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MARTINGALE CONVERGENCE THEOREM IN QUANTUM LOGICS

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Introduction

It is clear that both martingales and quantum logics belong to the topical themes of today's probability theory. The central result of the martingale theory is the martingale convergence theorem. This theorem was generalized into quantum logics (by supposing the distributivity) by B. Harman and B. Riečan in [3].

O. Nánásiová and S. Pulmannová have shown in [5] the possibility of a further generalization of the conditional expectation, the construction of which is fundamentally used in the proof of the martingale convergence theorem.

This paper follows the mentioned works and shows the possibility of the generalization of the martingale convergence theorem into quantum logics without supposing the distributivity, which means the generalization in line with the main purpose of the quantum – logics – theory.

The terms in this paper are used according to [7]; som preliminaries are even verbatim quoted.

Preliminaries

In the mathematical description of quantum experiments, a generalization of the classical probability theory is needed. While in the classical probability theory the set of all "experimentally verifiable propositions" of the physical system (or, equivalently, the set of all random events) can be mathematically described as a Boolean σ-algebra, in the quantum case a more general algebraic structure is needed. The reason for this is the fact that there exist pairs of physical quantities (e.g. position and momentum of a particle) which cannot be measured simultaneously with an arbitrary accuracy — as it can be seen in the case of the well-known Heisenberg uncertainty principle. In quantum logics approach to quantum mechanics the basic concepts are the set L of all experimentally verifiable propositions of the physical system and the set M of the physical states. L is usually supposed to be a partially ordered set with the greatest element 1 and the least element 0, with the orthocomplementation $\perp : L \to L$ such that

(i) $(a^{\perp})^{\perp} = a$, $a \in L$

(ii) $a \leqslant b$ if and only if $b^{\perp} \leqslant a^{\perp}$, $a, b \in L$

(iii) $a \vee a^{\perp} = 1$ for all $a \in L$

(where we denote by $x \wedge y$, resp. $x \vee y$, the infimum, resp. supremum of x and y of L if they exist); and with the orthomodular property

(iv) $a \leqslant b$ $(a, b \in L)$ implies $\exists c \in L$: $c \leqslant a^{\perp}$ and $b = a \vee c$; and which is closed under the formation of the suprema $\vee a_i$ for any sequence $\{a_i\} \subset L$ such that $a_i \leqslant a_j^{\perp}$, $i \neq j$. A set L with the properties just described is called a *logic*.

The elements $a, b \in L$ are called *orthogonal* $(a \perp b)$ if $a \leqslant b^{\perp}$, and they are *compatible* $(a \leftrightarrow b)$ if there exist elements a_1, b_1, c in L mutually orthogonal and such that $a = a_1 \vee c$ and $b = b_1 \vee c$. *The sets* $A \subset L$, $B \subset L$ are called *compatible* $(A \leftrightarrow B)$ if $a \leftrightarrow b$ for all $a \in A$, $b \in B$; A is a *comparible set* $(A \leftrightarrow)$ if $a \leftrightarrow b$ for all $a, b \in A$.

If for any $0 \neq a \in L$ and a set $M \subset L$ there holds:

(i) $M \leftrightarrow a$

(ii) $(M \wedge a) \leftrightarrow$,

we define M as *partial compatible with a* (M p.c. $[a]$).

A *state* on the logic L is a probability measure on L, i.e. a map m: $L \to [0, 1]$ such that:

(i) $m(1) = 1$

(ii) $m(\vee a_i) = \Sigma\, m(a_i)$

for any sequence $\{a_i\}$ of mutually orthogonal elements of L.

Let L_1 and L_2 be two logics. The mapping $h : L_1 \to L_2$ is called a *σ-homomorphism* if

(i) $h(1_1) = 1_2$ (where 1_1 and 1_2 are the greatest elements in L_1 and L_2, (respectively),

(ii) $p \perp q$, $p, q \in L_1$ implies $h(p) \perp h(q)$,

(iii) $h(\vee p_i) = \vee h(p_i)$ for any sequence $\{p_i\}$ of mutually orthogonal elements of L_1.

With the help of the concept of σ-homomorphism we introduce observables (corresponding to physical quantities). If R is the real line and $B(R)$ is the σ-algebra of all Borel sets, then any σ-homomorphism of $B(R)$ into L is called *observable* on L.

If x is an observable and f: $R \to R$ is a Borel measurable function, then the map $x \circ f^{-1}: E \mapsto x(f^{-1}(E))$ is also an observable, which is called the function f of the observable x.

Two observables x and y are called*compatible* if $x(E) \leftrightarrow y(E)$ for all $E, F \in B(R)$ $(x \leftrightarrow y)$; x is compatible with a if $x(E) \leftrightarrow a$ for all $E \in B(R)$.

If x is an observable and m is a state on L, then the map $m_x\colon E \mapsto m(x(E))$ is a probability measure on $B(R)$. It is called the *probability distribution of the observable* x. For $a \in L$ we define

$$\int_a x \, \mathrm{d}m = \int_R t m(x(\mathrm{d}t) \wedge a) = \int_R t \, \mathrm{d}v$$

where $v\colon E \mapsto m(x(E) \wedge a)$ if such an integral exists.

Remark: For the existence of this integral we need firstly to make sure that $m(x(E) \wedge a)$ is a measure; the sufficient condition for that is $x \leftrightarrow a$.

The *expectation of the observable* x in the state m can be defined by

$$m(x) = \int_R t m_x(\mathrm{d}t)$$

if the integral exists. The subset L_o of a logic L is called a *sublogic* of L if

(i) $a \in L_o$ implies $a^\perp \in L_o$

(ii) $a_1, a_2, \ldots \in L_o$ with $a_i \perp a_j$ implies $\vee a_i \in L_o$.

It can be easily verified that a sublogic of a logic is a logic itself. A sublogic $L_o \subset L$ is called a *lattice sublogic* provided that $a, b \in L_o$ implies $a \vee b$ exists in L and is in L_o. In this case L_o is a lattice. If a lattice sublogic is distributive, then it is a *Boolean σ-algebra* and it is called a sub-σ-algebra of L. A logic L is called *separable* if for any Boolean sub-σ-algebra B_o of L there exists a countable subset D of L such that B_o is the smallest Boolean sub-σ-algebra of L containing D.

Some important statements

Let us mention some of the generally known statements, which we shall use further in a relevant way. Their proof is to be found in [9] or [5].

Theorem 1: *Let x be an observable associated with a logic L. The range of the observable $x(R(x))$ is always a Boolean sub-σ-algebra of the logic L. If L_o is a sublogic of a logic L, then $x^{-1}(L_o)$ is a Boolean sub-σ-algebra $B(R)$ as well.*

Theorem 2: *Let L be a logic which is simultaneously a lattice. Then a necessary and sufficient condition for the existence a Boolean sub-σ-algebra B such that $A \subset B \subset L$ is that A is a compatible set (in L).*

Theorem 3: *Two observables are compatible if and only if such a Boolean sub-σ-algebra B exists that $R(x) \subset B$ and also $R(y) \subset B$.*

Theorem 4: *Let $x\colon B(R) \to L_o$ be an observable associated with L_o, where L_o is a countable generated Boolean sub-σ-algebra of the logic L. Then there exists* *an observable y: $B(R) \to L$ and a Borel measurable function f: $R \to R$ so that $L_o = \{y(E): E \in B(R)\}$ and $x(E) = y \circ f^{-1}(E)$ for all $E \in B(R)$.*

Theorem 5: *Let $\{x_n\}$ be maximally a countable system of mutually compatible observables associated with a lattice-logic L. Then there exists an observable y associated with L and a Borel measurable functions f_n so that $x_n = y\ f_n^{-1}$.*

Theorem 6: *Let L be a logic, $a \in L$. Then $L_{[0, a]} = \{b \in L: b \leqslant a\}$ is also a logic with the greatest element a and an ortho-complement $b' = b^{\perp} \wedge a$.*

Theorem 7: *Let x be an observable associated with L and $x \leftrightarrow a$, $a \in L$. Then $x \wedge a$ $(E \mapsto x(E) \wedge a)$ is also an observable associated with $L_{[0\ a]}$.*

Theorem 8: *Let m be a state on L. Let $a \in L$ be such an element that $m(a) = 1$. Then m is also a state on $L_{[0, a]}$.*

Theorem 9: *Let L be a logic, $M \subset L$, $M \leftrightarrow a$. $M \wedge a$ is a compatible set in* L *if and only if $M \wedge a$ is a compatible set in $L_{[0, a]}$.*

Theorem 5 enables us to define the *calcul* of compatible observables. If x and y are compatible observables, then let us define $y - x = z \circ (f_1 - f_2)^{-1}$, where $y = z \circ f_1^{-1}$, $x = z \circ f_2^{-1}$.

In the same way we can define the *sum* and the *product* of compatible observables. Another way how to find the calcul is available in [2].

Let $\{x_n\}$ be a sequence of the observables associated with L, x be an observable associated with L, m be a state and $x_n \leftrightarrow x$. We can say that $\{x_n\}$ *converges almost everywhere to x according to the state m* ($x_n \to x$ a.e. $[m]$) if

$$m(\lim_n \sup (x_n - x)(-\varepsilon, \varepsilon)^c) = 0 \text{ for all } \varepsilon > 0.$$

Conditional expectation of an observable

Before we state and prove the martingale convergence theorem, it is necessary to realize the construction of the conditional expectation of observable.

For $a, b \in L$ put $a \triangle b = (a^{\perp} \wedge b) \vee (a \wedge b^{\perp})$. For the observables x and y associated with L we shall write $x \approx y(m)$ if

$$m(x(E) \triangle y(E)) = 0 \quad \text{for any} \quad E \in B(R).$$

In [5] is proved the following lemma:

Lemma 1: *Let x, y, z be observables associated with the logic L such that* $(R(x) \cup R(y) \cup R(z))$ p.c. $[a]$ *for some $a \in L$; and let $m(a) = 1$. Then $x \approx y(m)$, $y \approx z(m)$ implies $x \approx z(m)$.*

Proof: First we prove the lemma in the special case $a = 1$. If b, c, d are compatible elements of L, then $b \triangle d \leqslant (b \triangle c) \vee (c \triangle d)$, so that

$$m(x(E) \triangle y(E)) = m(z(E) \triangle y(E)) = 0 \quad \text{implies} \quad m(x(E) \triangle z(E)) = 0$$

for all $E \in B(R)$.

Let $0 < a < 1$. Then $x \wedge a, y \wedge a, z \wedge a$ are mutually compatible observables associated with $L_{[0, a]}$ so that, by the above part of proof, $x \approx y(m)$, $y \approx z(m)$ implies $m((x \wedge a)(E) \triangle (z \wedge a)(E)) = 0$ for all $E \in B(R)$. But

$$m(x(E) \triangle z(E)) = m((x(E) \triangle z(E)) \wedge a) = m((x \wedge a)(E) \triangle (z \wedge a)(E)) = 0.$$

Definition 1: Let x be an observable associated with the logic L. Let L_o be a sublogic of the logic L, m be a state on L.

The *conditional expectation of the observable x according to the sublogic* L_o (we shall write $E(x/L_o)$) can be defined as such an observable u, for which there holds:

(i) $u(E) \in L_o$ for all $E \in B(R)$

(ii) $\int_a u \, \mathrm{d}m = \int_a x \, \mathrm{d}m$ for all $a \in L_o$ i.e. both sides exist and are equal.

In case when L_1 is a Boolean sub-σ-algebra of the logic L and $x(B(R)) \cup \cup L_o \subset L_1$, the existence and almost everywhere uniqueness (in accordance $\approx (m)$) of the conditional expectation of x is guaranteed by the following theorem.

Theorem 10: *Let L_1 be a countably generated Boolean sub-σ-algebra of the logic L, m be a state on L. Let L_o be a subalgebra of L_1. Let $x: B(R) \to L_1$ be an integrable observable. Then there exists an integrable observable $u: B(R) \to L_1$ such that there holds*:

(i) $\int_a x \, \mathrm{d}m = \int_a u \, \mathrm{d}m$ *for all* $a \in L_o$

(ii) $u(E) \in L_o$ *for all* $E \in B(R)$.

Proof: see [2].

Definition 2: Let L be a logic, $\{L_n\}_1^\infty$ be a sequence of sublogics of the logic L, $\{x_n\}_1^\infty$ be a sequence of observables, m be a state on L. The sequence $(x_n, L_n)_1^\infty$ is called the *martingale on the logic L* if:

(i) $L_n \subset L_{n+1} \subset L$, $n = 1, 2, \ldots$

(ii) x_n is associated with L_n, $n = 1, 2, \ldots$

(iii) $E(x_{n+1}/L_n) \approx x_n(m)$, $n = 1, 2, \ldots$

Theorem 5 and 10 allow to make the following statement:

Theorem 11: *Let L be a logic and $\{L_n\}_1^\infty$ a sequence of a countable generated Boolean sub-σ-algebras of the logic L. Let $(x_n, L_n)_1^\infty$ be a martingale on L and* $\sup_n m(|x_n|) < \infty$, *m is a state on L. Then there exists such an observable x* *associated with* $\mathscr{L} = \sigma_B\left(\bigcup_{n=1}^{\infty} L_n\right)$ *(the smallest Boolean σ-algebra such that* $\mathscr{L} \supset \bigcup_{n=1}^{\infty} L_n$*) that* $x_n \to x$ a.e. $[m]$.

Proof: see [3].

But this understandig of the conditional expectation is not too generalized, because it supposes the distributivity of L. Our purpose is, however, to avoid this presumption. Another conditional expectation version of an observable can be found by the following construction. Let L be a separable lattice–logic, L_o its sublogic, x an integrable observable associated with L, m a state on L. Let us have an element $0 \neq a \in L$, so that

(i) $m(a) = 1$

(ii) $(R(x) \cup L_o)$ p.c. $[a]$.

Because $(R(x) \cup L_o) \wedge a$ is the compatible set, there exists such a Boolean σ-algebra B that $(R(x) \cup L_o) \wedge a \subset B \subset L_{[0,a]}$, an observable $y\colon B(R) \to B$ and a Borel measurable function $f\colon R \to R$ so that $x \wedge a = y \circ f^{-1}$. To $L_o \wedge a$ there exists a Boolean σ-algebra B_o so that $L_o \wedge a \subset B_o \subset B$. Let us define $S_o = \{E \in B(R)\colon y(E) \in B_o\}$. It is clear that S_o is a sub-σ-algebra of the σ-algebra $S = y^{-1}(B)$. Therefore there exists $g = Em_y(f/S_o)$ (the "classical" conditional expectation of the function f according to S_o). Now let us define $z^+ = y \circ g^{-1}$.

It is clear that z^+ is an observable associated with $L_{[0,a]}$ whereby $R(z^+) \subset B_o$ (because g is S_o-measurable). In order to "extend" z^+ to the whole logic L, let us define

$$z = y \circ g^{-1} \vee (w \wedge a^{\perp}) \quad \text{where} \quad w(E) = \begin{cases} 0 & 0 \notin E \\ 1 & 0 \in E \end{cases}$$

Now it is still valid that

$$z(R) = z^+(R) \vee (w(R) \wedge a^{\perp}) = z^+(R) \vee a^{\perp} = a \vee a^{\perp} = 1.$$

Let us show that z is a conditional expectation version of the observable x according to L_o. Let $b \in L_o$.

Because $m(a) = 1$

$$\int_b x \, \mathrm{d}m = \int tm(x(\mathrm{d}t) \wedge b) =$$

$$= \int tm(x \wedge a(\mathrm{d}t) \wedge (b \wedge a)) = \int tm(y \circ f^{-1}(\mathrm{d}t) \wedge y(A)) =$$

$$= \int tm_y(f^{-1}(\mathrm{d}t) \cap A) = \int_A f(\omega)\, m_y(\mathrm{d}\omega) = \int_A g(\omega)\, m_y(\mathrm{d}\omega) =$$

$$= \int tm_y(g^{-1}(\mathrm{d}t) \cap A) = \int tm(y \circ g^{-1}(\mathrm{d}t) \wedge y(A)) =$$

$$= \int tm(y \circ g^{-1}(\mathrm{d}t) \wedge (b \wedge a)) = \int_b z \, \mathrm{d}m.$$

Let us prove yet the uniqueness (in accordance with $\approx (m)$) of this conditional expectation.

Let z_1, z_2 be two versions of a conditional expectation $E(x/L_o)$ for the same required element $a \in L$. We have $z_1 \leftrightarrow a$, $z_2 \leftrightarrow a$ and $R(z_1) \wedge a \subset L_o \wedge a$, $R(z_2) \wedge \wedge a \subset L_o \wedge a$.

As $L_o \wedge a \subset B_o \subset L_{[0,a]}$ and B_o is a Boolean sub-σ-algebra, then $z_1 \wedge a \leftrightarrow \leftrightarrow z_2 \wedge a$. Let $g_1: \Omega \to R$, $g_2: \Omega \to R$ be S_o-measurable functions such that $z_1 = y \circ g_1^{-1}$ and $z_2 = y \circ g_2^{-1}$. Then, as $z_1(E) \triangle z_2(E) \leftrightarrow a$ for any $E \in B(R)$,

$$m((z_1 \wedge a)(E) \triangle (z_2 \wedge a)(E)) = m(y \circ g_1^{-1}(E) \triangle y \circ g_2^{-1}(E)) =$$
$$= m(y \circ (g_1^{-1}(E) \triangle g_2^{-1}(E))) = m_y(g_1^{-1}(E) \triangle g_2^{-1}(E))$$

From that there holds for g_1 and $g_2 \int_F g_1 \, \mathrm{d}m_y = \int_F g_2 \, \mathrm{d}m_y$, for all $F \in S_o$ we get $m_y(g_1^{-1}(E) \triangle g_2^{-1}(E)) = 0$ for any $E \in B(R)$. Indeed, put

$$F_1 = \{\omega \in \Omega; \quad g_1(\omega) > g_2(\omega)\},$$
$$F_2 = \{\omega \in \Omega; \quad g_1(\omega) < g_2(\omega)\}.$$

As $F_1 \cup F_2 \in S_o$, $\int_{F_o} (g_1 - g_2) \, \mathrm{d}m_y = 0$ for any $F_o \subset F_1 \cup F_2$, $F_o \in S_o$, hence

$$m_y(F_1 \cup F_2) = m_y\{\omega; \quad g_1(\omega) \neq g_2(\omega)\} = 0.$$

As $g_1^{-1}(E) \triangle g_2^{-1}(E) \subset F_1 \cup F_2$, we obtain

$$m_y(g_1^{-1}(E) \triangle g_2^{-1}(E)) = 0$$

for any $E \in B(R)$.

Now $0 = m((z_1 \wedge a)(E) \triangle (z_2 \wedge a)(E)) = m(z_1(E) \triangle z_2(E))$ and this concludes the proof of uniqueness.

Let us notice the conditions we require for the element $0 \neq a \in L$. We have to recognize that if we have to use $1 \in L$ instead of this element, then the second condition actually says that L_o has to be a Boolean σ-algebra, which means that we get the same conditional expectation version as in theorem 10. On the other hand, it is clear that the conditions for the required $a \in L$ are "strong enough". One of the possibilities, how to seek such an $a \in L$ is to use the set-comutator qualities (see [5]). Furthermore we shall require $a \in L_o$, which is a sufficient condition for the L_o-measurability of the conditional expectation.

In concluding this part of the paper let us make a generalization of the previously defined calculus of observables.

Let x, y be two observables associated with the logic L, m is a state. Let there exist $0 = a \in L$ such that $(R(x) \cup R(y))$ p.c. $[a]$ and $m(a) = 1$. Then we can define $x - y = z \circ (f_1 - f_2)^{-1} \vee (w \wedge a^{\perp})$ where $x \wedge a = z \circ f_1^{-1}$, $y \wedge a = z \circ f_2^{-1}$ and w is the observable $\left(w = \left\langle \begin{matrix} 0 & 0 \notin E \\ 1 & 0 \in E \end{matrix} \right. \right)$.

The martingale convergence theorem

Theorem 12: *Let $(x_n, L_n)_1^{\infty}$ be a martingale on the separable lattice-logic L. Let there exist such a $0 \neq a \in L$ that*

(i) $m(a) = 1$

(ii) $(R(x_{n+1}) \cup L_n)$ p.c. $[a]$, $n = 1, 2, \ldots$

Let $\sup_n m(|x_n|) < \infty$.

Then there exists such an observable x that $x_n \to x$ a.e. $[m]$.

Proof: For each $n = 1, 2, \ldots$ $(R(x_{n+1}) \cup L_n) \wedge a$ are compatible sets and $(R(x_{n+1}) \cup L_n) \wedge a \subset L_{[0, a]}$, which means that there exists the Boolean sub-σ-algebras B_1, B_2, ... such that

$$(R(x_{n+1}) \cup L_n)) \wedge a \subset B_n, \; B_n \subset B_{n+1} \subset L_{[0, a]}, \quad n = 1, 2, \ldots$$

Because $\{B_n\}$ is an upper bounded sequence by $L_{[0, a]}$, there exists a Boolean sub-σ-algebra B of $L_{[0, a]}$ such that for each n there holds $(R(x_{n+1}) \cup L_n) \wedge a \subset$ $\subset B \subset L_{[0, a]}$. Therefore the observables $x_n \wedge a$, $n = 1, 2, \ldots$ are compatible (because $R(x_n \wedge a) \subset R(x_n) \wedge a \subset B$) and this means that there exists an observable y: $B(R) \to B$ and Borel measurable functions f_n: $R \to R$ that $x_n \wedge a =$ $= y \circ f_n^{-1}$. Similarly $L_n \wedge a$, $n = 1, 2, \ldots$ are compatible sets, too and therefore there exists a sequence B_1^+, B_2^+, ... of the Boolean sub-σ-algebras of $L_{[0, a]}$ such that $L_n \wedge a \subset B_n^+$.

Let us take instead of $\{B_n^+\}$ the smallest Boolean σ-algebras with this property.

Let us prove that $(x_n \wedge a, B_n^+)$ is a martingale on $L_{[0, a]}$. Evidently:

(i) $B_1^+ \subset B_2^+ \subset \ldots \subset B$ because $L_1 \subset L_2 \subset \ldots$ and $L_1 \wedge a \subset L_2 \wedge a \subset \ldots$

(ii) because $x_n(E) \in L_n$, therefore also $x_n(E) \wedge a \in L_n \wedge a \subset B_n^+$ as well

(iii) it is also necessary the prove that $E(x_{n+1} \wedge a / B_n^+) \approx x_n \wedge a (m)$.

Let us define:

$$S_n = \{A \in B(R) \colon y(A) \in B_n^+\} = y^{-1}(B_n^+) \quad \text{and} \quad g_n = Em_y(f_{n+1}/S_n).$$

From the assumption that $(x_n, L_n)_1^{\infty}$ is a martingale and from the conditional-expectation-construction of an observable we know that $E(x_{n+1}/L_n) =$ $= y \circ g_n^{-1} \vee (w \wedge a^{\perp}) \approx x_n$, therefore $x_n \wedge a \approx y \circ g_n^{-1} \vee (w \wedge a^{\perp}) \wedge a = y \circ g_n^{-1}$ (since $m(a) = 1$). Because $x_{n+1} \wedge a = y \circ f_{n+1}^{-1}$, from the construction g_n it is clear that $E(x_{n+1} \wedge a / B_n^+) \approx x_n \wedge a$ (with respect to lemma 1, too).

Now we can finally use theorem 11, according to which $x_n \wedge a \to h = y \circ \circ g^{-1}$ a.e. $[m]$, where g is such a Borel measurable function that $g_n \to g$ a.e. $[m_y]$.

Let us define $x = y \circ g^{-1} \vee (w \wedge a^{\perp})$.

It is also necessary to prove that

$$m(\limsup_n (x - x_n)(-\varepsilon, \varepsilon)^c) = 0.$$

Indeed,

$$m(\limsup_n (y \circ (g - g_n)^{-1}(-\varepsilon, \varepsilon)^c) \vee (w(-\varepsilon, \varepsilon)^c \wedge a^{\perp})) =$$

$$= m\left(\bigwedge_{i=1}^{\infty} \bigvee_{n=1}^{\infty} [(y \circ (g - g_n^{-1}(-\varepsilon, \varepsilon)^c) \vee (w(-\varepsilon, \varepsilon)^c \wedge a^{\perp})]\right) =$$

$$= \lim_i m\left(\bigvee_{n=1}^{\infty} [(y \circ (g - g_n)^{-1}(-\varepsilon, \varepsilon)^c) \vee (w(-\varepsilon, \varepsilon)^c \wedge a^{\perp})]\right) =$$

$$= \lim_i m\left(\bigvee_{n=1}^{\infty} [y \circ (g - g_n)^{-1}(-\varepsilon, \varepsilon)^c]\right) + [m(w(-\varepsilon, \varepsilon)^c \wedge a^{\perp})] =$$

$$= m(\limsup_n y \circ (g - g_n)^{-1}(-\varepsilon, \varepsilon)^c) + m(w(-\varepsilon, \varepsilon)^c \wedge a^{\perp}) = 0$$

because $g_n \to g$ a.e. $[m_y]$ and $m(0) = 0$, $m(a^{\perp}) = 0$ (from the condition $m(a) = = 1$), too.

This concludes the proof of the theorem.

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МАРТИНГАЛЬНАЯ СХОДИМОСТНАЯ ТЕОРЕМА В КВАНТОВЫХ ЛОГИКАХ

Ján Bán

Резюме

В работе приведено обобщение так называемой мартингальной сходимостной теоремы для квантовых логик.