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COVERING CONDITION IN THE LATTICE OF RADICAL CLASSES OF LINEARLY ORDERED GROUPS

GABRIELA PRINGEROVÁ

Existence of covers in lattices of varieties and quasivarieties was studied by A. N. Trachtman [10], V. A. Gorbunov [11] and G. Pollák [9]; for the case of varieties of lattice ordered groups cf. N. Ja. Medvedev [8]. Analogous questions concerning covers of torsion classes and radical classes of lattice ordered groups were studied by J. Jakubík [4], [3].

Radical classes of linearly ordered groups were investigated by C. G. Chehata and R. Wiegandt [1]. J. Jakubík [5] studied radical classes of abelian linearly ordered groups.

All linearly ordered groups dealt with in this note are assumed to be abelian; thus the words linearly ordered group will in the following always mean abelian linearly ordered group.

Let $\mathcal{R}_a$ be the lattice of all radical classes of linearly ordered groups (for definitions, cf. § 1 below). For each $X \in \mathcal{R}_a$ we denote by $a(X)$ the collection of all $Y \in \mathcal{R}_a$ such that Y covers X (i.e., the interval $[X, Y]$ of the lattice $\mathcal{R}_a$ is prime). Put

$$\mathcal{R}_1 = \{X \in \mathcal{R}_a \colon X \text{ is principal and } a(X) = \emptyset\}.$$

In [5] it was proved that whenever X is a principal radical class generated by an archimedean linearly ordered group, then X belongs to $\mathcal{R}_1$. From this it follows that the class $\mathcal{R}_1$ is infinite. In this note it will be shown that $\mathcal{R}_1$ is a large class (in the sense that there exists an injective mapping of the class of all cardinals into $\mathcal{R}_1$).

1. Preliminaries

For the basic definitions concerning linearly ordered groups cf. Fuchs [2] and Kokorin and Kopytov [6]. The group operation in a linearly ordered group will be written additively. In this paragraph we recall for the sake of completeness some definitions and results from [5] concerning radical classes of linearly ordered groups.

Let $\mathcal{G}_a$ be the class of all linearly ordered groups, $G \in \mathcal{G}_a$. Let

$$\{0\} = G_1 \subseteq G_2 \subseteq \dots \subseteq G_\alpha \subseteq \dots \quad (\alpha < \delta)$$

be an ascending chain of convex subgroups of G. Put

$$H = \cup_{\alpha<\delta} G_\alpha .$$

For each $\beta < \delta$ let G'_β be a linearly ordered group isomorphic to $G_\beta / \cup_{\gamma<\beta} G_\gamma$. Then H is said to be a transfinite extension of linearly ordered groups G'_β $(\beta < \delta)$. Let $\{G_i\}_{i \in I}$ be the set of those G'_β which are distinct from $\{0\}$; if this set is nonempty, then we also say that H is a transfinite extension of linearly ordered groups $G_i (i \in I)$.

1.1. Definition. *A nonempty class X of linearly ordered groups is called a radical class if*

(a) *X is closed under homomorphisms, and*

(b) *X is closed with respect to transfinite extensions.*

Let X be a radical class and $G \in \mathcal{G}_a$. Further let $\{H_\alpha\}$ be the set of all convex subgroups of G belonging to X. We put

$$X(G) = \cup_\alpha H_\alpha .$$

1.2. Proposition. (Cf. [5].) *Let X be a radical class and $G \in \mathcal{G}_a$. Then $X(G)$ belongs to X.*

Let us denote by $\mathcal{R}_a$ the collection of all radical classes of linearly ordered groups. For $X, Y \in \mathcal{R}_a$ we put $X \leqq Y$ if X is a subclass of Y. Then $\mathcal{G}_a$ is the greatest element of $\mathcal{R}_a$ and $R_0 = \{\{0\}\}$ is the least element of $\mathcal{R}_a$. Moreover, under the partial order $\leqq$, $\mathcal{R}_a$ is a complete lattice.

For each nonempty subclass Y of $\mathcal{G}_a$ let us denote by

TY — the intersection of all radical classes R with $Y \subseteq R$;

Hom Y — the class of all homomorphic images of linearly ordered groups belonging to Y;

Ext Y — the class of all transfinite extensions of linearly ordered groups belonging to Y.

TY is the least radical class containing Y as a subclass; it is said to be the radical class generated by Y. If $Y = \{G\}$ is a one-element set, then we also write $TY = T(G)$; the radical class $T(G)$ is called principal.

1.3. Proposition. (Cf. [5].) *Let Y be a nonempty subclass of $\mathcal{G}_a$. Then* TX = Ext Hom X.

1.4. Proposition. (Cf. [5].) *Let J be a nonempty class and for each $j \in J$ let X_j be a radical class. Then $\vee_{j \in J} X_j = \operatorname{Ext} \cup_{j \in J} X_j$.*

1.5. Proposition. (Cf. [5], 4.6.) *Let H be an archimedean linearly ordered group, $X = T(H)$. Then $X \in \mathscr{R}_1$.*

1.6. Corollary. *The class $\mathscr{R}_1$ is infinite.*

Proof. It is easy to verify that there exists an infinite set $S = \{G_i\}_{i \in I}$ of archimedean linearly ordered groups such that whenever i, j are distinct elements of I, then G_i fails to be isomorphic to G_j. Put $X_i = T(G_i)$ for each $i \in I$. Let $i, j \in I$, $i \neq j$ and assume that $G_i \in T(G_j)$. Hence in view of 1.3 we have $G_i \in \mathrm{Ext\,Hom}\{G_j\}$. Because G_i is o-simple, G_i belongs to $\mathrm{Hom}\{G_j\}$. Since G_j is o-simple, G_i is isomorphic either to G_j or to $\{0\}$; thus $G_i = \{0\}$. Therefore for each $k \in I$ with $k \neq i$ we have $G_k \notin T(G_i)$. Hence the set $\{T(G_k)\}_{k \in I}$ is infinite. According to 1.5 each $T(G_k)$ belongs to $\mathscr{R}_1$.

2. Lexicographic products

We recall the notion of the lexicographic product of linearly ordered groups. Let I be a linearly ordered set and for each $i \in I$ let G_i be a linearly ordered group. The lexicographic product $H = \Gamma_{i \in I} G_i$ of the system $\{G_i\}_{i \in I}$ is defined to be the set of all functions $f\colon I \to \cup_{i \in I} G_i$ such that (i) $f(i) \in G_i$ for each $i \in I$, and (ii) the set $\{i \in I\colon f(i) \neq 0\}$ is either empty or is dually well-ordered; the group operation in H is defined coordinate-wise and for $0 \neq f \in H$ we put $f > 0$ if $f(j) > 0$, where j is the greatest element of the set $\{i \in I\colon f(i) \neq 0\}$. In the case $I = \{1, 2, \ldots, n\}$ we write also $H = G_1 \circ G_2 \circ \ldots \circ G_n$.

For each linearly ordered group G we have $G = G \circ \{0\} = \{0\} \circ G$. If from G, G_1, $G_2 \in \mathscr{G}_a$, $G = G_1 \circ G_2$ it follows that $G_1 = \{0\}$ or $G_2 = \{0\}$, then G is said to be lexicographically indecomposable.

Let $H = \Gamma_{i \in I} G_i$. We shall apply the following denotations: For each $i \in I$ we denote by $\bar{G}_i$ the set of all $h \in H$ such that $h(j) = 0$ for each $j \in I$ with $j > i$; then $\bar{G}_i$ is a convex subgroup of H. For $K \subset H$ and $i \in I$, $K(G_i)$ is defined to be the set $\{k(i)\colon k \in K\}$; if K is a subgroup of H, then $K(G_i)$ is a subgroup of G_i. Further, we identify G_i with the set $\{h \in H\colon h(j) = 0$ for each $j \in I$ with $j \neq i\}$.

Let $\bar{G}_i^0$ be the set of all elements $h \in \bar{G}_i$ with $h(i) = 0$; $\bar{G}_i^0$ is a convex subgroup of $\bar{G}_i$ and the factor linearly ordered group $\bar{G}_i / \bar{G}_i^0$ is isomorphic with G_i. Thus if I is a well-ordered set, then H is a transfinite extension of linearly ordered groups G_i ($i \in I$).

Let us consider the case when there are given two lexicographic decompositions of H:

$$H = \Gamma_{i \in I} G_i, \quad H = \Gamma_{j \in J} K_j.$$

The following lemma is a corollary of the Malcev Theorem ([7]; cf. also [2]) on the existence of isomorphic refinements of two lexicographic decompositions.

2.1. Lemma. *Let H, G_i and K_j be as above. Assume that all G_i are non-zero and lexicographically indecomposable. Then there is a (unique) partition $I = \cup_{j \in J} I_j$ such that*

(i) if $j_1, j_2 \in J$, $j_1 < j_2$, $i_1 \in I_{j_1}$, $i_2 \in I_{j_2}$, then $i_1 < i_2$;

(ii) for each $j \in J$ there exists a lexicographic decomposition $K_j = \Gamma_{i \in I_j} G'_i$ such that G'_i is isomorphic with G_i for each $i \in I_j$.

2.2. Lemma. *Let H, G_i and K_j be as in 2.1. Assume that*

(i) there exists a partition $I = I' \cup I''$ with $I' \neq \emptyset \neq I''$ such that if $i_1, i_2 \in I'$, i_3, $i_4 \in I''$, then G_{i_1} and G_{i_2} are isomorphic, G_{i_3} and G_{i_4} are isomorphic, G_{i_1} fails to be isomorphic with G_{i_3} and $i_1 < i_3$;

(ii) all K_j are isomorphic.

Then J is a one-element set.

Proof. From 2.1 it follows that there are $j', j'' \in J$ with $I' \cap I_{j'} \neq \emptyset$, $I'' \cap I_{j''} \neq \emptyset$. If there exists $j \in J$ such that either (a) $I_j \subset I'$, or (b) $i_j \subset I''$, then we have a contradiction with 2.1 (since K_j is isomorphic to $K_{j'}$ and to $K_{j''}$). Hence for each $j \in J$ both $I_j \cap I'$ and $I_j \cap I''$ are nonempty. Assume that $\operatorname{card} J > 1$. Let j_1 and j_2 be distinct elements of J with $j_1 < j_2$. Then there are elements $i'' \in I_{j_1} \cap I''$, $i' \in I_{j_2} \cap I'$.

In view of $i' \in I_{j_2}$, $i'' \in I_{j_1}$ we have $i'' < i'$; on the other hand, from $i' \in I'$, $i'' \in I''$ we obtain $i' < i''$, which is a contradiction. Hence $\operatorname{card} J = 1$.

3. The class $\mathcal{R}_1$

In the following lemmas 3.1—3.3 we assume that $G \in \mathcal{G}_a$, $B = T(G)$, $A \in \mathcal{R}_a$ and that A covers B in the lattice $\mathcal{R}_a$.

3.1. Lemma. *There exists $H \in A \backslash B$ such that $B(H) = \{0\}$ and $B \vee T(H) = A$.*

Proof. Since B is covered by A, there is $H_1 \in A \backslash B$. Hence (cf. 1.2) $B(H_1) \neq H_1$ and $B(H_1) \in B$. Denote $H = H_1/B(H_1)$. Then $H \in A$. From $H \in B$ it would follow $H_1 \in B$ (with regard to 1.1, (b)), which is a contradiction; therefore $H \in A \backslash B$. In view of 1.4 we have $H_1 \in T(H) \vee B$, hence $B < T(H) \vee B \leqq A$ and thus $T(H) \vee B = A$. Moreover, from 1.2 we easily obtain that $B(H) = \{0\}$ is valid.

Let α be an infinite cardinal. We denote by $\omega(\alpha)$ the least ordinal having the property that the power of the set of all ordinals less than $\omega(\alpha)$ is α. For each $G \in \mathcal{G}_a$ we put

$$G_\alpha = \Gamma_{i \in I(\alpha)} G_i,$$

where $I(\alpha)$ is a linearly ordered set isomorphic with $\omega(\alpha)$ and G_i is a linearly ordered group isomorphic with G for each $i \in I(\alpha)$.

3.2. Lemma. *Let H be as in 3.1 and let $\alpha > \operatorname{card} H$. Then $H_\alpha \in B$.*

Proof. By way of contradiction, suppose that H_α does not belong to B. Since $H \in A$ and because H_α is a transfinite extension of H, we have $H_\alpha \in A$. From this and from the fact that B is covered by A we infer that $T(H_\alpha) \vee B = A$ is valid. Hence $H \in T(H_\alpha) \vee B$. In view of 1.4, $H \in \mathrm{Ext}(\mathrm{Hom}\{H_\alpha\} \cup B)$. If $G_1 \in \mathrm{Hom}\{H_\alpha\}$, $G_1 \neq \{0\}$, then $\mathrm{card}\, G_1 > \mathrm{card}\, H$. Hence $H \in \mathrm{Ext}\, B = B$, which is a contradiction.

3.3. Lemma. *There exists $G_1 \in \mathrm{Hom}(G)$, $G_1 \neq \{0\}$ such that G_1 can be expressed as a lexicographic product $G_1 = \Gamma_{j \in J} H_j$ of factors H_j isomorphic to H, where J is a well-ordered set.*

Proof. According to 3.2 we have $H_\alpha \in B = T(G)$. Hence is view of 1.3 $H_\alpha \in \mathrm{Ext\ Hom}\{G\}$. Thus there exists a convex subgroup G_1 of H_α with $G_1 \neq \{0\}$ such that $G_1 \in \mathrm{Hom}(G)$.

From the definition of H_α it follows that G_1 can be expressed uniquely as

$$G_1 = P \circ Q$$

such that (*i*) either $P = \{0\}$ or $P = \Gamma_{j \in J} H_j$, where J is a well-ordered set and each H_j is isomorphic to H, and (*ii*) either $Q = \{0\}$, or Q is isomorphic to a convex subgroup of H and Q is not isomorphic to H.

Suppose that $Q \neq \{0\}$. Since $G_1 \in \mathrm{Hom}(G)$, we have $G_1 \in B$ and hence $Q \in B$ (because of $Q \in \mathrm{Hom}(G_1)$). Thus $B(Q) = Q$ and hence $B(H) \neq \{0\}$; in view of 3.1, this is a contradiction. Hence $Q = \{0\}$ and therefore $P \neq \{0\}$, completing the proof.

Now let K_1 and K_2 be non-zero archimedean linearly ordered groups such that K_1 is not isomorphic to K_2. Let β and γ be infinite cardinals. Denote

$$G(\beta, \gamma) = (K_1)_\beta \circ (K_2)_\gamma \circ K_2.$$

3.4. Lemma. $T(G(\beta, \gamma))$ has no cover in the lattice $\mathcal{R}_a$.

Proof. Put $G(\beta, \gamma) = G$, $B = T(G)$ and assume that A is a cover of B. Let H be as in 3.1 and let G_1 be as in 3.3. Since both K_1 and K_2 are archimedean, G_1 can be expressed in the form

$$G_1 = P_1 \circ Q_1$$

such that (*i*) either $P_1 = \{0\}$ or $P_1 = \Gamma_{i \in I} K_i$, where I is a well-ordered set and each K_i is isomorphic to K_1; (*ii*) $Q_1 = \Gamma_{s \in S} K'_s$, where S is a well-ordered set and each K'_s is isomorphic to K_2.

First assume that $P_1 \neq \{0\}$. Then from 3.3 and 2.2 we obtain that $G_1 = H$ implying $H \in B$, which is a contradiction.

Now suppose that $P_1 = \{0\}$. Thus $G_1 = Q_1$. From this and from the Malcev Theorem on the existence of isomorphic rafinements (cf. [7] or [2]) it follows that H can be written as $H = \Gamma_{m \in M} K_m$, where M is a well-ordered set and each K_m is isomorphic to K_2. Hence $H \in T(K_2)$. From the definition of G we infer that $K_2 \in T(G)$, hence $H \in T(G)$, which is impossible.

3.5. Lemma. *Let β_1, β_2 be distinct infinite cardinals, $\beta_i > \operatorname{card} K_1$ $(i = 1, 2)$. Then $T(G(\beta_1, \gamma)) \neq T(G(\beta_2, \gamma))$.*

Proof. Let $\beta_1 < \beta_2$. It suffices to verify that $G(\beta_1, \gamma)$ does not belong to $T(G(\beta_2, \gamma))$. By way of contradiction assume that $G(\beta_1, \gamma) \in T(G(\beta_2, \gamma))$. Thus $G(\beta_1, \gamma) \in \operatorname{Ext}\operatorname{Hom}(G(\beta_2, \gamma))$. Hence there exists a convex subgroup $G_1 \neq \{0\}$ of $G(\beta_1, \gamma)$ and a homomorphic image G_2 of $G(\beta_2, \gamma)$ such that G_1 is isomorphic to G_2.

From the structure of $G(\beta_1, \gamma)$ and $G(\beta_2, \gamma)$ it follows that

$$G_1 = P_1 \circ Q_1,$$

such that either (i) $P_1 = (K_1)_{\beta_1}$ and Q_1 is a convex subgroup of $(K_2)_\gamma \circ K_2$, or (ii) P_1 is a convex subgroup of $(K_1)_{\beta_1}$ and $Q_1 = \{0\}$; further G_2 is isomorphic to

$$P_2 \circ Q_2$$

such that either (i') P_2 is a homomorphic image of $(K_1)_{\beta_2}$ and Q_2 is isomorphic to $(K_2)_\gamma \circ K_2$, or (ii') $P_2 = \{0\}$ and Q_2 is a homomorphic image of $(K_2)_\gamma \circ K_2$.

In view of (i) and (ii), the condition (ii') cannot hold. From (i') it follows that the condition (i) must be valid and that G_1 is isomorphic to $G(\beta_1, \gamma)$; this implies that P_1 is isomorphic to P_2. However, this is impossible, because $\operatorname{card} P_1 = \beta_1 < \beta_2 = \operatorname{card} P_2$.

From 3.4, 3.5 and from the fact that K_2 belongs to $T(G(\beta, \gamma)$ we obtain:

3.6. Theorem. Let $K \neq \{0\}$ be an archimedean linearly ordered group. There exists $C \subset \mathcal{R}_a$ such that:

(i) if $X \in C$, then X is principal and has no cover in $\mathcal{R}_a$;

(ii) $T(K) < X$ for each $X \in C$;

(iii) there exists an injective mapping of the class of all cardinals into C.

3.7. Corollary. *There exists an injective mapping of the class of all cardinals into $\mathcal{R}_1$.*

We conclude by remarking that for each $X \in C$ there exists $X' \in \mathcal{R}_a$ such that X covers X'; moreover, X' is the join of all radical classes less than X (the proof is analogous to that of Propos. 4.7, [5] and therefore it will be omitted).

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ПОКРЫТИЯ В РЕШЕТКЕ РАДИКАЛЬНЫХ КЛАССОВ ЛИНЕЙНО УПОРЯДОЧЕННЫХ ГРУПП

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Резюме

Пусть $\mathcal{R}_a$ — решетка всех радикальных классов абелевых линейно упорядоченных групп. Пусть, далее, $\mathcal{R}_1$ — класс всех $X \in \mathcal{R}_a$ таких, что а) X является главным радикальным классом, и б) X не имеет покрытий в $\mathcal{R}_a$. В статье доказано, что $\mathcal{R}_1$ — собственный класс.