Ladislav Nebeský
On squares of complementary graphs

Mathematica Slovaca, Vol. 30 (1980), No. 3, 247--249

Persistent URL: http://dml.cz/dmlcz/130715

Math. Slovaca 30, 1980, No. 3, 247—249

ON SQUARES OF COMPLEMENTARY GRAPHS

LADISLAV NEBESKÝ

By a graph we mean a graph in the sense of [1] or [7]. Let G be a graph. We denote by $V(G)$, $E(G)$, $\bar{G}$, $L(G)$, and $d(G)$ its vertex set, edge set, complement, line graph, and diameter, respectively. If G is disconnected, then we put $d(G)=\infty$. The cardinality of $V(G)$ is called the order of G. We say that G is hamiltonian-connected if for every pair of distinct vertices u, $v \in V(G)$, there exists a hamiltonian path which connects u and v.

If G is a graph, then by the square G^2 of G we mean the graph with $V(G^2)=V(G)$ and

$$E(G^2)=\{uv\,;\, u, v \in V(G) \text{ and } 1 \leqq d_G(u, v) \leqq 2\},$$

where $d_G(u, v)$ denotes the distance between u and v in G.

Squares of graphs have been studied intensively, first of all from the point of view of their hamiltonian properties. Fleischner [4] has proved that if G is a 2-connected graph, then G^2 is hamiltonian. This result was improved in [2], [8], and [3]; in [2] Chartrand, Hobbs, Jung, and Nash—Williams have proved that if G is a 2-connected graph, then G^2 is hamiltonian-connected. Hamiltonian properties of squares of trees were studied in [11]. For some further results concerning hamiltonian properties of squares of graphs the reader is referred to [5] and [6].

The following theorem gives a sufficient condition for the square of a graph to be hamiltonian-connected. Note that K_p denotes the complete graph of order p, and $K_p - e$ denotes the graph obtained from K_p by deleting exactly one edge.

Theorem. *Let G be a graph of order $p \geqq 2$. If $K_p \neq (\bar{G})^2 \neq K_p - e$, then G^2 is hamiltonian-connected.*

Proof. Assume that $K_p \neq (\bar{G})^2 \neq K_p - e$. Since $(\bar{G})^2 \neq K_p$, we have that $d(\bar{G}) > 2$. Let $d(\bar{G}) = \infty$. Then $\bar{G}$ is disconnected, and therefore $d(G) \leqslant 2$. This means that G^2 is complete, and thus hamiltonian-connected.

We shall assume that $d(\bar{G}) < \infty$. Then $\bar{G}$ is connected. Since $d(\bar{G}) > 2$, there exist $u_1, u_2 \in V(\bar{G})$ such that $d_{\bar{G}}(u_1, u_2) = 3$. Hence, $p \geqq 4$. For $i = 1, 2$ we denote

$$V_i = \{v \in V(\bar{G} - u_1 - u_2)\,;\, u_i v \in E(\bar{G})\}.$$

Since $\bar{G}$ is connected and $d_G(u_1, u_2)>1$, we have that $V_1 \neq \emptyset \neq V_2$. Since $d_G(u_1, u_2)>2$, we have that $V_1 \cap V_2 = \emptyset$.

We shall distinguish two cases:

Case 1. $V_1 \cup V_2 = V(\bar{G} - u_1 - u_2)$ If for every $v_1 \in V_1$ and $v_2 \in V_2$ there holds that $v_1 v_2 \in E(\bar{G})$, then for every pair of distinct vertices u' and u'' with the property that $\{u', u''\} \neq \{u_1, u_2\}$ there holds that $d_{\bar{G}}(u', u'') \leqq 2$, and thus $(\bar{G})^2 = K_p - e$, which is a contradiction. This means that there exist $v' \in V_1$ and $v'' \in V_2$ such that $v'v'' \notin E(\bar{G})$. We denote by F_1 the graph with $V(F_1) = V(G)$ and

$$E(F_1) = \{u_1u_2, u_2v', v'v'', v''u_1\} \cup$$
$$\cup \{u_1 w''; w'' \in V_2\} \cup \{u_2 w'; w' \in V_1\}.$$

Obviously, F_1 is a connected graph which contains exactly one cycle. Since $V_1 \cap V_2 = \emptyset$, we have that F_1 is a subgraph of G. It is easy to see that $(F_1)^2$ is hamiltonian-connected. Since $V(F_1) = V(G)$, we have that G^2 is hamiltonian-connected.

Case 2. $V_1 \cup V_2 \neq V(G - u_1 - u_2)$. Consider an arbitrary vertex $v \in V(G - u_1 - u_2) - (V_1 \cup V_2)$. We denote by F_2 the graph with $V(F_2) = V(G)$ and

$$E(F_2) = \{v_0u_1, u_1u_2, u_2v_0\} \cup$$
$$\cup \{u_1 w_2; w_2 \in V(G - v_0 - u_1 - u_2) - V_1\} \cup \{u_2 w_1; w_1 \in V_1\}.$$

Obviously, F_2 is a connected graph which contains exactly one cycle. It is easy to see that $(F_2)^2$ is hamiltonian-connected. Since F_2 is a spanning subgraph of G, we have that G^2 is hamiltonian-connected, which completes the proof.

We denote by P_4 the path of order four. Obviously, $\bar{P}_4 = P_4$, and $(P_4)^2 = K_4 - e$.

Corollary. *Let G be a graph different from P_4. Then G^2 or $(\bar{G})^2$ is hamiltonian-connected.*

Remark 1. A graph of order $p \geqq 1$ is called panconnected if for every pair of distinct vertices $u, v \in V(G)$ and for every integer j with the property that $d_G(u, v) \leqq j \leqq p-1$, there exists a path of length j which connects u and v in G. Fleischner [5] has proved that if G is a graph, then G^2 is panconnected if and only if G^2 is hamiltonian-connected.

Remark 2. In [9] it was proved that if G is a graph of order $\geqq 5$, then there exists $G' \in \{G, \bar{G}\}$ such that G' is connected and $L(G')$ is hamiltonian. This result was improved in [10], where it was also shown that for every integer $p \geqq 1$, there exists a graph G_p of order p such that neither $L(G_p)$ nor $L(\bar{G}_p)$ is hamiltonian-connected.

REFERENCES

[1] BEHZAD, M., CHARTRAND, G.: Introduction to the Theory of Graphs. Allyn and Bacon, Boston 1971.

[2] CHARTRAND, G., HOBBS, A. M., JUNG, H. A., NASH-WILLIAMS, C. St. J. A.: The square of a block is hamiltonian connected. J. Comb. Theory 16B, 1974, 290—292.

[3] FAUDREE, R. J., SCHELP, R. H.: The square of a block is strongly path connected. J. Comb. Theory 20B, 1976, 47—61.

[4] FLEISCHNER, H.: The square of every two-connected graph is hamiltonian. J. Comb. Theory 16B, 1974, 29—34.

[5] FLEISCHNER, H.: In the square of graphs, hamiltonicity and pancyclicity, hamiltonian connectedness and panconnectedness are equivalent concepts. Monatshefte Math. 82, 1976, 125—149.

[6] FLEISCHNER, H., HOBBS, A. M.: A necessary condition for the square of a graph to be hamiltonian. J. Comb. Theory 19, 1975, 97—118.

[7] HARARY, F.: Graph Theory. Addison-Wesley, Reading (Mass.) 1969.

[8] HOBBS, A. M.: The square of a block is vertex pancyclic. J. Comb. Theory 20B, 1976, 1—4.

[9] NEBESKÝ, L.: A theorem on hamiltonian line graphs, Comment. Math. Univ. Carolinae 14, 1973, 107—111.

[10] NEBESKÝ, L.: On pancyclic line graphs. Czechoslovak Mat. J. 28(103), 1978, 650—655.

[11] NEUMANN, F.: On a certain ordering of the vertices of a tree. Časopis pěst. mat. 89, 1964, 323—339

Received July 3, 1978

Filozofická fakulta Univerzity Karlovy
nám. Krásnoarmějců 2
116 38 Praha 1

О КВАДРАТАХ ДОПОЛНИТЕЛНЫХ ГРАФОВ

Ладислав Небеский

Резюме

Доказывается следующая теорема: *Пусть G – граф с $p \geqq 2$ вершинами. Если $K_p \neq (\bar{G})^2 \neq K_p - e$, то G^2 – гамильтоново связный.* ($\bar{G}$ обозначает дополнение графа G, K_p – полный граф с p вершинами и $K_p - e$ – граф, полученный из K_p удалением одного ребра).