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TWO-STAGE REGRESSION MODEL

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Introduction

A mixed linear model is characterized by the relations $E(\boldsymbol{Y}|\boldsymbol{\beta}) = \mathbf{X}\boldsymbol{\beta}$, $\mathrm{Var}(\boldsymbol{Y}|\boldsymbol{\vartheta}) = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$, where $\boldsymbol{Y}$ is an n-dimensional random vector, $\boldsymbol{\beta}$ is an unknown k-dimensional parameter, $\boldsymbol{\beta} \in \mathscr{R}^k$ (k-dimensional Euclidean space), $\mathbf{X}$ is a known $n \times k$ matrix, $\boldsymbol{\vartheta}$ is a p-dimensional vector of variance components (usually unknown), $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_p)' \in \underline{\boldsymbol{\vartheta}} \subset \mathscr{R}^p$, $\underline{\boldsymbol{\vartheta}}$ is an open set, $\mathbf{V}_i$, $i = 1, \ldots, p$, are known symmetric $n \times n$ matrices; E and Var denote mean value and covariance matrix, respectively.

$$\text{If } \boldsymbol{Y} = (\boldsymbol{Y}_1', \boldsymbol{Y}_2')', \ \mathbf{X} = \begin{bmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{C}, & \mathbf{X}_2 \end{bmatrix}, \ \mathbf{C} \neq \mathbf{0}, \ \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11}, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}_{22} \end{bmatrix},$$

where the dimensions of the vectors $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ are n_1 and n_2 ($n_1 + n_2 = n$), the matrices $\mathbf{X}_1$, $\mathbf{X}_2$ are of the types $n_1 \times k_1$, $n_2 \times k_2$ and the matrices $\boldsymbol{\Sigma}_{11}$, $\boldsymbol{\Sigma}_{22}$ are of the types $n_1 \times n_1$, $n_2 \times n_2$, respectively, then the regression model $(\boldsymbol{Y}, \mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$ is called the two-stage regression model ([1], [6], [7]).

The aim of the paper is to find the locally (or uniformly) best estimators of the parameters $\boldsymbol{\beta}$ and $\boldsymbol{\vartheta}$ under some, in the following more excactly specified, conditions.

1. Notations, definitions and auxiliary statements

Definition 1.1. *The two-stage regression model is regular if the ranks of the matrices* $\mathbf{X}_1$, $\mathbf{X}_2$, $\boldsymbol{\Sigma}_{11}$, $\boldsymbol{\Sigma}_{22}$ *are*: $R(\mathbf{X}_1) = k_1 \leqslant n_1$, $R(\mathbf{X}_2) = k_2 \leqslant n_2$, $R(\boldsymbol{\Sigma}_{11}) = n_1$, $R(\boldsymbol{\Sigma}_{22}) = n_2$.

In the following the matrices $\boldsymbol{\Sigma}_{11}$ and $\boldsymbol{\Sigma}_{22}$ are considered in the form $\boldsymbol{\Sigma}_{11} = \sigma_1^2 \mathbf{H}_1$ and $\boldsymbol{\Sigma}_{22} = \sigma_2^2 \mathbf{H}_2$, where $\sigma_i^2 \in (0, \infty)$, $i = 1, 2$, are variance components; thus $\boldsymbol{\vartheta} = (\sigma_1^2, \sigma_2^2)' \in (0, \infty) \times (0, \infty) = \underline{\boldsymbol{\vartheta}}$. It is obvious that the matrix $\boldsymbol{\Sigma}(\boldsymbol{\vartheta})$ is regular for each $\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}$. The variance components are considered to be a) known, b) unknown, their ratio $\varrho = \sigma_1^2/\sigma_2^2$ is known and c) unknown with the unknown ratio ϱ.

The LBLUE (locally best linear unbiased estimator) of the parameter $\boldsymbol{\beta}_i$, $i = 1, 2$, based on the vector $\boldsymbol{Y}_i$ is denoted by $\hat{\boldsymbol{\beta}}_i(\boldsymbol{Y}_i)$ (if it exists); thus, e.g., $\hat{\boldsymbol{\beta}}_2[\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1),\ \boldsymbol{Y}_2]$ is the LBLUE of $\boldsymbol{\beta}_2$ which is based on $\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1)$ and $\boldsymbol{Y}_2$. UBLUE means the uniform BLUE (with respect to the variance components).

The LMVQUIE (locally minimum variance quadratic unbiased invariant estimator) of the parameter σ_i^2 based on the vector $\boldsymbol{Y}_i$ is denoted as $\hat{\sigma}_i^2(\boldsymbol{Y}_i)$ (if it exists); estimators of the type $\boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$ are considered only; thus $\hat{\sigma}_2^2[\hat{\sigma}_1^2(\boldsymbol{Y}_1), \hat{\boldsymbol{\beta}}\,(\boldsymbol{Y}_1), \boldsymbol{Y}_2]$ is the LMVQUIE of σ_2^2 based on $\hat{\sigma}_1^2(\boldsymbol{Y}_1)$, $\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1)$ and $\boldsymbol{Y}_2$; here $\hat{\sigma}_2^2[\hat{\sigma}_1^2(\boldsymbol{Y}_1), \hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1),\ \boldsymbol{Y}_2] = k_1\hat{\sigma}_1^2(\boldsymbol{Y}_1) + k_2[\hat{\boldsymbol{\beta}}_1'(\boldsymbol{Y}_1),\ \boldsymbol{Y}_2']\mathbf{A}[\hat{\boldsymbol{\beta}}_1'(\boldsymbol{Y}_1),\ \boldsymbol{Y}_2']'$ (k_1, k_2 are properly chosen constants, $\mathbf{A}$ is a properly chosen matrix).

Within the two-stage regression model the estimators are permitted to be determined in the following sequence only:

$$\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1),\ \hat{\sigma}_1^2(\boldsymbol{Y}_1),\ \hat{\sigma}_2^2[\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1),\ \hat{\sigma}_1^2(\boldsymbol{Y}_1),\ \boldsymbol{Y}_2],\ \hat{\boldsymbol{\beta}}_2\{\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1),\ \boldsymbol{Y}_2|\hat{\sigma}_1^2(\boldsymbol{Y}_1),$$

$$\hat{\sigma}_2^2[\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1),\ \hat{\sigma}_1^2(\boldsymbol{Y}_1),\ \boldsymbol{Y}_2]\},\ \hat{\sigma}_2^2(\boldsymbol{Y}_1,\ \boldsymbol{Y}_2),\ \hat{\sigma}_1^2(\boldsymbol{Y}_1,\ \boldsymbol{Y}_2),\ \hat{\boldsymbol{\beta}}_2[\boldsymbol{Y}_1,\ \boldsymbol{Y}_2|\hat{\sigma}_1^2(\boldsymbol{Y}_1,\ \boldsymbol{Y}_2),$$

$$\hat{\sigma}_2^2(\boldsymbol{Y}_1,\ \boldsymbol{Y}_2)],\ \hat{\boldsymbol{\beta}}_1[\boldsymbol{Y}_1,\ \boldsymbol{Y}_2|\hat{\sigma}_1^2(\boldsymbol{Y}_1,\ \boldsymbol{Y}_2),\ \sigma_2^2(\boldsymbol{Y}_1,\ \boldsymbol{Y}_2)].$$

The notation $\hat{\boldsymbol{\beta}}_2[\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1),\ \boldsymbol{Y}_2|\sigma_1^2,\ \sigma_2^2]$ means the $(\sigma_1^2,\ \sigma_2^2)$-LBLUE.

In what follows the normality of the vector $\boldsymbol{Y}$ is assumed; $\boldsymbol{Y} \sim N(\mathbf{X}\boldsymbol{\beta}, \Sigma)$.

The symbol $\mathbf{A}^-$ means the generalized inverse (g-inverse) of the matrix $\mathbf{A}$, i.e. $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$; $\mathbf{A}^+$ is the Moore—Penrose g-inverse [4], $\mathbf{A}^-_{m(\mathbf{N})}$ is the minimum $\mathbf{N}$-seminorm g-inverse of the matrix $\mathbf{A}$ [4]. Ker$(\mathbf{A})$ denotes the null-space of the matrix $\mathbf{A}$ and $\mathcal{M}(\mathbf{A})$ denotes the column space of the matrix $\boldsymbol{A}$.

Lemma 1.1. *In the model* $\boldsymbol{Y} \sim N_n\left(\mathbf{X}\boldsymbol{\beta},\ \sum_{i=1}^{p} \vartheta_i\mathbf{V}_i\right)$ *the unbiased invariant estimator of the vector* $\boldsymbol{\vartheta}$ *exists if and only if the matrix* $\mathbf{K}^{(I)}$, $\{\mathbf{K}^{(I)}\}_{i,j} = \mathrm{Tr}(\mathbf{M}\mathbf{V}_i\mathbf{M}\mathbf{V}_j)$, $i, j = 1, \dots, p$, $\mathbf{M} = \mathbf{I} - \mathbf{X}\mathbf{X}^+$, *is regular. The matrix* $\mathbf{K}^{(I)}$ *is regular if and only if the matrices* $\mathbf{M}\mathbf{V}_1\mathbf{M}, \dots, \mathbf{M}\mathbf{V}_p\mathbf{M}$ *are linearly independent.*

Proof. See [5].

Lemma 1.2. *If* $\mathbf{K}^{(I)}$ *from Lemma* 1.1 *is regular, then the* $\boldsymbol{\vartheta}_0$-LMVQUIE *of the vector* $\boldsymbol{\vartheta}$ *is* $\hat{\boldsymbol{\vartheta}}(\boldsymbol{Y}) = \mathbf{S}^{-1}_{(\mathbf{M}\Sigma_0\mathbf{M})^+}\hat{\gamma}$, *where*

$$\{\mathbf{S}_{(\mathbf{M}\Sigma_0\mathbf{M})^+}\}_{i,j} = \mathrm{Tr}\,[(\mathbf{M}\Sigma_0\mathbf{M})^+\mathbf{V}_i(\mathbf{M}\Sigma_0\mathbf{M})^+\mathbf{V}_j], \quad i, j = 1, \dots, p,$$

$$\mathbf{M} = \mathbf{I} - \mathbf{X}\mathbf{X}^+,\ \Sigma_0 = \sum_{i=1}^{p} \vartheta_{0i}\mathbf{V}_i,\ (\vartheta_{01}, \dots, \vartheta_{0p})' = \boldsymbol{\vartheta}_0,\ \hat{\gamma} = (\hat{\gamma}_1, \dots, \hat{\gamma}_p)',$$

$$\hat{\gamma}_i = \boldsymbol{Y}'(\mathbf{M}\Sigma_0\mathbf{M})^+\mathbf{V}_i(\mathbf{M}\Sigma_0\mathbf{M})^+\boldsymbol{Y}, \quad i = 1, \dots, p.$$

Proof. See [5].

Lemma 1.3. *If* Σ_0 *in Lemma* 1.2 *is regular, then* $(\mathbf{M}\Sigma_0\mathbf{M})^+ = \Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{X}\cdot$ $\cdot(\mathbf{X}'\Sigma_0^{-1}\mathbf{X})^-\mathbf{X}'\Sigma_0^{-1}$.

Proof. The statement follows from the definition of the Moore—Penrose g-inverse.

2. Solution

Theorem 2.1. *If in the two-stage regression model the variance components σ_1^2 and σ_2^2 are known (i.e. the matrices Σ_{11} and Σ_{22} are known), then*

(1) $\hat{\beta}_1(Y_1) = (X_1'\Sigma_{11}^{-1}X_1)^{-1}X_1'\Sigma_{11}^{-1}Y_1$,

(2) $\hat{\beta}_2[\hat{\beta}_1(Y_1), Y_2] = (X_2'K_2^{-1}X_2)^{-1}X_2'K_2^{-1}[Y_2 - C\hat{\beta}_1(Y_1)]$,

where $K_2 = C(X_1'\Sigma_{11}^{-1}X_1)^{-1}C' + \Sigma_{22}$,

(3) $\hat{\beta}_2(Y_1, Y_2) = \hat{\beta}_2[\hat{\beta}_1(Y_1), Y_2]$,

(4) $\hat{\beta}_1(Y_1, Y_2) = \hat{\beta}_1(Y_1) + (X_1'\Sigma_{11}^{-1}X_1)^{-1}C'K_2^{-1}v_2$, *where*

$v_2 = Y_2 - C\hat{\beta}_1(Y_1) - X_2(X_2'K_2^{-1}X_2)^{-1}X_2'K_2^{-1}[Y_2 - C\hat{\beta}_1(Y_1)]$.

Proof. (1) is a well-known fact (see, e.g., [2]); (2) is a consequence of the fact that $Y_2 - C\hat{\beta}_1(Y_1) \sim N_{n_2}(X_2\beta_2, K_2)$; (3) is proved in [1]. As regards (4) it is sufficient to prove that a) the vector $(v_1', v_2')'$, $v_1 = Y_1 - X_1\hat{\beta}_1(Y_1)$, represents the class of all linear unbiased estimators of the function $g(\beta_1, \beta_2) = 0$, $\beta_1 \in \mathscr{R}^{k_1}$, $\beta_2 \in \mathscr{R}^{k_2}$, which are based on the vector $(Y_1', Y_2')'$ and b) $\hat{\beta}_1(Y_1, Y_2)$ arises from $\hat{\beta}_1(Y_1)$ using the covariance correction from the vector $(v_1', v_2')'$. The rest of the proof is then a consequence of the C. R. Rao fundamental lemma of locally best unbiased estimators [3, p. 257].

a) The class of all linear unbiased estimators of the function g is $\mathscr{U}_0 = \{L_1'Y_1 + L_2'Y_2 : L_1 \in \mathscr{R}^{n_1}, L_2 \in \mathscr{R}^{n_2}, E(L_1'Y_1 + L_2'Y_2 \mid \beta_1, \beta_2) = 0, \beta_1 \in \mathscr{R}^{k_1}, \beta_2 \in \mathscr{R}^{k_2}\} = \left\{L_1'Y_1 + L_2'Y_2 : \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \in \mathrm{Ker}\begin{bmatrix} X_1', & C' \\ 0, & X_2' \end{bmatrix}\right\}$. We shall prove that

$$(*) \qquad \mathrm{Ker}\begin{bmatrix} X_1', & C' \\ 0, & X_2' \end{bmatrix} = \mathscr{M}\begin{bmatrix} A_{11}, & A_{12} \\ A_{21}, & A_{22} \end{bmatrix},$$

$$A_{11} = I - (X_1')^-_{m(\Sigma_{11})}X_1', \quad A_{12} = -(X_1')^-_{m(\Sigma_{11})}C'[I - (X_2')^-_{m(K_2)}X_2'],$$

$$A_{21} = 0, \qquad A_{22} = I - (X_2')^-_{m(K_2)}X_2'.$$

$(*)$ follows from the inclusion

$$\mathscr{M}\begin{bmatrix} A_{11}, & A_{12} \\ A_{21}, & A_{22} \end{bmatrix} \subset \mathrm{Ker}\begin{bmatrix} X_1', & C' \\ 0, & X_2' \end{bmatrix} (\Leftarrow \mathscr{M}(C') \subset \mathscr{M}(X_1') = \mathscr{R}^{k_1})$$

and from the equality

$$\dim \operatorname{Ker}\begin{bmatrix} \mathbf{X}_1', & \mathbf{C}' \\ \mathbf{0}, & \mathbf{X}_2' \end{bmatrix} = n_1 + n_2 - (k_1 + k_2) = \dim \mathscr{M}\begin{bmatrix} \mathbf{A}_{11}, & \mathbf{A}_{12} \\ \mathbf{A}_{21}, & \mathbf{A}_{22} \end{bmatrix}.$$

Each vector $\begin{bmatrix} \boldsymbol{L}_1 \\ \boldsymbol{L}_2 \end{bmatrix}$ from $\operatorname{Ker}\begin{bmatrix} \mathbf{X}_1', & \mathbf{C}' \\ \mathbf{0}, & \mathbf{X}_2' \end{bmatrix}$ can be expressed in the form

$$(\boldsymbol{L}_1', \boldsymbol{L}_2') = (\boldsymbol{u}_1', \boldsymbol{u}_2')\begin{bmatrix} \mathbf{A}_{11}', & \mathbf{A}_{21}' \\ \mathbf{A}_{12}', & \mathbf{A}_{22}' \end{bmatrix}, \quad \boldsymbol{u}_1 \in \mathscr{R}^{n_1}, \ \boldsymbol{u}_2 \in \mathscr{R}^{n_2},$$

i.e. the element from $\mathscr{U}_0$ is of the form $(\boldsymbol{u}_1', \boldsymbol{u}_2')\begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}$, because of $\mathbf{A}_{11}'\boldsymbol{Y}_1 = \boldsymbol{v}_1 (\mathbf{A}_{21} = \mathbf{0})$, and $\mathbf{A}_{12}'\boldsymbol{Y}_1 + \mathbf{A}_{22}'\boldsymbol{Y}_2 = \boldsymbol{v}_2$.

b) The estimator

$$(**) \qquad \hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1) - \operatorname{cov}\left[\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1), \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}\right]\left[\operatorname{Var}\begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}\right]^- \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} = \hat{\hat{\boldsymbol{\beta}}}_1$$

possesses the property

$$(***) \qquad \operatorname{cov}\left[\hat{\hat{\boldsymbol{\beta}}}_1, \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}\right] = \mathbf{0}.$$

Here the equality

$$\operatorname{cov}\left[\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1), \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}\right]\left[\operatorname{Var}\begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}\right]^- \operatorname{Var}\begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} =$$

$$= \operatorname{cov}\left[\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1), \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}\right]\left(\Leftarrow \mathscr{M}\left(\left\{\operatorname{cov}\left[\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1), \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}\right]\right\}'\right) \subset \mathscr{M}\left[\operatorname{Var}\begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}\right]\right)$$

was applied.

(***) is the necessary and sufficient condition for (**) to be the LBLUE (see [3, p. 257]).

Because of $\operatorname{cov}\left[\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1), \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}\right] = [\mathbf{0}, -(\mathbf{X}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}'\mathbf{M}_{\mathbf{X}_2}']$

$$\operatorname{Var}\begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{\mathbf{X}_1}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{M}_{\mathbf{X}_2} \end{bmatrix}\begin{bmatrix} \boldsymbol{\Sigma}_{11}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{K}_2 \end{bmatrix}, \text{ where } \mathbf{M}_{\mathbf{X}_1} = \mathbf{I} - \mathbf{P}_{\mathbf{X}_1},$$

$\mathbf{P}_{\mathbf{X}_1} = \mathbf{X}_1(\mathbf{X}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}_{11}^{-1}$, $\mathbf{M}_{\mathbf{X}_2} = \mathbf{I} - \mathbf{P}_{\mathbf{X}_2}$, $\mathbf{P}_{\mathbf{X}_2} = \mathbf{X}_2(\mathbf{X}_2'\mathbf{K}_2^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{K}_2^{-1}$

and because it can be easily shown that

$$\left[\operatorname{Var}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right]^- = \begin{bmatrix} \boldsymbol{\Sigma}_{11}^{-1}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{K}_2^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{\mathbf{X}_1}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{M}_{\mathbf{X}_2} \end{bmatrix},$$

we see that (**) after some rearrangement is $\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1, \boldsymbol{Y}_2)$ from (4).

Remark 2.1. Theorem 2.1 can be easily modified for the case of the known ratio $\varrho = \sigma_1^2/\sigma_2^2$ of the unknown covariance components.

Theorem 2.2. *Consider a regular two-stage regression model with* $\boldsymbol{\Sigma}_{11} = \sigma_1^2 \mathbf{H}_1$ *and* $\boldsymbol{\Sigma}_{22} = \sigma_2^2 \mathbf{H}_2$ *when the ratio* $\varrho = \sigma_1^2/\sigma_2^2$ *of the unknown covariance components* $(\sigma_1^2, \sigma_2^2) \in (0, \infty) \times (0, \infty)$ *is unknown. Then*

1. *If* $\mathscr{M}(\mathbf{C}) \not\subset \mathscr{M}(\mathbf{X}_2)$ *and* $n_2 > k_2$, *then there exist* LMVQUIEs $\hat{\sigma}_1^2(\boldsymbol{Y}_1, \boldsymbol{Y}_2)$ and $\hat{\sigma}_2^2(\boldsymbol{Y}_1, \boldsymbol{Y}_2)$.

2. *If* $\mathscr{M}(\mathbf{C}) \subset \mathscr{M}(\mathbf{X}_2)$, $n_1 > k_1$, $n_2 > k_2$, *then there exist* UMVQUIEs $\hat{\sigma}_1^2(\boldsymbol{Y}_1, \boldsymbol{Y}_2)$ *and* $\hat{\sigma}_2^2(\boldsymbol{Y}_1, \boldsymbol{Y}_2)$.

Proof. Without any loss of generality we can consider

$$\boldsymbol{\Sigma}_{11} = \hat{\sigma}_1^2 \mathbf{I},\ \boldsymbol{\Sigma}_{22} = \sigma_2^2 \mathbf{I}. \text{ If } \mathbf{V}_1 = \begin{bmatrix} \mathbf{I}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0} \end{bmatrix},\ \mathbf{V}_2 = \begin{bmatrix} \mathbf{0}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{I} \end{bmatrix} \text{ and } \mathbf{X} = \begin{bmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{C}, & \mathbf{X}_2 \end{bmatrix},$$

then

$$\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \begin{bmatrix} \mathbf{P}_{11}, & \mathbf{P}_{12} \\ \mathbf{P}_{21}, & \mathbf{P}_{22} \end{bmatrix},$$

where

$$\mathbf{P}_{11} = \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1' - \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{C}'\varkappa\mathbf{C}(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1',$$

$$\varkappa = \mathbf{K}^{-1} - \mathbf{K}^{-1}\mathbf{X}_2(\mathbf{X}_2'\mathbf{K}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{K}^{-1}$$

$\Big(= \mathbf{K}^{-1}\mathbf{M}_{\mathbf{X}_2}$, $\mathbf{M}_{\mathbf{X}_2}$ is the $\mathbf{K}^{-1}$ — projector onto the $\mathbf{K}^{-1}$-orthogonal complement of the subspace $\mathscr{M}(\mathbf{X}_2)\Big)$,

$$\mathbf{K} = \mathbf{C}(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{C}' + \mathbf{I},$$

$$\mathbf{P}_{12} = \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{C}'\varkappa = \mathbf{P}_{21}',$$

$$\mathbf{P}_{22} = \mathbf{I} - \varkappa.$$

Here the relationships

$$\begin{bmatrix} \mathbf{X}_1'\mathbf{X}_1 + \mathbf{C}'\mathbf{C}, & \mathbf{C}'\mathbf{X}_2 \\ \mathbf{X}_2'\mathbf{C}, & \mathbf{X}_2'\mathbf{X}_2 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}, & \mathbf{B} \\ \mathbf{B}', & \mathbf{D} \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{A}^{-1}, & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1} \\ -(\mathbf{D} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{A}^{-1}, & (\mathbf{D} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1} \end{bmatrix},$$

$$(\mathbf{X}_1'\mathbf{X}_1 + \mathbf{C}'\mathbf{C})^{-1} = (\mathbf{X}_1'\mathbf{X}_1)^{-1} - (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{C}'[\mathbf{I} + \mathbf{C}(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{C}']^{-1}\mathbf{C}(\mathbf{X}_1'\mathbf{X}_1)^{-1} =$$
$$= (\mathbf{X}_1'\mathbf{X}_1)^{-1} - (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{C}'\mathbf{K}^{-1}\mathbf{C}(\mathbf{X}_1'\mathbf{X}_1)^{-1}$$

and $\mathbf{C}(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{C}' = \mathbf{K} - \mathbf{I}$ were utilized. In accordance with Lemma 1.1 it can be easily shown that

$$\mathbf{MV}_1\mathbf{M} = \begin{bmatrix} (\mathbf{I} - \mathbf{P}_{11})^2, & -(\mathbf{I} - \mathbf{P}_{11})\mathbf{P}_{12} \\ -\mathbf{P}_{21}(\mathbf{I} - \mathbf{P}_{11}), & \mathbf{P}_{21}\mathbf{P}_{12} \end{bmatrix},$$

$$\mathbf{MV}_2\mathbf{M} = \begin{bmatrix} \mathbf{P}_{12}\mathbf{P}_{21}, & -\mathbf{P}_{12}(\mathbf{I} - \mathbf{P}_{22}) \\ -(\mathbf{I} - \mathbf{P}_{22})\mathbf{P}_{21}, & (\mathbf{I} - \mathbf{P}_{22})^2 \end{bmatrix},$$

where $\mathbf{M} = \mathbf{I} - \mathbf{P}$. The diagonal submatrices of the matrices $\mathbf{MV}_1\mathbf{M}$ and $\mathbf{MV}_2\mathbf{M}$ can be expressed as follows:

$$(\mathbf{I} - \mathbf{P}_{11})^2 = \mathbf{M}^*_{\mathbf{X}_1} + \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{C}'(\varkappa - \varkappa^2)\mathbf{C}(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1',$$
$$\mathbf{M}^*_{\mathbf{X}_1} = \mathbf{I} - \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1',$$
$$\mathbf{P}_{21}\mathbf{P}_{12} = \varkappa - \varkappa^2,$$
$$\mathbf{P}_{12}\mathbf{P}_{21} = \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{C}'\varkappa^2\mathbf{C}(\mathbf{X}_1'\mathbf{x}_1)^{-1}\mathbf{X}_1',$$
$$(\mathbf{I} - \mathbf{P}_{22})^2 = \varkappa^2.$$

Furthermore, $\varkappa \neq \mathbf{0}$ ($\Leftarrow n_2 > k_2$) and $\varkappa^2 \neq \varkappa$ ($\Leftarrow \mathscr{M}(\mathbf{C}) \not\subset \mathscr{M}(\mathbf{X}_2)$). The first implication is obvious. The other can be proved by contradiction. Let $\varkappa = \varkappa^2$. Then

$$\mathbf{K}^{-1}\mathbf{M}_{\mathbf{X}_2} = \mathbf{K}^{-1}\mathbf{M}_{\mathbf{X}_2}\mathbf{K}^{-1}\mathbf{M}_{\mathbf{X}_2}(\Leftrightarrow \mathbf{M}_{\mathbf{X}_2}\mathbf{K}^{-1}\mathbf{M}_{\mathbf{X}_2} = \mathbf{M}_{\mathbf{X}_2})$$

and

$$\mathbf{K}^{-1} = [\mathbf{I} + \mathbf{C}(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{C}']^{-1} = \mathbf{I} - \mathbf{C}(\mathbf{X}_1'\mathbf{X}_1 + \mathbf{C}'\mathbf{C})^{-1}\mathbf{C}' \Rightarrow \mathbf{M}_{\mathbf{X}_2}\mathbf{K}^{-1}\mathbf{M}_{\mathbf{X}_2} =$$
$$= \mathbf{M}_{\mathbf{X}_2} - \mathbf{M}_{\mathbf{X}_2}\mathbf{C}(\mathbf{X}_1'\mathbf{X}_1 + \mathbf{C}'\mathbf{C})^{-1}\mathbf{C}'\mathbf{M}_{\mathbf{X}_2} \text{ thus } \mathbf{M}_{\mathbf{X}_2}\mathbf{K}^{-1}\mathbf{M}_{\mathbf{X}_2} = \mathbf{M}_{\mathbf{X}_2} \Rightarrow$$
$$\Rightarrow \mathbf{M}_{\mathbf{X}_2}\mathbf{C}(\mathbf{X}_1'\mathbf{X}_1 + \mathbf{C}'\mathbf{C})^{-1}\mathbf{C}'\mathbf{M}_{\mathbf{X}_2} = \mathbf{0} \Rightarrow \mathbf{M}_{\mathbf{X}_2}\mathbf{C} = \mathbf{0} \Rightarrow \mathscr{M}(\mathbf{C}) \subset \mathscr{M}(\mathbf{X}_2).$$

As $\varkappa$ is positive semidefinite $\varkappa \neq \mathbf{0} \Rightarrow \varkappa^2 \neq \mathbf{0}$. If $\forall\{k \in \mathscr{R}^1\}\varkappa - \varkappa^2 \neq k\varkappa^2$, then $\mathbf{P}_{21}\mathbf{P}_{12}$ and $(\mathbf{I} - \mathbf{P}_{22})^2$ are linearly independent and thus $\mathbf{MV}_1\mathbf{M}$ and $\mathbf{MV}_2\mathbf{M}$ are linearly independent. If $\exists\{k_0 \in \mathscr{R}^1\}\varkappa - \varkappa^2 = k_0\varkappa^2$, then

$$(\mathbf{I} - \mathbf{P}_{11})^2 = \mathbf{M}^*_{\mathbf{X}_1} + k_0\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{C}'\varkappa^2\mathbf{C}(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'$$

and

$$\mathbf{P}_{12}\mathbf{P}_{21} = \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{C}'\varkappa^2\mathbf{C}(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'$$

are nonzero matrices ($(\mathbf{I} - \mathbf{P}_{11})^2 \neq \mathbf{0}$ is obvious; $\mathbf{P}_{12}\mathbf{P}_{21} \neq \mathbf{0} \Leftarrow$ the rank

$$R(\mathbf{P}_{21}) = R[\varkappa\mathbf{C}(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'] = R[\mathbf{K}^{-1}\mathbf{M}_{\mathbf{X}_2}\mathbf{C}(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'] \geqslant$$

$$\geqslant R[\mathbf{M}_{\mathbf{X}_2}\mathbf{C}(\mathbf{X}_1'\mathbf{X}_1^{-1}\mathbf{X}_1'\mathbf{X}_1] = R(\mathbf{M}_{\mathbf{X}_2}\mathbf{C}) > 0$$

because of the assumption $\mathcal{M}(\mathbf{C}) \not\subset \mathcal{M}(\mathbf{X}_2)$) and they are linearly independent. It is a consequence of the fact that the column space of the matrix $\mathbf{M}^*_{\mathbf{X}_1}$ is orthogonal to the column space of the matrix $\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{C}'\varkappa^2\mathbf{C}(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'$, thus they are linearly independent. This implies the linear independence of the matrices $\mathbf{M}\mathbf{V}_1\mathbf{M}$ and $\mathbf{M}\mathbf{V}_2\mathbf{M}$. With respect to Lemma 1.1 the matrix $\mathbf{K}^{(I)}$ is regular.

The proof of the assertion 2 see in [6].

Theorem 2.3. *In the regular two-stage regression model from Theorem 2.2 it is valid that*

1. *If* $n_1 > k_1$, *then the* UMVQUIE (*with respect to* σ_1^2) *of the variance component* σ_1^2 *based on the vector* $\boldsymbol{Y}_1$ *is*

$$\hat{\sigma}_1^2(\boldsymbol{Y}_1) = \boldsymbol{v}_1'\mathbf{H}_1^{-1}\boldsymbol{v}_1/(n_1 - k_1),\ \boldsymbol{v}_1 = \boldsymbol{Y}_1 - \mathbf{X}_1\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1).$$

2. *If* $\mathcal{M}(\mathbf{C}) \not\subset \mathcal{M}(\mathbf{X}_2)\ \&\ n_2 > k_2$, *then*

$$\hat{\sigma}_1^2(\boldsymbol{Y}_1, \boldsymbol{Y}_2|\sigma_{01}^2, \sigma_{02}^2) = \frac{1}{\Delta}\{[(n_1 - k_1)\hat{\sigma}_1^2(\boldsymbol{Y}_1) + \sigma_{01}^2\boldsymbol{v}_2'\mathbf{K}_2^{-1}\boldsymbol{v}_2][n_2 - k_2 - 2\mathrm{Tr}(\mathbf{R}) +$$

$$+ \mathrm{Tr}(\mathbf{R}^2)] - \sigma_{01}^2\sigma_{02}^2\boldsymbol{v}_2'\mathbf{K}_2^{-1}\mathbf{H}_2\mathbf{K}_2^{-1}\boldsymbol{v}_2[n_2 - k_2 - \mathrm{Tr}(\mathbf{R})]\}$$

is the $(\sigma_{01}^2, \sigma_{02}^2)$-LMVQUIE *of the variance component* σ_1^2 *and*

$$\hat{\sigma}_2^2(\boldsymbol{Y}_1, \boldsymbol{Y}_2|\sigma_{01}^2, \sigma_{02}^2) = \frac{1}{\Delta}\{\boldsymbol{v}_2'\mathbf{K}_2^{-1}\mathbf{H}_2\mathbf{K}_2^{-1}\boldsymbol{v}_2\sigma_{02}^2[n_1 - k_1 + \mathrm{Tr}(\mathbf{R})] -$$

$$- \sigma_{02}^2[\sigma_{01}^{-2}(n_1 - k_1)\hat{\sigma}_1^2(\boldsymbol{Y}_1) + \boldsymbol{v}_2'\mathbf{K}_2^{-1}\boldsymbol{v}_2][\mathrm{Tr}(\mathbf{R}) - \mathrm{Tr}(\mathbf{R}^2)]\}$$

is the $(\sigma_{01}^2, \sigma_{02}^2)$-LMVQUIE *of the variance component* σ_2^2. *Here*

$$\Delta = \det\begin{bmatrix} n_1 - k_1 + \mathrm{Tr}(\mathbf{R}^2), & \mathrm{Tr}(\mathbf{R}) - \mathrm{Tr}(\mathbf{R}^2) \\ \mathrm{Tr}(\mathbf{R}) - \mathrm{Tr}(\mathbf{R}^2), & n_2 - k_2 - 2\mathrm{Tr}(\mathbf{R}) + \mathrm{Tr}(\mathbf{R}^2) \end{bmatrix}$$

$$\mathbf{K}_2 = \sigma_{01}^2\mathbf{C}(\mathbf{X}_1'\mathbf{H}_1^{-1}\mathbf{X}_1)^{-1}\mathbf{C}' + \sigma_{02}^2\mathbf{H}_2,$$

$$\mathbf{R} = \mathbf{C}(\mathbf{X}_1'\mathbf{H}_1^{-1}\mathbf{X}_1)^{-1}\mathbf{C}'\sigma_{01}^2[\mathbf{K}_2^{-1} - \mathbf{K}_2^{-1}\mathbf{X}_2(\mathbf{X}_2'\mathbf{K}_2^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{K}_2^{-1}].$$

3. *If* $\mathcal{M}(\mathbf{C}) \subset \mathcal{M}(\mathbf{X}_2)\ \&\ n_1 > k_1\ \&\ n_2 > k_2$, *then the* UMVQUIE *of* σ_1^2 *and* σ_2^2 *are*

$$\hat{\sigma}_1^2(\boldsymbol{Y}_1, \boldsymbol{Y}_2) = \hat{\sigma}_1^2(\boldsymbol{Y}_1) = \{1/\mathrm{Tr}[(\mathbf{M}_1\mathbf{H}_1\mathbf{M}_1)^+\mathbf{H}_1(\mathbf{M}_1\mathbf{H}_1\mathbf{M}_1)^+]\}\, \boldsymbol{Y}_1'(\mathbf{M}_1\mathbf{H}_1\mathbf{M}_1)^+\mathbf{H}_1 \cdot \cdot (\mathbf{M}_1\mathbf{H}_1\mathbf{M}_1)^+\boldsymbol{Y}_1,$$

$$\hat{\sigma}_2^2(\boldsymbol{Y}_1, \boldsymbol{Y}_2) = \hat{\sigma}_2^2(\boldsymbol{Y}_2) = \{1/\mathrm{Tr}[(\mathbf{M}_2\mathbf{H}_2\mathbf{M}_2)^+\mathbf{H}_2(\mathbf{M}_2\mathbf{H}_2\mathbf{M}_2)^+]\}\, \boldsymbol{Y}_2'(\mathbf{M}_2\mathbf{H}_2\mathbf{M}_2)^+\mathbf{H}_2 \cdot \cdot (\mathbf{M}_2\mathbf{H}_2\mathbf{M}_2)^+\boldsymbol{Y}_2,$$

where $\mathbf{M}_i = \mathbf{I} - \mathbf{X}_i\mathbf{X}_i^+$, $i = 1, 2$.

Proof. 1. The statement is a well-known fact (see, e.g., [2, Section 5.4]).

2. Let us denote $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{C}, & \mathbf{X}_2 \end{bmatrix}$, $\Sigma_0 = \begin{bmatrix} \sigma_{01}^2\mathbf{H}_1, & \mathbf{0} \\ \mathbf{0}, & \sigma_{02}^2\mathbf{H}_2 \end{bmatrix} = \sigma_{01}^2\mathbf{V}_1 + \sigma_{02}^2\mathbf{V}_2$, $\mathbf{M} = \mathbf{I} - \mathbf{X}\mathbf{X}^+$. Then by Lemma 1.3

$$(\mathbf{M}\Sigma_0\mathbf{M})^+ = \Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{X}(\mathbf{X}'\Sigma_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma_0^{-1} = \begin{bmatrix} ①, & ② \\ ②', & ③ \end{bmatrix}$$

where ①, ②, ③ are obtained analogously as $\mathbf{P}_{11}$, $\mathbf{P}_{12}$, $\mathbf{P}_{22}$ in the proof of Theorem 2.2

$$① = \mathbf{C}_{1,\sigma_{01}^2\mathbf{H}_1,\mathbf{X}_1} + \mathbf{H}_1^{-1}\mathbf{X}_1(\mathbf{X}_1'\mathbf{H}_1^{-1}\mathbf{X}_1)^{-1}\mathbf{C}'\mathbf{C}_{1,\sigma_{01}^2\mathbf{H}_1,\mathbf{X}_1}\mathbf{C}(\mathbf{X}_1'\mathbf{H}_1^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{H}_1^{-1},$$

$$\mathbf{C}_{1,\sigma_{01}^2\mathbf{H}_1,\mathbf{X}_1} = \sigma_{01}^{-2}(\mathbf{H}_1^{-1} - \mathbf{H}_1^{-1}\mathbf{X}_1(\mathbf{X}_1'\mathbf{H}_1^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{H}_1^{-1})(= \sigma_{01}^{-2}\mathbf{C}_{1,\mathbf{H}_1,\mathbf{X}_1}),$$

$$\mathbf{C}_{1,\mathbf{K}_2,\mathbf{X}_2} = \mathbf{K}_2^{-1} - \mathbf{K}_2^{-1}\mathbf{X}_2(\mathbf{X}_2'\mathbf{K}_2^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{K}_2^{-1},$$

$$\mathbf{K}_2 = \sigma_{01}^2\mathbf{C}(\mathbf{X}_1'\mathbf{H}_1^{-1}\mathbf{X}_1)^{-1}\mathbf{C}' + \sigma_{02}^2\mathbf{H}_2,$$

$$② = -\mathbf{H}_1^{-1}\mathbf{X}_1(\mathbf{X}_1'\mathbf{H}_1^{-1}\mathbf{X}_1)^{-1}\mathbf{C}'\mathbf{C}_{1,\mathbf{K}_2,\mathbf{X}_2},$$

$$③ = \mathbf{C}_{1,\mathbf{K}_2,\mathbf{X}_2}.$$

Furthermore

$$\left.\begin{aligned} (\mathbf{M}\Sigma_0\mathbf{M})^+\mathbf{V}_1(\mathbf{M}\Sigma_0\mathbf{M})^+ &= \begin{bmatrix} ①\mathbf{H}_1①, & ①\mathbf{H}_1② \\ ②'\mathbf{H}_1①, & ②'\mathbf{H}_1② \end{bmatrix} \\ (\mathbf{M}\Sigma_0\mathbf{M})^+\mathbf{V}_2(\mathbf{M}\Sigma_0\mathbf{M})^+ &= \begin{bmatrix} ②\mathbf{H}_2②', & ②\mathbf{H}_2③ \\ ③\mathbf{H}_2②', & ③\mathbf{H}_2③ \end{bmatrix} \end{aligned}\right\} \Rightarrow$$

$$\hat{\gamma}_1 = \boldsymbol{Y}'(\mathbf{M}\Sigma_0\mathbf{M})^+\mathbf{V}_1(\mathbf{M}\Sigma_0\mathbf{M})^+\boldsymbol{Y} = \sigma_{01}{}^4(n_1 - k_1)\,\hat{\sigma}_1^2(\boldsymbol{Y}_1) + [\boldsymbol{Y}_2 - \mathbf{C}\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1)]' \cdot \cdot \mathbf{C}_{1,\mathbf{K}_2,\mathbf{X}_2}\mathbf{C}(\mathbf{X}_1'\mathbf{H}_1^{-1}\mathbf{X}_1)^{-1}\mathbf{C}'\mathbf{C}_{1,\mathbf{K}_2,\mathbf{X}_2}[\boldsymbol{Y}_2 - \mathbf{C}\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1)],$$

$$\hat{\gamma}_2 = \boldsymbol{Y}'(\mathbf{M}\Sigma_0\mathbf{M})^+\mathbf{V}_2(\mathbf{M}\Sigma_0\mathbf{M})^+\boldsymbol{Y} = [\boldsymbol{Y}_2 - \mathbf{C}\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1)]'\mathbf{C}_{1,\mathbf{K}_2,\mathbf{X}_2}\mathbf{H}_2\mathbf{C}_{1,\mathbf{K}_2,\mathbf{X}_2} \cdot \cdot [\boldsymbol{Y}_2 - \mathbf{C}\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1)],$$

$$\mathrm{Tr}[(\mathbf{M}\boldsymbol{\Sigma}_0\mathbf{M})^+\mathbf{V}_1(\mathbf{M}\boldsymbol{\Sigma}_0\mathbf{M})^+\mathbf{V}_1] = \mathrm{Tr}[①\mathbf{H}_1①\mathbf{H}_1] = \sigma_{01}^{-4}(n_1 - k_1) + \sigma_{01}^{-4}\mathrm{Tr}(\mathbf{R}^2),$$

$$\mathrm{Tr}[(\mathbf{M}\boldsymbol{\Sigma}_0\mathbf{M})^+\mathbf{V}_1(\mathbf{M}\boldsymbol{\Sigma}_0\mathbf{M})^+\mathbf{V}_2] = \mathrm{Tr}[②'\mathbf{H}_1②\mathbf{H}_2] = \sigma_{01}^{-2}\sigma_{02}^{-2}[\mathrm{Tr}(\mathbf{R}) - \mathrm{Tr}(\mathbf{R}^2)],$$

$$\mathrm{Tr}[(\mathbf{M}\boldsymbol{\Sigma}_0\mathbf{M})^+\mathbf{V}_2(\mathbf{M}\boldsymbol{\Sigma}_0\mathbf{M})^+\mathbf{V}_2] = \mathrm{Tr}[③\mathbf{H}_2③\mathbf{H}_2] = $$
$$= \sigma_{02}^{-4}[n_2 - k_2 - 2\mathrm{Tr}(\mathbf{R}) + \mathrm{Tr}(\mathbf{R}^2)],$$

$$\mathbf{S}_{(\mathbf{M}\boldsymbol{\Sigma}_0\mathbf{M})^+} = \begin{bmatrix} \sigma_{01}^{-4}[n_1 - k_1 + \mathrm{Tr}(\mathbf{R}^2)], & \sigma_{01}^{-2}\sigma_{02}^{-2}[\mathrm{Tr}(\mathbf{R}) - \mathrm{Tr}(\mathbf{R}^2)] \\ \sigma_{01}^{-2}\sigma_{02}^{-2}[\mathrm{Tr}(\mathbf{R}) - \mathrm{Tr}(\mathbf{R}^2)], & \sigma_{02}^{-4}[n_2 - k_2 - 2\mathrm{Tr}(\mathbf{R}) + \mathrm{Tr}(\mathbf{R}^2)] \end{bmatrix}.$$

With respect to Lemma 1.2 the $(\sigma_{01}^2, \sigma_{02}^2)$-LMVQUIE of the vector $(\sigma_1^2, \sigma_2^2)'$ is

$$\begin{bmatrix} \hat{\sigma}_1^2(\boldsymbol{Y}_1, \boldsymbol{Y}_2) \\ \hat{\sigma}_2^2(\boldsymbol{Y}_1, \boldsymbol{Y}_2) \end{bmatrix} = \mathbf{S}^{-1}_{(\mathbf{M}\boldsymbol{\Sigma}_0\mathbf{M})^+} \begin{bmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{bmatrix}.$$

After substituting and rearranging this we obtain the assertion 2.

For statement 3 see [6].

Remark 2.2 Let the ratio $\varrho = \sigma_1^2/\sigma_2^2$ be known. Then the UMVQUIEs $\hat{\sigma}_1^2(\boldsymbol{Y}_1, \boldsymbol{Y}_2)\ (= \varrho\hat{\sigma}_2^2(\boldsymbol{Y}_1, \boldsymbol{Y}_2))$ and $\hat{\sigma}_2^2(\boldsymbol{Y}_1, \boldsymbol{Y}_2)$ can be easily derived. The expression for $\sigma_2^2(\boldsymbol{Y}_1, \boldsymbol{Y}_2)$ is an analogy of the expression for $\hat{\sigma}_1^2(\boldsymbol{Y}_1)$ from 1 of Theorem 2.3.

Remark 2.3. If $n_1 = k_1 \,\&\, \mathcal{M}(\mathbf{C}) \not\subset \mathcal{M}(\mathbf{X}_2) \,\&\, n_2 > k_2$, then the relations for $\hat{\sigma}_1^2(\boldsymbol{Y}_1, \boldsymbol{Y}_2 | \sigma_{01}^2, \sigma_{02}^2)$ and $\hat{\sigma}_2^2(\boldsymbol{Y}_1, \boldsymbol{Y}_2 | \sigma_{01}^2, \sigma_{02}^2)$ in Theorem 2.3 do not contain the expression $\hat{\sigma}_1^2(\boldsymbol{Y}_1)$ which is impossible to be determined. Nevertheless, the estimators $\hat{\sigma}_1^2(\boldsymbol{Y}_1, \boldsymbol{Y}_2 | \sigma_{01}^2, \sigma_{02}^2)$ and $\sigma_2^2(\boldsymbol{Y}_1, \boldsymbol{Y}_2 | \sigma_{01}^2, \sigma_{02}^2)$ exist.

Remark 2.4. The matrices $\mathbf{K}^{(I)}$ from Lemma 1.1 and $\mathbf{S}_{(\mathbf{M}\boldsymbol{\Sigma}_0\mathbf{M})^+}$ from Lemma 1.2 are simultaneously either regular or singular. The regularity of the matrix $\mathbf{K}^{(I)}$ was proved in Theorem 2.2. The regularity of the matrix $\mathbf{S}_{(\mathbf{M}\boldsymbol{\Sigma}_0\mathbf{M})^+}$ (from Theorem 2.3) can be directly proved if $n_1 > k_1$.

Denote $\mathbf{S}_1 = \sigma_{01}^2\mathbf{C}(\mathbf{X}_1'\mathbf{H}_1^{-1}\mathbf{X}_1)^{-1}\mathbf{C}'(\neq \mathbf{0})$, $\mathbf{S}_2 = \sigma_{02}^2\mathbf{H}_2$ ($\mathbf{S}_2$ is regular), $\mathbf{K}_2 = \mathbf{S}_1 + \mathbf{S}_2$ ($\mathbf{K}_2$ is regular). Express the matrix $\mathbf{C}_{1,\mathbf{K}_2,\mathbf{X}_2}$ in its factorized form $\mathbf{C}_{1,\mathbf{K}_2,\mathbf{X}_2} = \mathbf{J}\mathbf{J}'$, where $\mathbf{J}$ is of the type $n_2 \times R(\mathbf{C}_{1,\mathbf{K}_2,\mathbf{X}_2})$ and $R(\mathbf{C}_{1,\mathbf{K}_2,\mathbf{X}_2}) = n_2 - k_2$. Then in the Hilbert space $\mathscr{S}_{n_2 - k_2}$ of symmetric $(n_2 - k_2) \times (n_2 - k_2)$ matrices with the inner product $\langle \mathbf{A}, \mathbf{B} \rangle = \mathrm{Tr}(\mathbf{AB})$, $\mathbf{A}, \mathbf{B} \in \mathscr{S}_{n_2 - k_2}$, the Gram matrix $\mathbf{G}$ of the couple $\mathbf{J}'\mathbf{S}_1\mathbf{J}$ and $\mathbf{J}'\mathbf{S}_2\mathbf{J}$ is

$$\mathbf{G} = \begin{bmatrix} \langle \mathbf{J}'\mathbf{S}_1\mathbf{J}, \mathbf{J}'\mathbf{S}_1\mathbf{J} \rangle, & \langle \mathbf{J}'\mathbf{S}_1\mathbf{J}, \mathbf{J}'\mathbf{S}_2\mathbf{J} \rangle \\ \langle \mathbf{J}'\mathbf{S}_2\mathbf{J}, \mathbf{J}'\mathbf{S}_1\mathbf{J} \rangle, & \langle \mathbf{J}'\mathbf{S}_2\mathbf{J}, \mathbf{J}'\mathbf{S}_2\mathbf{J} \rangle \end{bmatrix}.$$

It can be easily proved that

$$\mathbf{G} + \begin{bmatrix} n_1 - k_1, & 0 \\ 0, & 0 \end{bmatrix} = \mathbf{S}_{(\mathbf{M}\boldsymbol{\Sigma}_0\mathbf{M})^+}.$$

The regularity of the matrix $\mathbf{J}'\mathbf{S}_2\mathbf{J}$ implies $\langle \mathbf{J}'\mathbf{S}_2\mathbf{J}, \mathbf{J}'\mathbf{S}_2\mathbf{J}\rangle = \sigma_{02}^{-4}[n_2 - k_2 - - 2\mathrm{Tr}(\mathbf{R}) + \mathrm{Tr}(\mathbf{R}^2)] > 0$. The matrix $\mathbf{G}$ is always positive semidefinite and $n_1 - k_1 > 0$, thus the matrix $\mathbf{S}_{(\mathbf{M}\Sigma_0\mathbf{M})^+}$ is regular.

Remark 2.5. Theorems 2.1, 2.2 and 2.3 enable us to determine the sequence

$$\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1),\ \hat{\sigma}_1^2(\mathbf{Y}_1),\ \sigma_2^2[\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1),\ \hat{\sigma}_1^2(\boldsymbol{Y}_1),\ \boldsymbol{Y}_2] =$$
$$= \hat{\sigma}_2^2[\boldsymbol{Y}_1, \boldsymbol{Y}_2|\hat{\sigma}_1^2(\boldsymbol{Y}_1), \sigma_{02}^2],\ \hat{\boldsymbol{\beta}}_2\{\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1),\ \boldsymbol{Y}_2,\ \hat{\sigma}_1^2(\boldsymbol{Y}_1),\ \hat{\sigma}_2^2[\boldsymbol{Y}_1, \boldsymbol{Y}_2|\hat{\sigma}_1^2(\boldsymbol{Y}_1), \sigma_{02}^2]\}, \ldots.$$

Instead of the estimator $\hat{\sigma}_2^2[\boldsymbol{Y}_1, \boldsymbol{Y}_2|\hat{\sigma}_1^2(\boldsymbol{Y}_1), \sigma_{02}^2]$ the estimator obtained iteratively can be used; for the value σ_{02}^2 we substitute $\hat{\sigma}_2^2[\boldsymbol{Y}_1, \boldsymbol{Y}_2|\hat{\sigma}_1^2(\boldsymbol{Y}_1), \sigma_{02}^2]$ and repeat this procedure several times.

Remark 2.6 The mean values of the following quadratic forms of the vectors $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$, frequently occurring in practice, are interesting (the notation from Theorem 2.3 is used):

(a) $E(\boldsymbol{v}_2'\mathbf{K}_2^{-1}\boldsymbol{v}_2|\sigma_1^2, \sigma_2^2) = \sigma_1^2\,\mathrm{Tr}(\mathbf{R})/\sigma_{01}^2 + \sigma_2^2[n_2 - k_2 - \mathrm{Tr}(\mathbf{R})]/\sigma_{02}^2$,

(b) $E(\boldsymbol{v}_2'\mathbf{K}_2^{-1}\mathbf{H}_2\mathbf{K}_2^{-1}\boldsymbol{v}_2|\sigma_1^2, \sigma_2^2) = \sigma_1^2[\mathrm{Tr}(\mathbf{R}) - \mathrm{Tr}(\mathbf{R}^2)]/(\sigma_{01}^2\sigma_{02}^2) +$

$+ \sigma_2^2[n_2 - k_2 - 2\mathrm{Tr}(\mathbf{R}) + \mathrm{Tr}(\mathbf{R}^2)]/\sigma_{02}^4$.

Denote $\tilde{\boldsymbol{\beta}}_2 = (\mathbf{X}_2'\mathbf{H}_2^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{H}_2^{-1}[\boldsymbol{Y}_2 - \mathbf{C}\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1)]$, $\tilde{\boldsymbol{v}}_2 = \boldsymbol{Y}_2 - \mathbf{C}\hat{\boldsymbol{\beta}}_1(\boldsymbol{Y}_1) - \mathbf{X}_2\tilde{\boldsymbol{\beta}}_2$. Then

(c) $E(\tilde{\boldsymbol{v}}_2'\mathbf{H}_2^{-1}\tilde{\boldsymbol{v}}_2|\sigma_1^2, \sigma_2^2) = \sigma_1^2\mathrm{Tr}[\mathbf{C}_{1,\mathbf{H}_2,\mathbf{X}_2}\mathbf{C}(\mathbf{X}_1'\mathbf{H}_1^{-1}\mathbf{X}_1)^{-1}\mathbf{C}'] + \sigma_2^2(n_2 - k_2)$.

This shows that none of the forms (a), (b) and (c) can be used alone for the estimation of the variance component σ_2^2. An exception is the case $\mathcal{M}(\mathbf{C}) \subset \subset \mathcal{M}(\mathbf{X}_2) \Rightarrow \mathbf{R} = \mathbf{0}\ \&\ \mathbf{C}_{1,\mathbf{H}_2,\mathbf{X}_2}\mathbf{C}(\mathbf{X}_1'\mathbf{H}_1^{-1}\mathbf{X}_1)^{-1}\mathbf{C}' = \mathbf{0}$ (in detail see [6] and [7]).

Remark 2.7. Estimates of σ_1^2 and σ_2^2 from theorem 2.3, cases 1. and 3, are always positive. This is not true in the case 2. The probability of obtaining the negative estimates in this case decreases with increasing n_1 and n_2. As an evaluation of the exact value of this probability in an actual case is difficult, a simulation study was made. It was found that $n_i - k_i > 20$, $i = 1, 2$, was sufficient for obtaining an acceptable small value of this probability.

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ДВУХЭТАПНАЯ РЕГРЕССИОННАЯ МОДЕЛЬ

Lubomír Kubáček

Резюме

Регрессионная модель $\boldsymbol{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \Sigma)$ называется регулярной двухэтапной, если $\boldsymbol{Y} = (\boldsymbol{Y}_1', \boldsymbol{Y}_2')'$, $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{C}, & \mathbf{X}_2 \end{bmatrix}$, $\boldsymbol{\beta} = (\boldsymbol{\beta}_1', \boldsymbol{\beta}_2')'$, $\Sigma = \sigma_1^2 \begin{bmatrix} \mathbf{H}_1, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0} \end{bmatrix} + \sigma_2^2 \begin{bmatrix} \mathbf{0}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{H}_2 \end{bmatrix}$; здесь $\mathbf{X}_i$ — $n_i \times k_i$ матрица, имеющая полный ранг в столбцах, $i = 1, 2$, $\mathbf{C} \neq \mathbf{0}$, а $\mathbf{H}_i$ — $n_i \times n_i$ положительно определенная матрица, $i = 1, 2$. Существует только одна последовательность, допустимая для определения оценок неизвестных параметров $\boldsymbol{\beta}_i \in \mathscr{R}^{k_i}$ (пространство Евклида размерности k_i), $\sigma_i^2 \in (0, \infty)$, $i = 1, 2$; эта последовательность указана в статье. Получены локально (или равномерно) наилучшие оценки этих параметров и показаны условия их существования.