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ON INTERVAL SUBALGEBRAS OF GENERALIZED MV-ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$ be a generalized MV-algebra with the underlying set A. Under the well-known notation, there exists a unital lattice ordered group (G,u) such that $\mathcal{A} = \Gamma(G,u)$. By applying the fundamental operations of $\mathcal{A}$ we can define a partial order $\leqq$ on A. Let $a,b \in A$, $a \leqq b$ and let $A_1 = [a,b]$ be the interval of $(A;\leqq)$. In this paper we prove that there exists a generalized MV-algebra $\mathcal{A}_1$ with the underlying set A_1 such that the fundamental operations of $\mathcal{A}_1$ are induced by certain polynomial functions over G.

I. Introduction

The notion of a generalized MV-algebra was introduced by Rachůnek [10], and by Georgescu and Iorgulescu [7], [8]. In [7] and [8], the term "pseudo MV-algebra" was applied.

A generalized MV-algebra is an algebraic system $\mathcal{A} = (A;\oplus,\neg,\sim,0,1)$ of type $(2,1,1,0,0)$ satisfying certain axioms; the definition is recalled in Section 2 below. If the operation $\oplus$ is commutative, then $\mathcal{A}$ is an MV-algebra; in this case, the operation $\sim$ coincides with the operation $\neg$. (Cf. Cignoli, D'Ottaviano and Mundici [5].)

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For each generalized MV-algebra $\mathcal{A}$ there exists a lattice ordered group G with a strong unit u such that, under the well-known notation, $\mathcal{A} = \Gamma(G, u)$. (Cf. Dvurečenskij [6].)

The main result of Chajda and Kühr [3] is the following theorem:

(α) *Let $\mathcal{A}$ be an MV-algebra and let $[a, b]$ be an interval of $(A; \leqq)$. Denote $[a, b] = A^*$ and for each $x, y \in A^*$ put*

$$x \oplus^* y = \big(\neg(\neg x \oplus a) \oplus y\big) \wedge b\,,$$
$$\neg^* x = \neg(x \oplus \neg b) \oplus a\,.$$

Then $\mathcal{A}^* = (A^*, \oplus^*, \neg^*, a, b)$ is an MV-algebra.

In proving (α), the authors applied the relation $\mathcal{A} = \Gamma(G, u)$, and the results of their earlier papers [1] and [2].

In the present paper we prove:

(β) *Let $\mathcal{A}$ be a generalized MV-algebra and let $[a, b]$ be an interval of $(A; \leqq)$. Denote $[a, b] = A_1$ and for each $x, y \in A_1$ put*

$$x \oplus_1 y = (x - a + y) \wedge b\,,$$
$$\neg_1 x = b - x + a\,, \qquad \sim_1 x = a - x + b\,.$$

Then we have

(i) *$\mathcal{A}_1 = (A_1; \oplus_1, \neg_1, \sim_1, a, b)$ is a generalized MV-algebra.*

(ii) *If $\mathcal{A}$ is an MV-algebra, then $\mathcal{A}_1$ is an MV-algebra as well; moreover, the operation $\oplus_1$ coincides with $\oplus^*$, and operation $\neg_1$ coincides with $\neg^*$.*

By proving (β), we do not apply (α). In view of (ii), the assertion (β) is a generalization of (α).

Some further results on a generalized MV-algebra $\mathcal{A}_1$ are proved; they concern Boolean elements of $\mathcal{A}_1$.

The intervals of $(A; \leqq)$ having the form $[0, b]$ were studied in the author's paper [10].

For the definition of the binary operation $\odot$ in an MV-algebra, cf. Section 2 below.

Chajda and Kühr [4; Theorem 3.2] proved the following result:

(γ) *Let $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1)$ be a generalized MV-algebra and let $a, b \in A$ such that $a < b$. For $x, y \in [a, b]$ define $x \oplus_{ab} y = \big(x \oplus (y \odot \sim a)\big) \wedge b$, $\neg_{ab} x = (\neg x \odot b) \oplus a$ and $\sim_{ab} x = a \oplus (b \cdot \sim x)$. Then $\big([a, b]; \oplus_{ab}, \neg_{ab}, \sim_{ab}, a, b\big)$ is a generalized MV-algebra.*

We prove:

(δ) *Let us apply the notation as in (β) and (γ). Let $x, y \in [a, b]$. Then*

$$x \oplus_{ab} y = x \oplus_1 y\,, \qquad \neg_{ab} x = \neg_1 x\,, \qquad \sim_{ab} x = \sim_1 x\,.$$

2. Preliminaries

We recall the definition of a generalized MV-algebra.

DEFINITION 2.1. Let $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1)$ be an algebra of type $(2, 1, 1, 0, 0)$. For $x, y \in A$ we put $x \odot y = \sim(\neg x \oplus \neg y)$. Then $\mathcal{A}$ is called a *generalized MV-algebra* if the following identities are valid:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 \oplus x = 1$;
(A4) $\neg 1 = 0$; $\sim 1 = 0$;
(A5) $\neg(\sim x \oplus \sim y) = \sim(\neg x \oplus \neg y)$;
(A6) $x \oplus (y \odot \sim x) = y \oplus (x \odot \sim y) = (\neg y \odot x) \oplus y = (\neg x \odot y) \oplus x$;
(A7) $(\neg x \oplus y) \odot x = y \odot (x \oplus \sim y)$;
(A8) $\sim\neg x = x$.

Let $\mathcal{A}$ be a generalized MV-algebra. For $x, y \in A$ we set $x \leqq y$ if $\neg x \oplus y = 1$. Then $(A; \leqq)$ is a distributive lattice with the least element 0 and the greatest element 1. We put $(A; \leqq) = \ell(\mathcal{A})$ and we say that $\ell(\mathcal{A})$ is the *underlying lattice* of $\mathcal{A}$.

For $a, b \in A$ with $a \leqq b$, the interval $[a, b]$ has the usual meaning.

Let G be a lattice ordered group with a strong unit u. We put $A = [0, u]$ and for $x, y \in A$ we set

$$x \oplus y = (x + y) \wedge u\,, \qquad \neg x = u - x\,, \qquad \sim x = -x + u\,, \quad 1 = u\,.$$

Then $(A; \oplus, \neg, \sim, 0, 1)$ is a generalized MV-algebra. Analogously as in the case of MV-algebras (cf. [5]) it is denoted by $\Gamma(G, u)$.

THEOREM 2.2. (Dvurečenskij [6].) *For each generalized MV-algebra $\mathcal{A}$ there exists a lattice ordered group G with a strong unit u such that $\mathcal{A} = \Gamma(G, u)$.*

In what follows, when speaking about a generalized MV-algebra $\mathcal{A}$ we always assume that G and u are as in 2.2.

LEMMA 2.3. (Cf. [3], [10].) *Let $\mathcal{A}$ be a generalized MV-algebra and $a \in A$. For $x, y \in [0, a]$ we put*

$$x \oplus_a y = (x + y) \wedge a\,, \qquad \neg_a x = a - x\,, \qquad \sim_a x = -x + a\,.$$

Then the structure $\mathcal{A}_a = \big([0, a]; \oplus_a, \neg_a, \sim_a, 0, a\big)$ is a generalized MV-algebra.

In fact, $\mathcal{A}_a = \Gamma(G_a, u)$, where G_a is the convex ℓ-subgroup of G generated by the element a.

LEMMA 2.4. *Let G be a lattice ordered group and $a \in G$. For each $x, y \in G$ we put $x +_a y = x - a + y$. Then $(G; +_a, \leqq)$ is a lattice ordered group with the neutral element a.*

P r o o f. It is easy to verify that $(G; +_a)$ is a group. If $x, y, p, q \in A$ and $x \leqq y$, then clearly $p +_a x +_a q \leqq p +_a y +_a q$. □

For $x \in G$ we denote by $-_a x$ the inverse element of x with respect to the group $(G; +_a)$. We have $-_a x = a - x + a$.

3. Proof of (β)

Assume that $\mathcal{A}$ is a generalized MV-algebra and that $[a, b]$ is an interval of $\ell(\mathcal{A})$.

Under the notation as in 2.4 we put $(G; +_a, \leqq) = G^a$. Let H be the convex ℓ-subgroup of G^a which is generated by the element b. Then b is a strong unit of H and hence we can construct the generalized MV-algebra $\mathcal{A}_1 = \Gamma(H, b)$.

From the definition of $\mathcal{A}_1$ we immediately obtain:

LEMMA 3.1. $\ell(\mathcal{A}_1) = [a, b]$.

P r o o f. The corresponding operations on $\mathcal{A}_1$ will be denoted by $\oplus_1, \neg_1$ and $\sim_1$.

Let $x, y \in [a, b]$. Then we have

$$
\begin{aligned}
x \oplus_1 y &= (x +_a y) \wedge b = (x - a + y) \wedge b\,,\\
\neg_1 x &= b -_a x = b +_a (-_a x) = b - a + (a - x + a) = b - x + a\,,\\
\sim_1 x &= -_a x +_a b = (a - x + a) +_a b = (a - x + a) - a + b = a - x + b\,.
\end{aligned}
$$

Hence we verified that the assertion (i) of (β) is valid.

For verifying that the assertion (ii) of (β) holds, let us suppose that $\mathcal{A}$ is an MV-algebra and consider the operations $\oplus^*$ and $\neg^*$ as defined in (α).

Thus we deal with the operation

$$x \oplus^* y = \big(\neg(\neg x \oplus a) \oplus y\big) \wedge b\,,$$

where $x, y \in [a, b]$. We have $-x + a \leqq 0$, thus $u - x + a \leqq u$ and hence

$$
\begin{aligned}
\neg x \oplus a &= \big((u - x) + a\big) \wedge u = u - x + a\,,\\
\neg(\neg x \oplus a) &= u - (u - x + a) = x - a\,,\\
\neg(\neg x \oplus a) \oplus y &= \big((x - a) + y\big) \wedge u\,,\\
\neg\big((\neg x \oplus a) \oplus y\big) \wedge b &= (x - a + y) \wedge u \wedge b = (x - a + y) \wedge b\,.
\end{aligned}
$$

We obtain $x \oplus^* y = x \oplus_1 y$.

Further, consider the operation

$$\neg^* x = \neg(x \oplus \neg b) \oplus a$$

for $x, y \in [a,b]$. We get

$$x \oplus \neg b = \big(x + (u - b)\big) \wedge u = (x - b + u) \wedge u\,.$$

Since $x - b \leqq 0$, we obtain $x - b + u \leqq u$, hence

$$\begin{aligned} x \oplus \neg b &= x - b + u\,,\\ \neg(x \oplus \neg b) &= u - (x - b + u) = b - x\,,\\ \neg(x \oplus \neg b) \oplus a &= (-x + b + a) \wedge u\,. \end{aligned}$$

Since $x \geqq a$, we get $-x + a \leqq 0$ and then $-x + b + a = -x + a + b \leqq b$. In view of $b \leqq u$ we obtain $(-x + b + a) \wedge u = -x + b + a = b - x + a$. We conclude that $\neg^* x = \neg_1 x$, completing the proof of (β). □

Until now we supposed that G is a lattice ordered group, u is a strong unit of G, $\mathcal{A} = \Gamma(G,u)$ and $[a,b] \subseteq [0,u]$. Let us add two remarks on the role of the element u to the previous proof.

1) Let G be a lattice ordered group and let $[a,b]$ be an interval in the positive cone G^1 of G. Let u be any element of G with $b \leqq u$. We denote by G' the convex ℓ-subgroup of G generated by the element u. Then u is a strong unit of G'. Consider the generalized MV-algebra $\mathcal{A} = \Gamma(G',u)$. Working with G', we arrive to the same formulas as in (β). Hence u need not be a strong unit of G; it suffices that the condition $[a,b] \subseteq [0,u]$ is satisfied.

2) Let G, u and $[a,b]$ be as in 1). Further, let u^* be an element of G with $[a,b] \subseteq [0,u^*]$. Working with u^* instead of u, we arrive, again, to the same formulas for $\oplus_1$ and $\neg_1$ as those given in (β). Hence these operations remain valid by this change of u.

Let $\mathcal{A}_1$ be as in (β); we say that $\mathcal{A}_1$ is an *interval subalgebra* of $\mathcal{A}$.

P r o o f o f (δ) . We recall that for $x, y \in [a,b]$ we have

$$\begin{gathered} x \oplus_1 y = (x - a + y) \wedge b\,, \quad \neg_1 x = b - x + a\,, \quad \sim_1 x = a - x + b\,,\\ x \oplus_{ab} y = \big(x \oplus (y \odot \sim a)\big) \wedge b\,,\\ \neg_{ab} x = (\neg x \odot b) \oplus a\,, \quad \sim_{ab} x = a \oplus (b \odot \sim x)\,. \end{gathered}$$

Since

$$\neg x \oplus \neg y = \big((u - x) + (u - y)\big) \wedge u\,,$$

we obtain

$$x \odot y = -\big((u - x + u - y) \wedge u\big) + u = \big((y - u + x - u) \vee (-u)\big) + u\,,$$
$$x \odot y = (y - u + x) \vee 0\,. \tag{+}$$

a) In view of (+) we get

$$y \odot {\sim}a = y \odot (-a + u) = (-a + u - u + y) \vee 0 = (-a + y) \vee 0\,.$$

Since $y \geqq a$, we obtain $y \odot {\sim}a = -a + y$. Hence

$$x \oplus (y \odot {\sim}a) = (x - a + y) \wedge u\,.$$

Therefore

$$\big(x \oplus (y \odot {\sim}a)\big) \wedge b = (x - a + y) \wedge u \wedge b = (x - a + y) \wedge b\,.$$

Thus $x \oplus_{ab} y = x \oplus_1 y$.

b) The relation (+) yields

$$\neg x \odot b = (u - x) \odot b = (b - u + u - x) \vee 0\,.$$

Since $b - x \geqq 0$, we get $\neg x \odot b = b - x$. Hence

$$(\neg x \odot b) \oplus a = (b - x + a) \wedge u\,.$$

In view of $x \geqq a$ we have $b - x + a \leqq b$, whence $(b - x + a) \wedge u = b - x + a$. Therefore $\neg_{ab} x = \neg_1 x$.

By applying analogous steps we obtain ${\sim}_{ab} x = {\sim}_1 x$. □

4. Some further properties of an interval subalgebra

We slightly modify the above notation. Let $\mathcal{G} = (G; +, \leqq)$ be a lattice ordered group and let $0 \leqq a \in G$. We put $\mathcal{G}_a = (G; +_a, \leqq)$, where $x +_a y = x - a + y$ for each $x, y \in G$. In view of 2.4, $\mathcal{G}_a$ is a lattice ordered group with the neutral element a.

For each $x \in G$ we put $\varphi(x) = x + a$. Then for any $x, y \in G$ we have

$$\varphi(x) +_a \varphi(y) = (x + a) - a + (y + a) = x + y + a = \varphi(x + y)\,.$$

We obtain:

Lemma 4.1. *The mapping φ is an isomorphism of the lattice ordered group $\mathcal{G}$ onto the lattice ordered group $\mathcal{G}_a$.*

We denote by φ_c the mapping φ reduced to the set $[0, c]$, where $c = b - a$.

LEMMA 4.2. *φ_c is an isomorphism of the generalized MV-algebra $\mathcal{A}_c$ onto the generalized MV-algebra $\mathcal{A}_1$.*

P r o o f . We have $\varphi_c(0) = a$ and $\varphi_c(c) = b$. Thus in view of 4.1, φ_c is an isomorphism of the lattice $\ell(\mathcal{A}_c)$ onto the lattice $\ell(\mathcal{A}_1)$.

We denote by $\oplus_c$, $\neg_c$ and $\sim_c$ the corresponding operations in the generalized MV-algebra $\mathcal{A}_c$. We have to verify that the relations

(1) $\varphi_c(x \oplus_c y) = \varphi_c(x) \oplus_1 \varphi_c(y)$,
(2) $\varphi_c(\neg_c x) = \neg_1 \varphi_c(x)$,
(3) $\varphi_c(\sim_c x) = \sim_1 \varphi_c(x)$

are valid for each $x, y \in [0, c]$.

From the definition of $\mathcal{A}_c$ we obtain

$$x \oplus_c y = (x + y) \wedge c\,, \qquad \neg_c x = c - x\,, \qquad \sim_c x = -x + c\,.$$

We get

$$\begin{aligned} \varphi_c(x \oplus_c y) &= \varphi_c\big((x+y) \wedge c\big) = \varphi_c(x+y) \wedge \varphi_c(c) = (x+y+a) \wedge b\,, \\ \varphi_c(x) \oplus_1 \varphi_c(y) &= \big(\varphi_c(x) +_a \varphi_c(y)\big) \wedge b = \big((x+a) - a + (y+a)\big) \wedge b \\ &= (x+y+a) \wedge b\,; \end{aligned}$$

thus (1) is valid.

Further, we have

$$\begin{aligned} \varphi_c(\neg_c x) &= \varphi_c(c - x) = c - x + a = b - a - x + a\,, \\ \neg_1\big(\varphi_c(x)\big) &= b - \varphi_c(x) + a = b - (x+a) + a = b - a - x + a\,, \end{aligned}$$

thus (2) holds. The proof of (3) is analogous to that of (2). □

As a corollary we obtain:

PROPOSITION 4.3. *If $\mathcal{A}_1 = \big([a,b], \oplus_1, \neg_1, \sim_1, a, b\big)$ is an interval subalgebra of an MV-algebra $\mathcal{A}$, then $\mathcal{A}_1$ is isomorphic to an interval subalgebra $\mathcal{A}_2$ of $\mathcal{A}$ such that the underlying set of $\mathcal{A}_2$ is the interval $[0, b-a]$ of the lattice $\ell(\mathcal{A})$.*

An element x_0 of a generalized MV-algebra is called *Boolean* if it has a complement in the lattice $\ell(\mathcal{A})$. We denote by $B(\mathcal{A})$ the set of all Boolean elements of $\mathcal{A}$.

From the fact that $\ell(\mathcal{A})$ is a distributive lattice we immediately obtain:

LEMMA 4.4. *$B(\mathcal{A})$ is a sublattice of $\ell(\mathcal{A})$ and it is a Boolean algebra.*

LEMMA 4.5. *Let $c \in A$ and let $x_0 \in B(\mathcal{A})$. Then $c \wedge x_0 \in B(\mathcal{A}_c)$.*

P r o o f . There exists a complement y_0 of x_0 in $\ell(\mathcal{A})$. Then we have

$$(c \wedge x_0) \wedge (c \wedge y_0) = 0\,, \qquad (c \wedge x_0) \vee (c \wedge y_0) = c \wedge (x_0 \vee y_0) = c \wedge u = c\,,$$

hence $c \wedge y_0$ is a complement of $c \wedge x_0$ in $\ell(\mathcal{A}_c)$. □

LEMMA 4.6. *Let a, b and c be as above. Let $x_0 \in B(\mathcal{A})$. Then $(c \wedge x_0) + a \in B(\mathcal{A}_1)$.*

Proof. This is a consequence of 4.2 and 4.5. □

For each $x_0 \in B(\mathcal{A})$ we put

$$\psi_1(x_0) = x_0 \wedge c, \qquad \psi_2(x_0) = (x_0 \wedge c) + a.$$

PROPOSITION 4.7. *Let $\mathcal{A}_1$ be an interval subalgebra of a generalized MV-algebra $\mathcal{A}$. Suppose that $\ell(\mathcal{A}_1) = [a, b]$; put $c = b - a$. Then ψ_2 is a homomorphism of $B(\mathcal{A})$ into $B(\mathcal{A}_1)$.*

Proof. Let ψ_1 be as above. In view of the distributivity of $\ell(\mathcal{A})$ and according to 4.6 we infer that ψ_1 is a homomorphism of $B(\mathcal{A})$ into $B(\mathcal{A}_c)$. Then by applying 4.1 we conclude that ψ_2 is a homomorphism of $B(\mathcal{A})$ into $B(\mathcal{A}_1)$. □

Internal direct factors of a generalized MV-algebras were studied in [9]. If c is an element of A, then $\mathcal{A}_c$ is an internal direct factor of $\mathcal{A}$ if and only if c is a Boolean element of $\mathcal{A}$. From this and from 4.5 and 4.6 we obtain:

PROPOSITION 4.8. *Let $\mathcal{A}^*$ be a direct factor of $\mathcal{A}$ and let x^* be the greatest element of $\ell(\mathcal{A}^*)$. Let $\mathcal{A}_1$ be an interval subalgebra of $\mathcal{A}$ with $\ell(\mathcal{A}_1) = [a, b]$, $c = b - a$. Put $(x^* \wedge c) + a = x^1$. Then there exists an internal direct factor $\mathcal{A}_1^*$ of $\mathcal{A}_1$ such that $\ell(\mathcal{A}_1^*) = [a, x^1]$.*

Under the notation as in 4.8, put $\psi_3(\mathcal{A}^*) = \mathcal{A}_1^*$. Then according to the relation between Boolean elements and internal direct factors of $\mathcal{A}$, and in view of 4.7 and 4.8 we conclude that ψ_3 is a homomorphism of the Boolean algebra of all internal direct factors of $\mathcal{A}$ into the Boolean algebra of all internal direct factors of $\mathcal{A}_1$.

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