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PERIODIC ORBITS OF CERTAIN HÉNON–LIKE MAPS

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ABSTRACT. The existence of periodic orbits for certain two-dimensional Hénon-like maps is shown. For this purpose, critical point theorems are used.

1. Introduction

The purpose of this brief report is to show the existence of periodic orbits of Hénon-like maps of the forms

$$r_p(x,y) = \big(b \cdot x + d \cdot y - f(p,x),\ c \cdot x\big) \tag{1.1}$$

and

$$r(x,y) = \big(b \cdot x + d \cdot y - q(x),\ c \cdot x\big)\,, \tag{1.2}$$

where b, d, c are constant satisfying $c \cdot d = -1$.

We assume $f \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $f(\cdot\,,0) = 0$. We shall study the existence of periodic orbits of (1.1) near $x = 0$, $y = 0$ considering p as a bifurcation parameter. Under additional conditions for f we show the existence of a closed interval I such that for $p \notin I$ the point $x = 0$, $y = 0$ is a hyperbolic fixed point of (1.1). Hence there is no periodic orbit of (1.1) near $(0,0)$. On the other hand, the set of bifurcation values p of periodic orbits of (1.1) near $(0,0)$ is dense in I. Thus for each open neighbourhood U of $(0,0)$ it holds: each $s \in I$ can be approximated by a sequence $\{p_n\}_{n=3}^{\infty} \subset I$, $p_n \to s$, such that the map (1.1) with $p = p_n$ has an n-periodic nontrivial orbit in U. (The trivial orbit is the fixed point $(0,0)$.)

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We study the map (1.2) globally when q is asymptotically linear at the infinity. We show the existence of an infinite number of periodic orbits whose minimal periods tend to the infinity.

We see that the orbit $\{(x_n, y_n)\}_{-\infty}^{\infty}$ of (1.1) satisfies

$$x_{n+2} - bx_{n+1} + x_n + f(p, x_{n+1}) = 0 \tag{1.3}$$

and similarly for (1.2). Hence we study the difference equation (1.3). Note that there is a relation between (1.3) and the area preserving twist maps (see Angenent [1]). Indeed, let us put

$$h(x,z) = -1/2\big(bx - 1/b\,z\big)^2 + \int_0^z f(p,s)\,\mathrm{d}s - (b - b^2 - 1/b^2)\,z^2/2\,,$$

and following [1, p. 355] we define a map F in the following way:

$$F(x,y) = (x_1, y_1) \iff y = \frac{\partial}{\partial x}h(x, x_1)\,, \quad y_1 = -\frac{\partial}{\partial z}h(x, x_1)\,.$$

Hence

$$\begin{aligned} y &= -b^2x + x_1\,,\\ y_1 &= -x + x_1/b^2 - f(p, x_1) + (b - b^2 - 1/b^2)x_1\,, \end{aligned}$$

and

$$\begin{aligned} x_1 &= y + b^2x\,,\\ y_1 &= y/b^2 - f(p, y + b^2x) + (b - b^2 - 1/b^2)\cdot(y + b^2x)\,. \end{aligned} \tag{1.4}$$

But the orbit $\{(x_n, y_n)\}_{-\infty}^{\infty}$ of (1.4) satisfies precisely the equation (1.3).

Essentially, our approach to the problem is similar to [1]. We shall define a functional as in [1, p. 354], whose critical points are periodic orbits of (1.3) or of a similar equation corresponding to (1.2). Then we apply theorems of [2] and [5] to prove our results. The author of this paper has recently used the same approach for studying discretizations of higher dimensional variational problems [4]. We note that for $b = 2$ the equation (1.3) is the Euler discretization of $z'' + f(p,z) = 0$.

2. Local results

We study the existence of periodic orbits of (1.1) near $(0,0)$. We assume $f \in C^2(\mathbb{R}\times\mathbb{R},\mathbb{R})$, $f(\cdot\,,0) = 0$ and $g(\cdot) = \frac{\partial f}{\partial x}(\cdot\,,0)$ satisfies $g'(\cdot) > 0$, $\inf_{\mathbb{R}} g < -2 + b < 2 + b < \sup_{\mathbb{R}} g$.

THEOREM 2.1. *For each* $s \in \langle g^{-1}(b-2), g^{-1}(b+2)\rangle$, $\delta > 0$ *there exists a sequence* $\{p_n\}_{n=2}^{\infty} \subset \langle g^{-1}(b-2) - \delta, g^{-1}(b+2) + \delta\rangle$ *with the properties:*

i) $p_n \to s$ *as* $n \to \infty$,
ii) *for* $p = p_n$ *the map* (1.1) *has a nontrivial* n*-periodic orbit* $\{y_1, \dots, y_n\}$ *such that* $\max_i |y_i| < \delta$.

We see that $\{x_1, \dots, x_{n+1}\}$, $x_{n+1} = x_1$ is the n-periodic orbit of (1.3) if and only if for $\tilde{f}(p,z) = (2-b)z + f(p,z)$ there holds:

$$\begin{aligned} x_2 - 2x_1 + x_n + \tilde{f}(p, x_1) &= 0, \\ &\vdots \qquad\qquad n \geqq 2. \\ x_1 - 2x_n + x_{n-1} + \tilde{f}(p, x_n) &= 0, \end{aligned}$$

We put

$$\mathbf{D}\colon \mathbb{R}^n \to \mathbb{R}^n, \quad \mathbf{D}(x_1, \dots, x_n) = (x_2 + x_n - 2x_1, \dots, x_1 + x_{n-1} - 2x_n),$$
$$\boldsymbol{F}(p, \cdot)\colon \mathbb{R}^n \to \mathbb{R}^n, \quad \boldsymbol{F}(p, x_1, \dots, x_n) = \big(\tilde{f}(p, x_1), \dots, \tilde{f}(p, x_n)\big).$$

Then the above equation has the form

$$\mathbf{D}\boldsymbol{x} + \boldsymbol{F}(p, \boldsymbol{x}) = \mathbf{0}, \qquad \boldsymbol{x} = (x_1, \dots, x_n). \tag{2.1}$$

Note that $\operatorname{grad}\big((\mathbf{D}\boldsymbol{x}, \boldsymbol{x})/2 + \tilde{q}(p, x_1) + \dots + \tilde{q}(p, x_n)\big) = \mathbf{D}\boldsymbol{x} + \boldsymbol{F}(p, \boldsymbol{x})$, where $\tilde{q}(p,z) = \int_0^z \tilde{f}(p,s)\, \mathrm{d}s$.

LEMMA 2.2. *The spectrum of* $\mathbf{D}$ *is* $\big\{-4\sin^2 \frac{\pi}{n} j, \ j = 0, \dots, n-1\big\}$.

Proof. See [3]. □

Proof of Theorem 2.1. The linearization of (2.1) at $x = 0$ has the form

$$\mathbf{A}(p) = \mathbf{D} + \big(2 - b + g(p)\big) \cdot \mathrm{Id}.$$

Hence the matrix $\mathbf{A}(p)$ has eigenvalues

$$\Big\{-4\sin^2 \frac{\pi}{n} j + 2 - b + g(p), \ j = 0, \dots, n-1\Big\}.$$

If $2 - b + g(p) \neq 4\sin^2 \frac{\pi}{n} j$ for each $j = 0, \dots, n-1$, then $\mathbf{A}(p)$ is invertible and we can define the positive Morse index (see [5, pp. 53]) $M(p)$ of $\mathbf{A}(p)$. Moreover, if p passes through the numbers $g^{-1}\left(4\sin^2\frac{\pi}{n}j+b-2\right)$, then there is a change of the numbers $M(p)$. Hence by a result of Chow and Lauterbach [2] the numbers $g^{-1}\left(4\sin^2\frac{\pi}{n}j+b-2\right)$ are bifurcation values of p for (2.1). Finally, we see that the set $\left\{g^{-1}\left(4\sin^2\frac{\pi}{n}j+b-2\right),\quad j\in\{0,\dots,n-1\},\ n\in\{2,3\dots\}\right\}$, is dense in $\left\langle g^{-1}(b-2),\, g^{-1}(b+2)\right\rangle$. Note that $b-2>\inf g$ and $\sup g>b+2$. □

It is clear that for $p\notin\left\langle g^{-1}(b-2),\, g^{-1}(b+2)\right\rangle$ the fixed point $(0,0)$ of (1.1) is hyperbolic, i.e., the eigenvalues of $\mathrm{D}r_p(0,0)$ lie off the unit circle. For $p\in\left\langle g^{-1}(b-2),\, g^{-1}(b+2)\right\rangle$ the eigenvalues of $\mathrm{D}r_p(0,0)$ lie on the unit circle. The following theorem is the consequence of this fact.

THEOREM 2.3. *For $p\notin\left\langle g^{-1}(b-2),\, g^{-1}(b+2)\right\rangle$ there is a $\delta>0$ such that for each $s\in(p-\delta,p+\delta)$ the map (1.1) with $p=s$ has no nontrivial periodic orbits $\{y_1,\dots,y_n\}$ satisfying $\max|y_i|<\delta$.*

3. A global result

We shall study the map (1.2). For this purpose we need the following result:

THEOREM A. (see Li and Liu [5]) *Let $\tilde{a}\colon\mathbb{R}^m\to\mathbb{R}$ be a C^2-function satisfying $|\operatorname{grad}\tilde{a}(\mathbf{x})-\mathbf{A}_\infty\mathbf{x}|/|\mathbf{x}|\to 0$ as $|\mathbf{x}|\to\infty$ for a symmetric nonsingular matrix $\mathbf{A}_\infty\in\mathcal{L}(\mathbb{R}^m)$. Suppose that $\tilde{a}$ has critical points $\tilde{\mathbf{x}}_1,\dots,\tilde{\mathbf{x}}_k$ and all of them are nondegenerate. If $M\left(\tilde{a}''(\tilde{\mathbf{x}}_i)\right)\neq M(\mathbf{A}_\infty)$ for each i, then $\tilde{a}$ has another critical point. Here $\tilde{a}''$ is the Hessian of $\tilde{a}$, $M(\mathbf{B})$ is the positive Morse index of the symmetric matrix $\mathbf{B}$.*

THEOREM 3.1. *Let us assume:*

i) $\lim\limits_{|x|\to\infty} q(x)/x=s$,

ii) *q has only a finite number of roots $x_1,\dots,x_m$, i.e., $q(x_i)=0$, $m\geq 1$,*

iii) *$s\in(b-2,\,b+2)$, $q'(x_i)\neq b-2$, $s\neq q'(x_i)$, for $i=1,\dots,m$.*

Then the map r has an infinite number of nontrivial periodic orbits whose minimal periods tend to ∞, i.e., there is a sequence of natural numbers $\{n_i\}_{i=1}^{\infty}$, $n_{i+1}>n_i$, such that r has a periodic orbit with the minimal period n_i for any i. (Here the trivial periodic orbits are fixed points of r.)

Proof. We take a sequence of prime numbers $\{p_t\}_{t=1}^{\infty}$ such that

$$2-b+s,\quad 2-b+q(x_j)\neq 4\sin^2\frac{\pi}{p_t}k,\qquad p_t>2,$$

for each natural number t and $j = 1, \dots, m$, $0 \le k \le p_t$. Then we solve (2.1) for $n = p_k$, $f(a,x) = q(x)$. We shall apply Theorem A with

$$\begin{aligned} \tilde{a}(\mathbf{x}) &= (\mathbf{D}\mathbf{x}, \mathbf{x})/2 + \tilde{q}(x_1) + \cdots + \tilde{q}(x_n), \\ \tilde{q}(z) &= (2-b)z^2/2 + \int_0^z q(s)\,\mathrm{d}s, \\ \tilde{\mathbf{x}}_i &= (x_i, \dots, x_i), \\ \mathbf{A}_\infty &= \mathbf{D} + (2 - b + s)\cdot \mathrm{Id}. \end{aligned}$$

In this case we have

$$\tilde{a}''(\tilde{\mathbf{x}}_i) = \mathbf{D} + \big(q'(x_i) + 2 - b\big)\cdot \mathrm{Id},$$

and eigenvalues of $\tilde{a}''(\tilde{\mathbf{x}}_i)$ and $\mathbf{A}_\infty$ are the following:

$$\Big\{-4\sin^2 \frac{\pi}{p_k} j + 2 - b + q'(x_i), \;\; 0 \leqq j \leqq p_k - 1\Big\}$$

and

$$\Big\{-4\sin^2 \frac{\pi}{p_k} j + 2 - b + s, \;\; 0 \leqq j \leqq p_k - 1\Big\},$$

respectively.

By the choice of $\{p_k\}$ we see that $\tilde{a}''(\tilde{\mathbf{x}}_i)$, $\mathbf{A}_\infty$ are nonsingular. We can define the positive Morse indexes $M\big(\tilde{a}''(\tilde{\mathbf{x}}_i)\big)$ and $M(\mathbf{A}_\infty)$. By [3] we know that 0, $-4\sin^2 \frac{\pi}{p_k} j$, $0 < j \le (p_k - 1)/2$ have the geometric multiplicities 1, 2 in $\mathbf{D}$, respectively. Hence

$$\begin{aligned} M\big(\tilde{a}''(\tilde{\mathbf{x}}_i)\big) &= 2\#\Big\{0 < j \le (p_k-1)/2, \;\; -4\sin^2 \frac{\pi}{p_k} j + 2 - b + q'(x_i) > 0\Big\} + 1, \\ M(\mathbf{A}_\infty) &= 2\#\Big\{0 < j \le (p_k-1)/2, \;\; -4\sin^2 \frac{\pi}{p_k} j + 2 - b + s > 0\Big\} + 1, \end{aligned}$$

($\#$ means the cardinality).

Using the assumption iii) we see that

$$M\big(\tilde{a}''(\tilde{\mathbf{x}}_i)\big) \ne M(\mathbf{A}_\infty), \qquad i = 1, \dots, m,$$

for p_k large. Hence Theorem A implies the existence of a critical point, i.e. a solution of (2.1) for our case $f(\cdot, x) = q(x)$, $n = p_k$, different from x_i, $i = 1, \dots, m$. This gives a p_k-periodic nontrivial orbit of r, for k large. Since $\{p_t\}_{t=1}^{\infty}$ is a sequence of prime numbers, we can conclude the proof. □

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