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Mathematica Slovaca, Vol. 31 (1981), No. 2, 115--120

Persistent URL: http://dml.cz/dmlcz/130960

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ON BASES AND MAXIMAL IDEALS IN SEMIGROUPS

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In [4] and [5] the structure of semigroups having both one-sided and two-sided bases has been investigated. The aim of the present paper is to show the mutual relationship between bases and maximal ideals, and some relationship between one-sided bases and two-sided bases in semigroups.

A subset A of a semigroup S is right (left, two-sided) base of S if A is a smalest subset of S for which $A \cup SA = S$ ($A \cup AS = S$, $A \cup SA \cup AS \cup SAS = S$) holds.

In our further considerations $(a)_L$, ($(a)_R$, $(a)_F$) will denote the principal left (right, two-sided) ideal, generated by an element $a \in S$.

Let $a, b \in S$. We say that $a \leqslant b$ iff $(a)_L \subset (b)_L$. We shall consider only right bases and left ideals, but the considerations would be completely analogous for left bases and right ideals, two-sided bases and two-sided ideals, respectively.

If L_a and L_b are $\mathscr{L}$-classes of a semigroup S, then we shall write $L_a < L_b$ iff $(a)_L \subset (b)_L$.

The following statement is known:

Lemma 1 [7]. *A non-empty subset A of S is a right base of S iff A satisfies the conditions:*

(1) *for any $x \in S$ there exists $a \in A$ such that $x \leqslant a$,*

(2) *for any two distinct elements a_1, $a_2 \in A$ neither $a_1 \leqslant a_2$ nor $a_2 \leqslant a_1$.*

From Lemma 1 we get that any two distinct elements of a right base A belong to two distinct $\mathscr{L}$-classes, and all elements of A belong to maximal $\mathscr{L}$-classes. Moreover, a base contains just one element from every maximal $\mathscr{L}$-class of S.

Lemma 2 ([1]). *Let a semigroups S contain left ideals. Then L is a maximal left ideal of S iff $S - L$ is a maximal $\mathscr{L}$-class of S.*

Lemma 3 ([2]). *If a semigroup S contains at least one right base, then S contains maximal left ideals.*

Remark. Lemmas 2 and 3 are proved in [1] and [2] for unary algebras. We can obtain them for semigroups if we assign to a semigroup S a unary algebra $\langle S, F\rangle$, where F is the set of all left translations of S. Then all subalgebras of the algebra $\langle S, F\rangle$ correspond to all left ideals of S.

Definition 1. *We say that a left (right, two-sided) ideal L (R, M) of S is a covered left (right, two-sided) ideal if $L\subset S(S-L)$ $(R\subset(S-R)S$, $M\subset S(S-M)S)$.*

Theorem 1. *Suppose that a semigroup S contains at least one right base. Then*

(1) *S contains maximal left ideals,*

(2) *Every maximal left ideal L_i is of the form: $L_i=S-L_{a_i}$,*

(3) *The intersection of all maximal left ideals of S is either the empty set, or a left covered ideal.*

Proof. (1) and (2) are obtained from Lemmas 2 and 3 and the Remark above.

(3) Let $\bigcap\limits_{i\in I} L_i=l\neq\emptyset$. Evidently, l is a left ideal. It is sufficient to show that l is covered. For maximal left ideals L_i the relation $L_i=S-L_{a_i}$, $i\in I$ holds. Then $l=\bigcap\limits_{i\in I} L_i = \bigcap\limits_{i\in I}(S-L_{a_i}) = S-\bigcup\limits_{i\in I} L_{a_i}$. Therefore, no element from the $\mathscr{L}$-classes L_{a_i} $(i\in I)$ belongs to l. Let us consider a right base A. (From every $\mathscr{L}$-class L_{a_i}, $i\in I$ just one element belongs to A). Then $A\cup SA=S$. This implies $S-A\subset SA$, $A\subset\bigcup\limits_{i\in I} L_{a_i}$. Hence, $S-A\supset S-\bigcup\limits_{i\in I} L_{a_i}$. Then $l=S-\bigcup\limits_{i\in I} L_{a_i}\subset S-A\subset SA\subset S(S-l)$ hence, l is covered, as $A\subset S-l$.

If a semigroup contains maximal left ideals, it does not mean that it must have a right base.

Example 1. Let $S=\{\langle 0, 1)\cup\{a\}\}$ and the multiplication is defined as follows:

(1) for r_1, $r_2\in\langle 0, 1)$ r_1r_2 is the product of real numbers.

(2) $ar=ra=0$ for any $r\in\langle 0, 1)$.

(3) $aa=0$.

$(a)_L=a\cup Sa=\{a\}\cup\{0\}=\{0, a\}$. All the $\mathscr{L}$-classes are one element subsets. $\langle 0, 1)$ is a proper left ideal of S and evidently it is a maximal one. No element $r\in\langle 0, 1)$ is contained in maximal $\mathscr{L}$-class. Hence, S cannot have a right base.

Theorem 2. *Let a semigroup S contain maximal left ideals. If the intersection of all maximal left ideals is either empty or a left covered ideal, then S contains at least one right base.*

Proof. Let

$$l=\bigcap_{i\in I} L_i=\bigcap_{i\in I}(S-L_{a_i})=S-\bigcup_{i\in I} L_{a_i} \tag{1}$$

Any L_{a_i} is a maximal $\mathscr{L}$-class of S. Let us construct A in the following way: From every L_{a_i} we put into A just one element. We shall show that A is a right base of S.

(a) If $l=\emptyset$, then $S-\bigcup_{i\in I} L_{a_i}=\emptyset$, so that $S=\bigcup_{i\in I} L_{a_i}$. And for A we get $A\cup SA=S$. However, we cannot omit even one element of A, if A should generate S, because any $\mathscr{L}$-class L_{a_i} is maximal. So, $A\cup SA=S$, i.e. A generates S and it is a smallest subset of S with the property: $A\cup SA=S$, hence, A is a right base.

(b) Let $l\neq\emptyset$ be a left covered ideal of S. This means that $l\subset S(S-l)$. However,

$$l\subset S(S-l)\subset A\cup SA \tag{$*$}$$

(because we put into A one element from every $\mathscr{L}$-class L_{a_i}). From the relation $(*)$ we get

$$A\cup SA=S,$$

so that A generates S. And again we cannot omit even one element from A, because any $\mathscr{L}$-class is a maximal one. Therefore, A is a smallest set generating S, hence a right base.

Corollary. Let L_i, $i\in I$, be all maximal left ideals of S. If $\bigcap_{i\in I} L_i=l$ is a minimal left ideal of S, then S contains at least one right base.

Proof. To prove our statement it is sufficient to show that the minimal left ideal, which is the intersection of all maximal left ideals, is a left covered ideal. First, the maximality of $\mathscr{L}$-classes L_{a_i} implies that

$$(a_i)_L\cap L_{a_j}=\emptyset \quad \text{for} \quad i\neq j.$$

The relation (1) implies that l is an $\mathscr{L}$-class of S, different from all $\mathscr{L}$-classes L_{a_i}, $i\in I$. If l were a maximal $\mathscr{L}$-class, then we would get another maximal left ideal $S-l$, different from L_i, $i\in I$, which contradicts the assumption that L_i, $i\in I$, are all maximal left ideals of S. So, there is some $i_0\in I$ such that for $a\in L_{a_{i0}}$, $l\subset(a)_L$, therefore $l\subset Sa\subset S(S-l)$. Hence, l is a left covered ideal of S.

2.

It is clear that if a subset $A\subset S$ is a right (left) base of S, i.e., $A\cup SA=S$ ($A\cup AS=S$), then it holds $A\cup SA\cup AS\cup SAS=S$. However, this does not mean that A is a two-sided base of S too, because we do not know, whether there is no proper subset $A_1\subset A$, $A_1\neq A$ such that $A_1\cup SA_1\cup A_1S\cup SA_1S=S$ and A_1 is a minimal one.

The question arises, whether the existence of a one-sided base in S does imply the existence of a two-sided base in S. We shall show by means of an example that in general this need not be so.

First, we give an evident statement.

Theorem 3. *Let a semigroup S contain a right (left) base A, which is finite. Then S contains a two-sided base too.*

Corollary. Any finite semigroup contains both a one-sided and a two-sided bases.

Example 2. Let S consist of all matrices A_{mn} of type $m \times n$, $m = 1, 2, \ldots$, $n = 1, 2, \ldots$ and of an element $z \neq A_{mn}$ for $m, n = 1, 2, \ldots$. The elements of A_{mn} are real numbers. The operation in S is defined as follows:

1. Let A_{km}, A_{rs} be any two elements of S different from z. Then

$$A_{km} \cdot A_{rs} = \begin{cases} A_{ks} & \text{if } m = r, \text{ where } A_{ks} \text{ is the usual product of matrices,} \\ z & \text{if } m \neq r. \end{cases}$$

2. Let A_{mn} be any matrix of S different from z. Then $A_{mn} \cdot z = z \cdot A_{mn} = z \cdot z = z$.

Evidently, S with the operation defined in this way, is a semigroup.

We shall show that $A = \{E_n / n = 1, 2, \ldots$, where E_n is the unit matrix of order $n\}$ is a right base for S. For that it is sufficient to show that the conditions (1), (2) of Lemma 1 are satisfied.

Let $X = A_{st}$. Then $X = A_{st} \in (E_t)_L = E_t \cup SE_t$, because $A_{st} = A_{st} \cdot E_t$, so that $A_{st} \in SE_t$. This implies $(A_{st})_L \subset (E_t)_L$, hence $A_{st} \leqslant E_t$. In the case $X = z$, $z \in (E_t)_L$ for $t = 1, 2, \ldots$ Hence, we have shown that for any $X \in S$, $X \leqslant E_n$ holds for some positive integer n.

Let E_t, E_s be two elements of A, $s \neq t$. We are going to show that neither $E_s \leqslant E_t$, nor $E_t \leqslant E_s$. First we shall show that $E_t \bar{\in} (E_s)_L$ (hence $(E_t)_L \subset (E_s)_L$ does not hold) and $E_s \bar{\in} (E_t)_L$ (i.e., $(E_s)_L \subset (E_t)_L$ does not hold). $(E_s)_L$ contains all matrices A_{ms}, where $m = 1, 2, \ldots$ and the element z. This is clear, because if $X = A_{mn}$ is any matrix of S, then

$$A_{mn} \cdot E_s = \begin{cases} A_{ms} & \text{if } n = s, \\ z & \text{if } n \neq s. \end{cases}$$

Since $t \neq s$ we get $E_t \bar{\in} (E_s)_L$. Analogously $E_s \bar{\in} (E_t)_L$ can be shown. Therefore, neither $E_s \leqslant E_t$ nor $E_t \leqslant E_s$. Hence $A = \{E_n\}_{n=1}^{\infty}$ is a right base of S. Now by Theorem 1 one can construct maximal left ideals.

In our further consideration we shall need the following

Lemma 2. *Let Y be any matrix of S. Then, for a matrix $X \in S$ we have $X \in (Y)_F$ ($(Y)_F = Y \cup SY \cup YS \cup SYS$) if and only if $h(X) \leqslant h(Y)$, where $h(X)$, $h(Y)$ denotes the rank of X, Y respectively.*

Proof. a. Let $X \in (Y)_F$. Then $X \in SYS$ (evidently, $(Y)_F = SYS$), so that there exist U, $V \in S$ such that $X = UYV$. This implies $h(X) \leqslant \min\{h(U), h(Y), h(V)\}$ (see [6], p. 101).

b. Let X, Y be matrices of S, X of type $m \times n$, Y of type $p \times q$, and let $h(X) \leqslant h(Y)$. Let us denote $h(X) = r$, $h(Y) = s$. Clearly $r \leqslant \min\{m, n, p, q\}$ $(*)$

X can be transformed by means of elementary operations to a matrix X' of the form

$$X' = \begin{pmatrix} E_r, & 0 \\ 0 & 0 \end{pmatrix},$$

where E_r is the unit matrix of order r and 0 are such zero matrices that X' is of the same type as X. It is known (see [3]) that X can be expressed in the following form: $X = U_1 X' V_1$, where U_1, V_1 are regular matrices of order m, n respectively. Then $X' = U_1^{-1} X V_1^{-1}$.

The same can be done for Y. We have $Y = U_2 Y' V_2$, and $Y' = U_2^{-1} Y V_2^{-1}$, where U_2, V_2 are regular matrices of order p, q respectively, and Y' is the matrix of the form

$$Y' = \begin{pmatrix} E_s, & 0 \\ 0, & 0 \end{pmatrix}$$

Let us denote

$$A' = \begin{pmatrix} E_r, & 0 \\ 0, & 0 \end{pmatrix} \qquad B' = \begin{pmatrix} E_r, & 0 \\ 0, & 0 \end{pmatrix}$$

where A' is of type $m \times p$, B' is of type $q \times n$. With regard to ($*$) this is possible. Then $X' = A' Y' B'$.

From that we get

$$X = U_1 X' V_1 = U_1 A' Y' B' V_1 = U_1 A' U_2^{-1} Y V_2^{-1} B' V_1.$$

The product $U_1 A' U_2^{-1}$ is a matrix of type $m \times p$, since U_1 is of type $m \times m$, A' of type $m \times p$, U_2^{-1} of type $p \times p$. Similarly, the product $V_2^{-1} B' V_1$ is a matrix of type $q \times n$. If we put $A = U_1 A' U_2^{-1}$, and $B = V_2^{-1} B' V_1$, we have

$$X = AYB,$$

therefore $X \in SYS$, and hence $X \in (Y)_F$. Lemma 2 is proved.

Now, we shall show that S does not contain maximal $\mathcal{F}$-classes and therefore S cannot have a two-sided base.

Let A be an arbitrary element of S. According to Lemma 2 there exists an element B of S such that $(A)_F \subset (B)_F$ and $(A)_F \neq (B)_F$. (For B we can choose an arbitrary matrix whose rank is greater than the rank of the matrix A). Hence S cannot contain maximal $\mathcal{F}$-classes and therefore S does not contain a two-sided base.

REFERENCES

[1] АБРГАН И., О максимальных подалгебрах в унарных алгебрах, Mat. Čas. 24, 1974, 113—128.
[2] АБРГАН И., О минимальных множествах образующих в унарных алгебрах, Mat. Čas. 25, 1975, 305—317.

[3] BIRKHOFF G., MAC LANE S., A survey of modern algebra, Third Edition, The Macmillan Company, New York.
[4] FABRICI I., One-sided bases of semigroups, Mat. Čas. 22, 1972, 286—290.
[5] FABRICI I., Two-sided bases of semigroups, Mat. Čas. 25, 1975, 173—178.
[6] КУРОШ А. Г., Алгебра, Гос. изд. физ.-мат. литератури, Москва, 1962.
[7] TAMURA T., One-sided bases and translations of a semigroup, Math. Japan 3, 1955, 137—141.

Received January 14, 1976

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О БАЗИСАХ И МАКСИМАЛЬНЫХ ИДЕАЛАХ В ПОЛУГРУППАХ

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Резюме

В первой части работы доказаны утверждения, касающиеся соотношения между базисами полугрупп и максимальными идеалами полугрупп.

Во второй части показано, что из существования одностороннего базиса не вытекает существование двустороннего базиса.