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ON FORCED FIRST ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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ABSTRACT. In this paper, sufficient conditions have been obtained for the oscillation of bounded (unbounded) solutions of a class of forced first order neutral differential equations with positive and negative coefficients. The techniques used here are different from those used to be employed for such equations earlier.

1.

In recent years, some authors (see [1] - [7]) have studied oscillatory behaviour of solutions of first order homogeneous neutral differential equations with positive and negative coefficients. These coefficients may be constants (see [3]) or functions of t (see [1], [4], [6], [7]). Some of the works with variable coefficients hold for constant coefficients ([1], [7]). In all these papers the problem has been reduced to the existence of a positive solution of certain first order delay-differential inequality. The assumptions are made conveniently so that the inequality does not admit a positive solution and hence a contradiction is obtained. It seems that no work has been done on oscillation of forced first order neutral differential equations with positive and negative coefficients. The present note is concerned with this problem for equations of the type

$$\left[x(t)+\sum_{i=1}^{l} c_i(t)x(t-\tau_i)\right]' + \sum_{j=1}^{m} p_j(t)x(t-\sigma_j) - \sum_{k=1}^{n} q_k(t)x(t-\alpha_k) = f(t)\,, \quad (1)$$

where $p_j, q_k \in C\big([t_0,\infty),[0,\infty)\big)$, $c_i \in C\big([t_0,\infty),\mathbb{R}\big)$, $t_0 \in \mathbb{R}$, $\tau_i \geq 0$, $\sigma_j \geq 0$ and $\alpha_k \geq 0$, $1 \leq i \leq l$, $1 \leq j \leq m$, $1 \leq k \leq n$. The method adopted in earlier

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papers for homogeneous equations does not work for (1). In this paper we have developed different techniques to study oscillation of (1). All the theorems in this paper hold for homogeneous equations. However, Theorems 5–7 hold for equations with constant coefficients.

In [3], Farrell et al considered first order homogeneous neutral differential equation with positive and negative coefficients of the form

$$\big[x(t) + cx(t-\tau)\big]' + px(t-\sigma) - qx(t-\alpha) = 0\,, \tag{2}$$

where p, q, c are constants. They proved that every solution of (2) is oscillatory or tends to zero as $t \to \infty$ if $-1 < c < 0$, $1+c-q(\sigma-\alpha) > 0$, $p > q > 0$, $\tau > 0$, $\sigma \geq 0$ and $\alpha \geq 0$. To some extent this result is comparable to Theorem 5 of this paper. We may note that we deal with several delays (see equation (1)). In the process we are able to show that every bounded solution of (1) is oscillatory or tends to zero as $t \to \infty$. The conditions in other theorems in [3] are not comparable to our conditions in Theorems 5–7.

Yu and Wang [6] obtained the following result for

$$\big[x(t) + c(t)x(t-\tau)\big]' + p(t)x(t-\sigma) - q(t)x(t-\alpha) = 0\,. \tag{3}$$

If $p, q, -c \in C\big([t_0,\infty),[0,\infty)\big)$, $\tau > 0$, $\sigma \geq 0$, $\alpha \geq 0$, $\sigma \geq \alpha$, $p(t) - q(t+\alpha-\sigma) \geq 0$ but $\neq 0$ and there hold

$$1 + c(t) - \int\limits_{t-(\sigma-\alpha)}^{t} q(s)\,\mathrm{d}s \geq 0 \qquad \text{for large } t\,,$$

and either

$$A > \frac{1}{\mathrm{e}}$$

or

$$A \leq \frac{1}{\mathrm{e}} \qquad \text{and} \qquad M > 1 - \frac{1}{2}\Big(1 - A - \sqrt{1 - 2A - A^2}\,\Big)$$

then every solution of (3) oscillates, where

$$A = \liminf_{t\to\infty} \int\limits_{t-\sigma}^{t} \big(p(s) - q(s+\alpha-\sigma)\big)\Bigg(1 - c(s-\sigma) + \int\limits_{s-\sigma+\alpha}^{s} q(u-\sigma)\,\mathrm{d}u\Bigg)\,\mathrm{d}s$$

and

$$M = \limsup_{t\to\infty} \int\limits_{t-\sigma}^{t} \big(p(s) - q(s+\alpha-\sigma)\big)\Bigg(1 - c(s-\sigma) + \int\limits_{s-\sigma+\alpha}^{s} q(u-\sigma)\,\mathrm{d}u\Bigg)\,\mathrm{d}s\,.$$

Although the coefficients in (3) are functions of t, the delays involved are single unlike in (1). These conditions are very complicated for verification through examples. On the other hand, our conditions in all the theorems are easy to verify. Section 2 deals with these results. By a solution x of (1) on $[t_x, \infty)$, $t_x \geq t_0$, we mean a real-valued continuous function x on $[t_x - T_0, \infty)$ such that $x(t) + \sum_{i=1}^{l} c_i(t)x(t-\tau_i)$ is once continuously differentiable for $t \geq t_x$ and (1) is satisfied identically for $t \geq t_x$, where $T_0 = \max\{\tau_i, \sigma_j, \alpha_k : 1 \leq i \leq l,\ 1 \leq j \leq m,\ 1 \leq k \leq n\}$. A solution of (1) is said to be *oscillatory* if and only if it has arbitrarily large zeros; otherwise, it is said to be *nonoscillatory*.

We assume that there exists $F \in C'\big([t_0, \infty), \mathbb{R}\big)$ such that $F'(t) = f(t)$.

2.

In this section we study oscillatory behaviour of solutions of (1) under certain conditions on coefficient functions. It may be noted that some of our results are not satisfactory for constant coefficients. However, all the results are true for homogeneous equations.

THEOREM 1. *Suppose that*

(A_1) $c_i \leq c_i(t) \leq 0$ *such that* $\sum_{i=1}^{l} c_i > -1$,

(A_2) *there exists a* $j^* \in \{1, \dots, m\}$ *such that* $\sigma_{j^*} > \max\{\alpha_k : 1 \leq k \leq n\}$, *and*

$$p_{j^*}(t) \geq \sum_{k=1}^{n} q_k(t - \sigma_{j^*} + \alpha_k) \qquad \text{for} \quad t \geq t_0 + T_0,$$

(A_3) $\int_{t_0}^{\infty} \Big(\sum_{k=1}^{n} q_k(t) \Big)\, \mathrm{d}t < \infty$,

(A_4) $F(t)$ *is bounded.*

Then every unbounded solution of (1) *oscillates.*

Proof. Let $x(t)$ be an unbounded nonoscillatory solution of (1) on $[t_x, \infty)$, $t_x \geq t_0$. Suppose that $x(t) > 0$ for $t \geq t_1 > t_x$. For $t \geq t_1 + T_0$, we set

$$z(t) = x(t) + \sum_{i=1}^{l} c_i(t)x(t - \tau_i) - \sum_{k=1}^{n} \int_{t-\sigma_{j^*}+\alpha_k}^{t} q_k(s)x(s - \alpha_k)\, \mathrm{d}s - F(t). \tag{4}$$

Thus, for $t \geq t_2 \geq t_1 + T_0$,

$$z'(t) \leq -\Big[p_{j^*}(t) - \sum_{k=1}^{n} q_k(t - \sigma_{j^*} + \alpha_k)\Big] x(t - \sigma_{j^*}) \leq 0. \tag{5}$$

Hence $\lim_{t\to\infty} z(t) = \mu$, $-\infty \le \mu < \infty$. Since $x(t)$ is unbounded, there exists a sequence $\langle t_w \rangle \subset [t_2, \infty)$ such that $\lim_{w\to\infty} t_w = \infty$, $\lim_{w\to\infty} x(t_w) = \infty$ and $x(t_w) = \max\{x(t):\ t_2 \le t \le t_w\}$. Thus (4) yields

$$z(t) \ge x(t) + \sum_{i=1}^{l} c_i x(t-\tau_i) - \sum_{k=1}^{n} \int_{t-\sigma_{j^*}}^{t-\alpha_k} q_k(s+\alpha_k) x(s)\, \mathrm{d}s - \beta ,$$

where we assume that $|F(t)| \le \beta$, $t \in [t_0, \infty)$, that is,

$$\begin{aligned} z(t) \ge x(t) &+ \sum_{i=1}^{l} c_i x(t-\tau_i) - \beta \\ &- \left(\sum_{k=1}^{n} \int_{t-\sigma_{j^*}}^{t-\alpha_k} q_k(s+\alpha_k)\, \mathrm{d}s \right) \max\{x(s):\ t-\sigma_{j^*} \le s \le t\} . \end{aligned}$$

From (A_3) it follows that

$$\lim_{t\to\infty} \sum_{k=1}^{n} \int_{t-\sigma_{j^*}}^{t-\alpha_k} q_k(s+\alpha_k)\, \mathrm{d}s = 0 .$$

Hence, for $0 < \varepsilon < 1 + \sum_{i=1}^{l} c_i$, we may find $t_3 > t_2$ such that

$$\sum_{k=1}^{n} \int_{t-\sigma_{j^*}}^{t-\alpha_k} q_k(s+\alpha_k)\, \mathrm{d}s < \varepsilon$$

for $t \ge t_3$. Choosing w sufficiently large such that $t_w > t_3 + T_0$, we obtain

$$\begin{aligned} z(t_w) &\ge x(t_w) + \sum_{i=1}^{l} c_i x(t_w-\tau_i) - \beta - \varepsilon \max\{x(s):\ t_w-\sigma_{j^*} \le s \le t_w\} \\ &\ge \left[1 + \sum_{i=1}^{l} c_i - \varepsilon \right] x(t_w) - \beta . \end{aligned}$$

Thus $\lim_{w\to\infty} z(t_w) = +\infty$ a contradiction.

If $x(t) < 0$ for $t \ge t_1 > t_x$, then we put $y(t) = -x(t)$ to obtain $y(t) > 0$ for $t \ge t_1$ and

$$\left[y(t) + \sum_{i=1}^{l} c_i(t) y(t-\tau_i) \right]' + \sum_{j=1}^{m} p_j(t) y(t-\sigma_j) - \sum_{k=1}^{n} q_k(t) y(t-\alpha_k) = -f(t) .$$

Setting $G(t) = -F(t)$ and proceeding as above we get a contradiction. Hence the theorem is proved.

EXAMPLE 1. Consider

$$\left[x(t) - \frac{1}{2t}x(t-\pi)\right]' + \left(1+\frac{1}{t}\right)x\left(t-\frac{\pi}{2}\right) + \frac{1}{4\pi t}x(t-2\pi) - \frac{1}{2t^2}x(t-\pi) = \left(1+\frac{1}{4\pi}\right)\sin t + \frac{\pi-1}{2}\cos t \tag{6}$$

for $t \geq 4\pi$. Here

$$F(t) = -\frac{(1+4\pi)}{4\pi}\cos t + \frac{\pi-1}{2}\sin t\,.$$

As all the conditions of Theorem 1 are satisfied, then it follows that all unbounded solutions of (6) are oscillatory. In particular, $x(t) = t\sin t$ is an unbounded oscillatory solution of (6).

It is interesting to note that equation (1) admits an unbounded nonoscillatory solution if $F(t)$ is unbounded notwithstanding the conditions (A_1) (A_3).

EXAMPLE 2. The equation

$$\left[x(t) - \frac{1}{2}x(t-2\pi)\right]' + tx(t-\pi) + tx(t-2\pi) - \frac{1}{t^2}x(t) - \frac{1}{t^2}x(t-\pi) = \left(\pi+\frac{t}{2}\right)\cos t + \left(\frac{1}{2}-\pi t\right)\sin t - \frac{\pi}{t^2}\sin t + 1 + 4t^2 + \frac{2\pi}{t^2} - 6\pi t - \frac{4}{t}$$

for $t \geq 3\pi$, admits an unbounded nonoscillatory solution $x(t) = t(\sin t + 2)$. Clearly, the assumptions (A_1)–(A_3) hold and

$$F(t) = \frac{t}{2}\sin t + \pi t\cos t + t + \frac{4t^3}{3} - 3\pi t^2 - 4\log t - \frac{2\pi}{t} - \pi\int_{3\pi}^{t}\frac{\sin\theta}{\theta^2}\,d\theta$$

is unbounded.

The following example demonstrates that the assumptions (A_1) (A_4) are not enough for the oscillation of all bounded solutions of (1).

EXAMPLE 3. Consider

$$\left[x(t) - \frac{1}{2}x(t-2\pi)\right]' + \frac{1}{t^2}x(t-2\pi) - \frac{1}{t^3}x(t-\pi) = \frac{2(t-1)}{t^3}$$

for $t \geq 3\pi$. In this case

$$F(t) = \frac{1-2t}{t^2}\,.$$

Although all the conditions of Theorem 1 are satisfied, the above equation admits a bounded nonoscillatory solution $x(t) = 2$.

THEOREM 2. *Suppose that the assumptions* (A_1)–(A_3) *hold and*

(A_4') $\lim\limits_{t\to\infty} F(t) = \beta, \ -\infty < \beta < \infty,$

(A_5) $\int\limits_{t_0}^{\infty} p_{j^*}(t)\,\mathrm{d}t = +\infty$

are satisfied. Then every bounded solution of (1) *oscillates or tends to zero as* $t \to \infty$.

P r o o f . Let $x(t)$ be a bounded nonoscillatory solution of (1) on $[t_x, \infty)$, $t_x \ge t_0$. Let $x(t) > 0$ for $t \ge t_1 \ge t_x$. The case $x(t) < 0$ for $t \ge t_1$ may similarly be dealt with. For $t \ge t_1 + T_0$, we set $z(t)$ as in (4) to obtain (5). Since $x(t)$ is bounded, then $z(t)$ is bounded and hence $\lim\limits_{t\to\infty} z(t) = \mu$, where $-\infty < \mu < \infty$. We consider three cases, viz,

(i) $\mu + \beta < 0$,
(ii) $\mu + \beta = 0$,
(iii) $\mu + \beta > 0$.

Since $x(t)$ is bounded, from (A_3) it follows that

$$\lim_{t\to\infty} \sum_{k=1}^{n} \int\limits_{t-\sigma_{j^*}+\alpha_k}^{t} q_k(s) x(s-\alpha_k)\,\mathrm{d}s = 0\,. \tag{7}$$

Thus $\mu + \beta < 0$ implies that

$$\begin{aligned} 0 &> \lim_{t\to\infty} \big[z(t) + F(t)\big] \\ &\ge \varlimsup_{t\to\infty} \Bigg[x(t) + \sum_{i=1}^{l} c_i x(t-\tau_i) - \sum_{k=1}^{n} \int\limits_{t-\sigma_{j^*}+\alpha_k}^{t} q_k(s) x(s-\alpha_k)\,\mathrm{d}s\Bigg] \\ &\ge \varlimsup_{t\to\infty} \Bigg[x(t) + \sum_{i=1}^{l} c_i x(t-\tau_i)\Bigg] \\ &\ge \varlimsup_{t\to\infty} x(t) + \varliminf_{t\to\infty} \sum_{i=1}^{l} c_i x(t-\tau_i) \\ &> \Bigg(1 + \sum_{i=1}^{l} c_i\Bigg) \varlimsup_{t\to\infty} x(t) \ge 0\,, \end{aligned}$$

a contradiction. From (5) we obtain, for $t \ge t_2 \ge t_1 + T_0$,

$$\int\limits_{t_2}^{\infty} \Bigg[p_{j^*}(t) - \sum_{k=1}^{n} q_k(t-\sigma_{j^*}+\alpha_k)\Bigg] x(t-\sigma_{j^*})\,\mathrm{d}t < \infty\,.$$

But (A_3) and (A_5) imply that

$$\int_{t_2}^{\infty}\left[p_{j^*}(t)-\sum_{k=1}^{n} q_k(t-\sigma_{j^*}+\alpha_k)\right] dt=\infty .$$

Hence $\varliminf_{t\to\infty} x(t)=0$. On the other hand, $\mu+\beta>0$ and (4) yield

$$0<\mu+\beta=\varliminf_{t\to\infty}\left[z(t)+F(t)\right]\le \varliminf_{t\to\infty} x(t),$$

a contradiction. Thus $\mu+\beta=0$. Consequently,

$$\begin{aligned}0&=\lim_{t\to\infty}\left[z(t)+F(t)\right]\\ &\ge \varlimsup_{t\to\infty}\left[x(t)+\sum_{i=1}^{l} c_i x(t-\tau_i)\right]\ge\left(1+\sum_{i=1}^{l} c_i\right)\varlimsup_{t\to\infty} x(t).\end{aligned}$$

Hence $\lim_{t\to\infty} x(t)=0$. This completes the proof of the theorem. □

EXAMPLE 4. From Theorem 2 it follows that all bounded solutions of

$$\begin{aligned}\left[x(t)-\tfrac{1}{2}x(t-1)\right]'+\frac{1}{t-1}x(t-1)-\frac{4}{(2t-1)^2}x\left(t-\frac{1}{2}\right)\\ =\frac{2}{(t-1)^3}-\frac{2}{t^3}-\frac{16}{(2t-1)^4}, \qquad t\ge 2,\end{aligned}$$

oscillate or tend to zero as $t\to\infty$. In particular, $x(t)=\frac{1}{t^2}$ is such a solution of the equation. It may be noted that (A_4') fails for Example 1 and (A_5) fails for Example 3.

COROLLARY. *Suppose that the conditions* (A_1)–(A_3), (A_4') *and* (A_5) *hold. Then every solution of* (1) *oscillates or tends to zero as* $t\to\infty$.

The proof follows from Theorems 1 and 2.

THEOREM 3. *Let* (A_2), (A_3) *and* (A_5) *hold. Suppose that*

(A_4'') $\lim_{t\to\infty} F(t)=0$,

(A_6) $0\le c_i(t)\le c_i$ *such that* $\sum_{i=1}^{l} c_i<1$.

Then every bounded solution of (1) *oscillates or tends to zero as* $t\to\infty$.

Proof. Let $x(t)$ be a bounded nonoscillatory solution of (1) on $[t_x,\infty)$, $t_x\ge t_0$, such that $x(t)>0$ for $t\ge t_1\ge t_x$. Setting $z(t)$ as in (4) for $t\ge t_1+T_0$, we obtain (5). Boundedness of $x(t)$ implies that $z(t)$ is bounded and hence

$-\infty < \mu < \infty$, where $\lim_{t\to\infty} z(t) = \mu$. Further, (A_3) and boundedness of $x(t)$ yield (7). Consequently, from (4) we get

$$\mu = \lim_{t\to\infty}\left[x(t) + \sum_{i=1}^{l} c_i(t)x(t-\tau_i)\right] \geq 0\,.$$

Since $z(t)$ is monotonic decreasing and $\mu \geq 0$, then $z(t) > 0$ for large t. Proceeding as in Theorem 2 one may obtain $\underline{\lim}_{t\to\infty} x(t) = 0$. Since

$$\begin{aligned}
&z(t) - \sum_{i=1}^{l} c_i(t)z(t-\tau_i)\\
&\leq x(t) + \sum_{i=1}^{l} c_i(t)\sum_{k=1}^{n}\int_{t-\sigma_{j^*}+\alpha_k-\tau_i}^{t-\tau_i} q_k(s)x(s-\alpha_k)\,\mathrm{d}s + \sum_{i=1}^{l} c_i(t)F(t-\tau_i) - F(t)
\end{aligned}$$

and

$$\underline{\lim}_{t\to\infty}\left[z(t) - \sum_{i=1}^{l} c_i(t)z(t-\tau_i)\right] \geq \underline{\lim}_{t\to\infty}\left[z(t) - \sum_{i=1}^{l} c_i z(t-\tau_i)\right] = \left(1 - \sum_{i=1}^{l} c_i\right)\mu\,,$$

then

$$\begin{aligned}
&\left(1 - \sum_{i=1}^{l} c_i\right)\mu\\
&\leq \underline{\lim}_{t\to\infty}\left[x(t) + \sum_{i=1}^{l} c_i\sum_{k=1}^{n}\int_{t-\sigma_{j^*}+\alpha_k-\tau_i}^{t-\tau_i} q_k(s)x(s-\alpha_k)\,\mathrm{d}s + \sum_{i=1}^{l} c_i|F(t-\tau_i)| + |F(t)|\right]\\
&\leq \underline{\lim}_{t\to\infty} x(t) + \overline{\lim}_{t\to\infty}\left[\sum_{i=1}^{l} c_i\sum_{k=1}^{n}\int_{t-\sigma_{j^*}+\alpha_k-\tau_i}^{t-\tau_i} q_k(s)x(s-\alpha_k)\,\mathrm{d}s\right]\\
&\qquad\qquad + \overline{\lim}_{t\to\infty}\left[\sum_{i=1}^{l} c_i|F(t-\tau_i)| + |F(t)|\right] \leq 0\,.
\end{aligned}$$

Hence $\mu = 0$. Consequently, $x(t) \leq x(t) + \sum_{i=1}^{l} c_i(t)x(t-\tau_i)$ implies that $\lim_{t\to\infty} x(t) = 0$. If $x(t) < 0$ for $t \geq t_1$, then we set $y(t) = -x(t)$ and proceed as above to obtain $\lim_{t\to\infty} y(t) = 0$, that is, $\lim_{t\to\infty} x(t) = 0$. Thus the theorem is proved. □

THEOREM 4. *Suppose that* (A_2), (A_3) *and* (A_4'') *hold. Let*

(A_7) $1 \le c_1(t) \le c_1$,

(A_8) $p_{j^*}(t)$ *is monotonic increasing and* $q_k(t)$ *is monotonic decreasing for* $k \in \{1, \dots, n\}$,

(A_9) $\int\limits_{t_0+\sigma_{j^*}}^{\infty} [p_{j^*}(t)/c_1(t+\tau_1-\sigma_{j^*})]\, dt = \infty$.

Then every bounded solution of (1) *with* $l = 1$ *oscillates or tends to zero as* $t \to \infty$.

Proof. Let $x(t)$ be a bounded nonoscillatory solution of (1) (with $l = 1$) on $[t_x, \infty)$, $t_x \ge t_0$, such that $x(t) > 0$ for $t \ge t_1 \ge t_x$. Setting $z(t)$ as in (4) for $t \ge t_1 + T_0$, (5) is obtained. Since $c_1(t) \le c_1$, then proceeding as in Theorem 3, we obtain

$$\infty > \mu = \lim_{t\to\infty} [x(t) + c_1(t)x(t-\tau_1)] \ge 0$$

and hence $z(t) > 0$ for $t \ge t_2 > t_1 + T_0$, where $\lim\limits_{t\to\infty} z(t) = \mu$. If possible, let $\mu > 0$. Then, for $0 < \varepsilon < \mu$ there exists $t_3 > t_2$ such that $x(t)+c_1(t)x(t-\tau_1) > \mu - \varepsilon$ for $t \ge t_3$. However, for $t \ge t_4 \ge t_3 + T_0$,

$$\begin{aligned}
&z'(t) + z'(t-\tau_1) \\
&\le \left[\sum_{k=1}^{n} q_k(t-\sigma_{j^*}+\alpha_k) - p_{j^*}(t)\right] x(t-\sigma_{j^*}) \\
&\quad + \left[\sum_{k=1}^{n} q_k(t-\sigma_{j^*}+\alpha_k-\tau_1) - p_{j^*}(t-\tau_1)\right] x(t-\sigma_{j^*}-\tau_1) \\
&\le \left[\sum_{k=1}^{n} q_k(t-\sigma_{j^*}+\alpha_k-\tau_1) - p_{j^*}(t-\tau_1)\right] \big(x(t-\sigma_{j^*}) + x(t-\sigma_{j^*}-\tau_1)\big) \\
&\le \frac{1}{c_1(t-\sigma_{j^*})} \left[\sum_{k=1}^{n} q_k(t-\sigma_{j^*}+\alpha_k-\tau_1) - p_{j^*}(t-\tau_1)\right] \cdot \\
&\qquad\qquad \cdot \big(x(t-\sigma_{j^*}) + c_1(t-\sigma_{j^*})x(t-\sigma_{j^*}-\tau_1)\big) \\
&< \frac{\mu-\varepsilon}{c_1(t-\sigma_{j^*})} \left[\sum_{k=1}^{n} q_k(t-\sigma_{j^*}+\alpha_k-\tau_1) - p_{j^*}(t-\tau_1)\right].
\end{aligned}$$

Integrating the above inequality from t_4 to t we obtain

$$\begin{aligned}
0 &< z(t) + z(t-\tau_1) \\
&\le \lambda + (\mu-\varepsilon)\int_{t_4}^{t}\left[\sum_{k=1}^{n} q_k(s-\sigma_{j^*}+\alpha_k-\tau_1) - p_{j^*}(s-\tau_1)\right]\Big/ c_1(s-\sigma_{j^*})\,\mathrm{d}s \\
&\le \lambda + (\mu-\varepsilon)\left[\int_{t_4}^{t}\left(\sum_{k=1}^{n} q_k(s-\sigma_{j^*}+\alpha_k-\tau_1)\right)\mathrm{d}s \right. \\
&\qquad\qquad \left. - \int_{t_4-\tau_1}^{t-\tau_1} p_{j^*}(s)/c_1(s+\tau_1-\sigma_{j^*})\,\mathrm{d}s\right],
\end{aligned}$$

where $\lambda = z(t_4) + z(t_4-\tau_1)$. Thus $0 < z(t) + z(t-\tau_1) < 0$ for large t, a contradiction due to (A_3) and (A_9). Hence $\mu = 0$. Consequently, $x(t) < x(t)+c_1(t)x(t-\tau_1)$ implies that $\overline{\lim\limits_{t\to\infty}}\, x(t) = 0$. Thus $\lim\limits_{t\to\infty} x(t) = 0$. If $x(t) < 0$ for $t \ge t_1$, then we put $y(t) = -x(t)$ and proceed as above to obtain $\lim\limits_{t\to\infty} x(t) = 0$. Hence the theorem is proved. □

Following examples illustrate above theorems.

EXAMPLE 5. Consider

$$\begin{aligned}
&\left[x(t) + \tfrac{1}{t}x(t-\pi)\right]' + \left(1+\tfrac{1}{t}\right)x\left(t-\tfrac{\pi}{2}\right) \\
&\quad + \frac{1}{t-\pi}x(t-2\pi) + \frac{1}{t-\pi}x(t-\pi) - \frac{1}{t^2}x(t-\pi) = \frac{2}{t^2}(\sin t - t\cos t), \\
&\hfill t \ge \pi + 1.
\end{aligned}$$

Here $F(t) = -\frac{2}{t}\sin t$. As all the conditions of Theorem 3 are satisfied, then every bounded solution of the equation oscillates or tends to zero as $t \to \infty$. In particular, $x(t) = \sin t$ is a bounded oscillatory solution of the equation.

EXAMPLE 6. From Theorem 4 it follows that all bounded solutions of

$$\begin{aligned}
&\left[x(t) + 2x(t-1)\right]' + (t-1)x(t-1) - \frac{4}{(2t-1)^2}x\left(t-\tfrac{1}{2}\right) \\
&\qquad = \frac{1}{(t-1)^2} - \frac{32}{(2t-1)^5} - \frac{3}{t^4} - \frac{6}{(t-1)^4},
\end{aligned}$$

$t \ge 1$ oscillate or tend to zero as $t \to \infty$. In particular, $x(t) = \frac{1}{t^3}$ is a bounded nonoscillatory solution of the equation which $\to 0$ as $t \to \infty$.

In the following we develop another technique to study oscillatory (asymptotic) behaviour of solutions of (1). This technique is found to be suitable for the study of similar problem for second order equation of the form (1). These results will be discussed in a different paper.

THEOREM 5. *Suppose that* (A_1), (A_4') *hold and*

(H_1) $m \geq n$, $p_k(t) \geq q_k(t)$ *and* $\sigma_k \geq \alpha_k$, $1 \leq k \leq n$, *such that* $p_k(t) > q_k(t)$ *for some* $k \in \{l, \dots, n\}$.

(H_2) $p_j(t)$ *is monotonic increasing and* $q_k(t)$ *is monotonic decreasing for* $1 \leq j \leq m$ *and* $1 \leq k \leq n$.

Then every bounded solution of (1) *oscillates or tends to zero as* $t \to \infty$.

Proof. Let $x(t)$ be a bounded nonoscillatory solution of (1) on $[t_x, \infty)$, $t_x \geq t_0$. Let $x(t) > 0$ for $t \geq t_1 \geq t_x$. Setting

$$z(t) = x(t) + \sum_{i=1}^{l} c_i(t)x(t-\tau_i) - \sum_{k=1}^{n} \int_{t-\sigma_k}^{t-\alpha_k} q_k(s)x(s)\,\mathrm{d}s - F(t) \tag{8}$$

for $t \geq t_1 + T_0$, we obtain, due to (H_1), that

$$\begin{aligned} z'(t) &\leq -\sum_{j=1}^{m} p_j(t)x(t-\sigma_j) + \sum_{k=1}^{n} q_k(t-\sigma_k)x(t-\sigma_k) \\ &\leq -\sum_{k=1}^{n} \big[p_k(t-\sigma_k) - q_k(t-\sigma_k)\big]x(t-\sigma_k) \leq 0. \end{aligned} \tag{9}$$

Since $x(t)$ is bounded, then (A_1), (A_4') and (H_2) imply that $z(t)$ is bounded. Hence $-\infty < \mu < \infty$ where $\lim\limits_{t\to\infty} z(t) = \mu$. From (9) it follows, due to (H_2), that $x \in L^1\big([t_1,\infty),\mathbb{R}\big)$ and hence $q_k x \in L^1\big([t_1,\infty),\mathbb{R}\big)$, $1 \leq k \leq n$. Thus

$$\lim_{t\to\infty}\left[x(t) + \sum_{i=1}^{l} c_i(t)x(t-\tau_i)\right] = \mu + \beta. \tag{10}$$

We consider three cases, viz., $\mu+\beta > 0$, $\mu+\beta < 0$ and $\mu+\beta = 0$. Let $\mu+\beta > 0$. Then, for $0 < \varepsilon < \mu+\beta$, there exists a $t_2 > t_1 + T_0$ such that

$$x(t) \geq x(t) + \sum_{i=1}^{l} c_i(t)x(t-\tau_i) > (\mu+\beta) - \varepsilon$$

for $t \geq t_2$ and hence $x \notin L^1\big([t_1,\infty),\mathbb{R}\big)$, a contradiction. If $\mu+\beta < 0$, then, for $0 < \varepsilon < -(\mu+\beta)$, we can find a $t_3 > t_1 + T_0$ such that

$$\mu + \beta + \varepsilon > x(t) + \sum_{i\ 1}^{l} c_i(t)x(t-\tau_i) > -\sum_{i=1}^{l} x(t-\tau_i).$$

Hence $x \notin L^1([t_1,\infty),\mathbb{R})$, a contradiction. Thus $\mu+\beta=0$. From (8) we get

$$\begin{aligned}
0=\mu+\beta &= \lim_{t\to\infty}\left[x(t)+\sum_{i=1}^{l} c_i(t)x(t-\tau_i)\right] \\
&\geq \overline{\lim_{t\to\infty}}\left[x(t)+\sum_{i=1}^{l} c_i x(t-\tau_i)\right] \\
&\geq \overline{\lim_{t\to\infty}}\, x(t)+\underline{\lim_{t\to\infty}}\sum_{i=1}^{l} c_i x(t-\tau_i) \\
&\geq \overline{\lim_{t\to\infty}}\, x(t)+\sum_{i=1}^{l} c_i \overline{\lim_{t\to\infty}}\, x(t-\tau_i) \\
&\geq \left(1+\sum_{i=1}^{l} c_i\right)\overline{\lim_{t\to\infty}}\, x(t)\,.
\end{aligned}$$

Thus $\lim_{t\to\infty} x(t)=0$. For $x(t)<0$, $t\geq t_1$, the proof is similar and hence is omitted. This completes the proof of the theorem. □

Remark. The assumption (H_1) means that out of m functions $p_j(t)$ it is possible to choose n functions $p_k(t)$ satisfying $p_k(t)\geq q_k(t)$, $1\leq k\leq n$, and the corresponding delays satisfying $\sigma_k\geq\alpha_k$, $1\leq k\leq n$. It is always possible to rearrange $p_j(t)$'s in $\sum_{j=1}^{m} p_j(t)x(t-\sigma_j)$ and rename them so that the first n number of $p_j(t)$ satisfy the condition.

THEOREM 6. *Let* (A_4'), (H_1) *and* (H_2) *hold. If* $0\leq c_i(t)\leq c_i$, $1\leq i\leq l$, *then every bounded solution of* (1) *oscillates or tends to zero as* $t\to\infty$.

Proof. Let $x(t)$ be a bounded nonoscillatory solution of (1) on $[t_x,\infty)$, $t_x\geq t_0$, such that $x(t)>0$ for $t\geq t_1\geq t_x$. Proceeding as in Theorem 5 we obtain (10). Since $c_i(t)\geq 0$, then $\lim_{t\to\infty}\left[x(t)+\sum_{i=1}^{l} c_i(t)x(t-\tau_i)\right]\geq 0$. Thus we consider two cases, viz., $\mu+\beta>0$ or $\mu+\beta=0$. However, $\mu+\beta>0$ implies that $x\notin L^1([t_1,\infty),\mathbb{R})$, a contradiction. Hence $\mu+\beta=0$. As $x(t)\leq x(t)+\sum_{i=1}^{l} c_i(t)x(t-\tau_i)$, then $\lim_{t\to\infty} x(t)=0$. The proof for the case $x(t)<0$ for $t\geq t_1$ is similar and is omitted. Thus the theorem is proved. □

EXAMPLE 7. Consider

$$\left[x(t)-\frac{1}{2}x(t-1)\right]'+(t-1)x(t-1)-\frac{2}{(2t-1)}x\left(t-\frac{1}{2}\right)$$
$$=\frac{1}{(t-1)^2}+\frac{3}{2(t-1)^4}-\frac{3}{t^4}-\frac{16}{(2t-1)^4}\,,\qquad t\geq 3\,.$$

Hence $F(t) = \frac{1}{t^3} + \frac{8}{3(2t-1)^3} - \frac{1}{2(t-1)^3} - \frac{1}{(t-1)}$. From Theorem 5 it follows that every bounded solution of the equation oscillates or tends to zero. Clearly, $x(t) = \frac{1}{t^3}$ is a bounded solution of the equation which tends to zero as $t \to \infty$.

EXAMPLE 8. From Theorem 6 it follows that all bounded solutions of

$$\big[x(t) + 2x(t-1)\big]' + 2x(t-1) - \mathrm{e}^{-t+1/2}\, x\Big(t - \frac{1}{2}\Big) = -\,\mathrm{e}^{-t} - \mathrm{e}^{-2t+1}\,, \qquad t \geq 1\,,$$

oscillate or tend to zero as $t \to \infty$. In particular, $x(t) = \mathrm{e}^{-t}$ is such a solution.

Remark. We may note that Theorem 2 cannot be applied to Example 7 as (A_3) fails to hold. On the other hand, Theorem 5 cannot be applied to Example 4 as (H_2) does not hold. Further, Theorem 6 cannot be applied to Example 5 as (H_2) fails to hold. Theorem 3 cannot be applied to Example 8 since (A_6) does not hold.

THEOREM 7. *Let (A_4'), (H_1) and (H_2) hold. Let $-\infty < c_i \leq c_i(t) \leq -1$. Then every bounded solution $x(t)$ of (1) oscillates or $\varliminf\limits_{t\to\infty} |x(t)| = 0$.*

Proof. One may proceed as in the proof of Theorem 5 to obtain $x \in L^1\big([t_1, \infty), \mathbb{R}\big)$. Hence $\varliminf\limits_{t\to\infty} |x(t)| = 0$. □

Remark. Theorems 5–7 hold for equations with constant coefficients.

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