring if and only if it satisfies the following conditions:*

(1) *for every $a \in R$ there is a $c \in \mathcal{S}(R)$ such that $\ell(c) \subseteq \ell(a)$;*
(2) *for every $a \in \mathcal{S}(R)$ and $r \in R$, $\big(\ell(a) + Ra\big)r = 0$ implies $r = 0$;*
(3) *for every $a, b \in \mathcal{S}(R)$ there exist $c \in \mathcal{S}(R)$ and $x, y \in R$ such that*

$$\ell(c) \subseteq \ell(a) \cap \ell(b), \qquad ca = xa^2, \qquad cb = yb^2;$$

(4) *For every $a, c \in \mathcal{S}(R)$ and $b \in R$ there exist $u \in \mathcal{S}(R)$ and $v, x \in R$ such that*

$$\ell(u) \subseteq \ell(a), \qquad ua = xa^2, \qquad xbc = vc^2.$$

Remarks.

1. Conditions (1) and (2) are relatively mild; unlike the case of classical quotient rings, they are needed here because $\mathcal{S}(R)$ may contain zero divisors. Condition (3) will ensure the common denominator property; it is needed because $\mathcal{S}(R)$ need not be multiplicatively closed. Condition (4) is a generalization

of the left Ore condition. In fact, if $\ell(u) \subseteq \ell(a)$ in the quotient ring Q with u and a square-cancellable, then $u^{\#}a = x$ if and only if $ua = u^2x$. Thus Condition (4) expresses $a^{\#}bc^{\#} = u^{\#}v$.

2. By applying Condition (3) several times, we see that this condition holds if instead of $a, b \in \mathcal{S}(R)$ we start from an arbitrary finite number of elements of $\mathcal{S}(R)$. Furthermore, $x \in Ra$, $y \in Rb$ such that $\ell(c) \subseteq \ell(x) \cap \ell(y)$ can be chosen in Condition (3). Indeed, we have $\ell(a) = \ell(a^2)$, $\ell(b) = \ell(b^2)$. Apply now the condition for $a^2, b^2 \in \mathcal{S}(R)$; then we get $ca^2 = xa^4$, hence (by $a \in \mathcal{S}(R)$) $ca = xa^3 = (xa)a^2$, and then $c^2a = c(ca) = (cxa)a^2$, and similarly $c^2b = (cyb)b^2$. Here $\ell(c^2) = \ell(c) \subseteq \ell(cxa) \cap \ell(cyb)$.

3. In Condition (4), u, v, x can be chosen so that $\ell(u) \subseteq \ell(v) \cap \ell(x)$. Indeed, in the same way as above, we first choose u, v, x and then replace them by u^2, uv, ux. Furthermore, v can be chosen from Rc as one can see in the same way as in the previous remark.

4. We have two proofs for Theorem 6. One of them is constructive, but it requires long pages of computations, hence we prefer giving only a sketch of this proof and present a full proof which makes use of the Utumi left quotient ring.

Proof.

Necessity: Suppose that R has a Fountain-Gould left quotient ring Q.

(1) Every $a \in R$ can be written in the form $c^{\#}d$, and here $\ell_R(c) \subseteq \ell_Q(c^{\#}d) \cap R = \ell_Q(a) \cap R = \ell_R(a)$.

(2) For $a \in \mathcal{S}(R)$ we have $Q\big(\ell_R(a) + Ra\big) = Q$, hence $\big(\ell_R(a) + Ra\big)r = 0$ implies $Qr = 0$, which implies in turn $r = 0$.

(3) Take any $a, b \in \mathcal{S}(R)$. By the Common Denominator Theorem, $a^{\#}, b^{\#} \in Q$ can be written in the form $a^{\#} = c^{\#}x$, $b^{\#} = c^{\#}y$ with $c \in \mathcal{S}(R)$, $x, y \in R$. Now we get $ca = ca^{\#}a^2 = cc^{\#}xa^2 = xa^2$, and similarly $cb = yb^2$. Clearly, $\ell(c) \subseteq \ell(a) \cap \ell(b)$.

(4) Take any $a, c \in \mathcal{S}(R)$ and $b \in R$. Again, by the Common Denominator Theorem, there exist $u \in \mathcal{S}(R)$, $v, x \in R$ such that $a^{\#}bc^{\#} = u^{\#}v$, $a^{\#} = u^{\#}x$. Clearly, $\ell(u) \subseteq \ell(a)$. Now we get, as above, $ua = xa^2$. Furthermore, $u^{\#}xbc^{\#} = a^{\#}bc^{\#} = u^{\#}v$, hence $xbc = uu^{\#}xbc^{\#}c^2 = uu^{\#}vc^2 = vc^2$.

Sufficiency: Suppose that R is a ring which satisfies conditions (1)–(4). According to Remarks 2 and 3, we may assume that $x \in Ra$, $y \in Rb$, $\ell(c) \subseteq \ell(x) \cap \ell(y)$ hold in Condition (3), and $\ell(u) \subseteq \ell(v) \cap \ell(x)$ holds in Condition (4).

Condition (2) implies that R is right faithful, hence it has an Utumi left quotient ring S.

Next we claim that, for every $a \in \mathcal{S}(R)$, R is a left quotient ring of $\ell(a)+Ra$. Take any $x, y \in R$ with $x \neq 0$. First, we are going to prove the existence of an $s \in R$ such that $sx \neq 0$, $sy \in \ell(a) + Ra$. By applying Condition (1) to x and y, and then Condition (3) to the elements of $\mathcal{S}(R)$ thus obtained, we find a $b \in \mathcal{S}(R)$ such that $\ell(b) \subseteq \ell(x) \cap \ell(y)$. Now we apply Condition (4) to $b, a \in \mathcal{S}(R)$ and $by \in R$ and find elements $u \in \mathcal{S}(R)$, $v, w \in R$ such that $\ell(u) \subseteq \ell(b)$, $ub = vb^2$ and $vbya = wa^2$. From $(u - vb)b = 0$ and $\ell(b) \subseteq \ell(x)$ we get $ux = vbx$. Here, $ux \neq 0$ because, in case $ux = 0$, we would have, in view of $\ell(u) \subseteq \ell(b) \subseteq \ell(x)$, $\big(\ell(u) + Ru\big)x = 0$, hence, by Condition (2), $x = 0$. Next, by $(vby - wa)a = 0$, we have $vby \in \ell(a) + Ra$. So, $s = vb$ does the job. Secondly, we apply what we have just proven to $sx \neq 0$ and s, and find a $t \in R$ such that $tsx \neq 0$ and $ts \in \ell(a) + Ra$. Then $r = ts$ is an element we have been looking for, since $sy \in \ell(a) + Ra$ implies $tsy \in \ell(a) + Ra$.

Since $a^2 \in \mathcal{S}(R)$, R is also a left quotient ring of $\ell(a^2) + Ra^2 = \ell(a) + Ra^2$. By [2; Lemma 3.12], we have now $\ell(a) \cap Ra = 0 = \ell(a) \cap Ra^2$. Denote by $a^\#$ the mapping $\ell(a) + Ra^2 \to R$ which sends $x + ra^2$ to ra, $x \in \ell(a)$. This is an R-homomorphism, hence $a^\# \in S$. In S, $a \in R$ can be represented as the R-homomorphism from $\ell(a) + Ra$ to R which sends $x + ra$ to ra^2, $x \in \ell(a)$. Now it is clear that $aa^\# = a^\# a$ is idempotent on $\ell(a) + Ra^2$, hence a lies in a multiplicative subgroup of S with unity $aa^\# = a^\# a$, and therefore $a^\#$ is the group inverse of a in S.

Thus we have shown that $a^\#$ exists in S for every $a \in \mathcal{S}(R)$.

For an arbitrary $r \in R$, we choose a $c \in \mathcal{S}(R)$ such that $\ell(c) \subseteq \ell(r)$. Then $\big(\ell(c) + Rc\big)(cc^\# r - r) = 0$, hence $c^\#(cr) = cc^\# r = r$ in S; in other words, every element r of R can be written in S in the form $c^\# cr$ for any $c \in \mathcal{S}(R)$ satisfying $\ell(c) \subseteq \ell(r)$.

Denote by Q the subring of S generated by R and the set $\{a^\# \mid a \in \mathcal{S}(R)\}$. Given any $a, b \in \mathcal{S}(R)$, Condition (3) and Remark 2 give $a^\# = c^\# ca(a^\#)^2 = c^\# xa^2(a^\#)^2 = c^\# x$ for $x \in Ra$, and similarly, $b^\# = c^\# y$. Next, given any $a, c \in \mathcal{S}(R)$ and $b \in R$, Condition (4) and Remark 3 yield $a^\# bc^\# = u^\# xbc(c^\#)^2 = u^\# vc^2(c^\#)^2 = u^\# v$. From these observations it follows that every element of Q can be written in the form $a^\# b$ for some $a \in \mathcal{S}(R)$, $b \in R$.

Thus R is a left order in Q.

Sketch of a constructive proof for the sufficiency of the conditions in Theorem 6

Given a ring R which satisfies these conditions, put

$$Q_1 = \{(a,b) \mid a \in \mathcal{S}(R)\,, \ b \in R\,, \ \ell(a) \subseteq \ell(b)\}$$

and

$$Q = Q_1/\!\sim\,,$$

where

$$(a,b) \sim (c,d) \iff \exists\, u \in \mathcal{S}(R)\ \exists\, x,y \in R \text{ with } \ell(u) \subseteq \ell(a) \cap \ell(c)\,.$$
$$ua = xa^2\,, \quad uc = yc^2\,, \quad xb = yd\,.$$

This $\sim$ can be shown to be an equivalence relation.

Definition of $+$ *:* Given (a,b), $(c,d) \in Q$, by Condition (3) and Remark 2. to $a, c \in \mathcal{S}(R)$ we find $u \in \mathcal{S}(R)$, $x, y \in R$ such that $\ell(u) \subseteq \ell(a) \cap \ell(c) \cap \ell(x) \cap \ell(y)$, $ua = xa^2$, $uc = yc^2$. Now we put

$$(a,b) + (c,d) = (u, xb + yd)\,.$$

Definition of $\cdot$ *:* Given (a,b), $(c,d) \in Q$, by Condition (4) and Remark 3. to $a, c \in \mathcal{S}(R)$, $b \in R$ we find $u \in \mathcal{S}(R)$, $x, v \in R$ such that $\ell(u) \subseteq \ell(a) \cap \ell(x) \cap \ell(v)$, $ua = xa^2$, $xbc = vc^2$. Now we put

$$(a,b) \cdot (c,d) = (u, vd)\,.$$

It can be shown that these operations are well defined and they turn Q into a ring.

For an arbitrary $a \in R$, by Condition (1), we find a $c \in \mathcal{S}(R)$ such that $\ell(c) \subseteq \ell(a)$. It can be shown that the mapping $a \mapsto (c, ca)$ is well defined and is an embedding of R into Q.

Finally, for any $c \in \mathcal{S}(R)$, $(c,c) \in Q$ is idempotent, and (c^2, c) is the group inverse of $(c, c^2) = c$ in Q. Next, given $(a,b) \in Q$, it is straightforward to check that $(a^2, a)(a,b) = (a,b)$ in Q.

Remark. In this proof we use only right square-cancellability of the elements of $\mathcal{S}(R)$. Hence this proof shows the validity of the following.

PROPOSITION 7. *Let $\mathcal{S}$ be a set of right square-cancellable elements of a ring R such that Conditions (1)–(4) of Theorem 6 are satisfied in R if $\mathcal{S}(R)$ is replaced by $\mathcal{S}$. Then R can be embedded in a ring Q such that*

(i) *the elements of $\mathcal{S}$ have group inverses in Q,*

(ii) *every element of Q can be written in the form $a^{\#}b$ with $a \in \mathcal{S}$, $b \in R$.*

This shows, in particular, that Q is a left quotient ring of R.

THEOREM 8. *Let R be a left order in Q, and $f\colon R \to S$ be a ring homomorphism such that, for every $a \in \mathcal{S}(R)$, $f(a)$ has a group inverse $f(a)^{\#}$ in S. Then f can be extended to a homomorphism $\overline{f}\colon Q \to S$ if and only if, for every $a \in \mathcal{S}(R)$ and $b \in R$, $\ell_R(a) \subseteq \ell_R(b)$ implies $\ell_S\big(f(a)\big) \subseteq \ell_S\big(f(b)\big)$. If $\overline{f}$ exists, then it is unique. If f is one-to-one, then so is $\overline{f}$.*

Proof. Let $a \in \mathcal{S}(R)$. Since the group inverses $a^{\#} \in Q$ and $f(a)^{\#} \in S$ are uniquely determined by a and f, the only possibility to define $\overline{f}$ is $\overline{f}(a^{\#}) = f(a)^{\#}$, and then $\overline{f}(a^{\#}b) = f(a)^{\#}f(b)$.

If $\overline{f}\colon Q \to S$ is an extension of f and $\ell_R(a) \subseteq \ell_R(b)$ for some $a \in \mathcal{S}(R)$ and $b \in R$, then we have $b = a^{\#}ab = aa^{\#}b$, hence $f(b) = f(a)f(a)^{\#}f(b)$, which shows that $\ell_S\big(f(a)\big) \subseteq \ell_S\big(f(b)\big)$.

Conversely, suppose that $\ell_S\big(f(a)\big) \subseteq \ell_S\big(f(b)\big)$ whenever $\ell_R(a) \subseteq \ell_R(b)$ for $a \in \mathcal{S}(R)$, $b \in R$. We have to show that the $\overline{f}$ above is well defined. Suppose that $a^{\#}b = c^{\#}d$ in Q. By the Common Denominator Theorem, there exist $u \in \mathcal{S}(R)$ and $x, y \in R$ such that $\ell_R(u) \subseteq \ell_R(a) \cap \ell_R(c)$, $a^{\#} = u^{\#}x$, $c^{\#} = u^{\#}y$, i.e. $ua = xa^2$ and $uc = yc^2$. In the same way as in Remark 2 to Theorem 6, we may choose $x \in Ra$, $y \in Rb$. Now $u^{\#}xb = a^{\#}b = c^{\#}d = u^{\#}yc$, hence $xb = uu^{\#}xb = uu^{\#}yd = yd$. Next, from $ua = xa^2$ we obtain $f(u)f(a) = f(x)f(a^2)$, hence

$$f(u)^{\#}f(u)f(a) = f(u)^{\#}f(x)f(a)^2 .$$

By the assumption, $\ell_R(u) \subseteq \ell_R(a)$ implies $\ell_S\big(f(u)\big) \subseteq \ell_S\big(f(a)\big)$, hence $f(u)^{\#}f(u)f(a) = f(a)$. Therefore we have

$$f(a)^{\#} = f(a)\big(f(a)^{\#}\big)^2 = f(u)^{\#}f(x)f(a)^2\big(f(a)^2\big)^{\#} = f(u)^{\#}f(x)$$

because $x \in Ra$. Similarly, $f(c)^{\#} = f(u)^{\#}f(y)$. In view of $xb = yd$, we have $f(x)f(b) = f(y)f(d)$, whence $f(a)^{\#}f(b) = f(u)^{\#}f(x)f(b) = f(u)^{\#}f(y)f(d) = f(c)^{\#}f(d)$, as was to be shown. By the Common Denominator Theorem, it is clear that $\overline{f}$ is additive.

Finally, $a^{\#}bc^{\#} = u^{\#}v$ with $\ell_R(u) \subseteq \ell_R(a)$, $v \in Rc$, means that $ua = xa^2$ and $xbc = vc^2$ for some $x \in Ra$. Since f takes this over to $f(u)f(a) = f(x)f(a)^2$ and $f(x)f(b)f(c) = f(v)f(c)^2$, $f(x) \in Sf(a)$, $f(v) \in Sf(c)$, $\ell_S(f(u)) \subseteq \ell_S(f(a))$, we obtain that

$$f(a)^{\#}f(b)f(c)^{\#} = f(u)^{\#}f(v).$$

It follows that f preserves multiplication.

COROLLARY. *If R is a left order in Q and a right order in S, then there is an isomorphism $Q \to S$ which is the identity on R.*

Proof. Suppose $\ell_R(a) \subseteq \ell_R(b)$ for some $a \in \mathcal{S}(R)$, $b \in R$. Then $aR \subseteq r_R(\ell_R(a))$ and $bR \subseteq r_R(\ell_R(a))$. Next, if $x \in r_R(\ell_R(a)) \cap r_R(a)$, then $\ell_R(a)x = 0 = ax$, hence $(\ell_R(a) + Ra)x = 0$, and then $x = 0$ by Theorem 6, Condition (2). Therefore $r_R(\ell_R(a)) + r_R(a)$ is a direct sum. On the other hand, we have $[aR + r_R(a)]S = aRS + r_R(a)S = aS + r_S(a)$ and the latter is also a direct sum for a has a group inverse. Thus we have $S = aS \oplus r_S(a) = [aR + r_R(a)]S \subseteq [r_R(\ell_R(a)) \oplus r_R(a)]S = r_R(\ell_R(a))S \oplus r_S(a)$. This implies that $aS = r_R(\ell_R(a))S \supseteq bS$, whence $\ell_S(a) \subseteq \ell_S(b)$.

By Theorem 8, the identity mapping of R extends now to a unique embedding of Q into S. Here $a^{\#}$ in Q goes to $a^{\#}$ in S for every $a \in \mathcal{S}(R)$, and since S is generated by R and the elements of the form $a^{\#}$, this embedding is onto.

Finally, we would like to say some words about the relation of classical left quotient rings and Fountain-Gould left quotient rings.

PROPOSITION 9. *Suppose that a ring R has a Fountain-Gould left quotient ring Q and an element which is not a zero-divisor. Then Q is also the classical left quotient ring of R.*

Proof. Let $a \neq 0$ be an element of R which is not a zero-divisor. Put $e = aa^{\#}$ in Q, then $\ell_Q(e) = Q\ell_R(a) = 0$. Since $(q - qe)e = 0$ for every $q \in Q$, we obtain that e is a right identity in Q. Take now any $r \in R$, $r \neq 0$. Then by $Qa = Qe = Q$, we have $Q(r - er) = Qa(r - er) = Q(ar - aer) = 0$, whence $r - er = 0$. For any $c \in \mathcal{S}(R)$, this implies that $c^{\#} = c(c^{\#})^2 = ec(c^{\#})^2 = ec^{\#}$, from which it follows that $eq = q$ for all $q \in Q$. Thus e is an identity of Q. By [1; Theorem 3.4], this implies that R is a classical left order in Q.

COROLLARY. *If a ring R has a Fountain-Gould left quotient ring R and a classical left quotient ring S, then Q and S are isomorphic over R.*

[1; Example 3.1] shows a ring which has a classical left quotient ring but no Fountain-Gould left quotient ring. On the other hand, a ring which has a

Fountain-Gould left quotient ring without identity, cannot have a classical left quotient ring.

REFERENCES

[1] FOUNTAIN, J.—GOULD, V.: *Orders in rings without identity*, Comm. Algebra **18** (1990), 3085–3110.

[2] FOUNTAIN, J.—GOULD, V.: *Orders in regular rings with minimal condition for principal right ideals*, Comm. Algebra **19** (1991), 1501–1527.

[3] GOULD, V.: *Semigroup of left quotients — the uniqueness problem*, Proc. Edinburgh Math. Soc. (2) **35** (1992), 213–226.

[4] JOHNSON, R. E.: *The extended centralizer of a ring over a module*, Proc. Amer. Math. Soc. **2** (1951), 891–895.

[5] LAMBEK, J.: *Lectures on Rings and Modules*, Blaisdell Publ. Co., Waltham-Toronto -London, 1966.

[6] UTUMI, Y.: *On quotient rings*, Osaka J. Math. **8** (1956), 1–18.

Received August 5, 1993

Mathematical Institute
Hungarian Academy of Sciences
H-1364 Budapest, Pf. 127
Hungary