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HYPERINVARIANT SUBSPACE LATTICE OF WEAK CONTRACTIONS

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1. Introduction

The present paper is a continuation of our preceding work [6]. We follow the notation of [6]. Recall that for the Hilbert space $\mathfrak{H}$, $\mathcal{S}(\mathfrak{H})$ denotes the lattice of all (closed) subspaces of $\mathfrak{H}$. If T is a bounded linear operator on $\mathfrak{H}$, $\mathrm{lat}(T)$ and $\mathrm{hyperlat}(T)$ will denote the invariant and the hyperinvariant subspace lattice of T, respectively.

Let $\{T\}'$ and $\{T\}''$ denote the commutant and the double commutant of T, respectively. Obviously for every $S \in \{T\}''$ $\ker S$ and $\overline{\mathrm{rng}}\, S$ are from $\mathrm{hyperlat}(T)$. In [6] we studied which contractions have the following property:

(L) $\mathrm{hyperlat}(T)$ *is the smallest complete sublattice of* $\mathcal{S}(\mathfrak{H})$ *which contains all subspaces that are of the form* $\ker u(T)$ *or* $\overline{\mathrm{rng}}\, v(T)$ *for* u *and* v *from* H^∞.

Here we shall study a more general property of T:

(L′) $\mathrm{hyperlat}(T)$ *is the smallest complete sublattice of* $\mathcal{S}(\mathfrak{H})$ *which contains all subspaces of the form* $\ker S$ *or* $\overline{\mathrm{rng}}\, V$ *for* S, V *from* $\{T\}''$.

Obviously (L) $\Rightarrow$ (L′).

Let E_n be the n-dimensional Euclidian space and let L^2_n and H^2_n denote the standard Lebesgue and Hardy spaces of E_n-valued functions defined on the unit circle C ($1 \leqq n \leqq \infty$). Instead of e^{it} we use t to denote the argument of a function defined on C. A statement involving t is said to be true if it holds for almost all t with respect to the Lebesgue measure. If F_1 and F_2 are Borel subsets of C, then $F_1 \subset F_2$ means that their difference $F_1 \setminus F_2$ is of the Lebesgue measure zero, $F_1 = F_2$ means that their symmetric difference has the measure zero.

2. C_{11} weak contractions

Let T be a completely non-unitary (c.n.u.) C_{11} weak contraction. As was shown in [1, chap. VIII] its defect indices are equal ($d_T = d_{T^*}$) and its characteristic function Θ_T admits a scalar multiple. We shall consider the functional model of such contraction defined on

$$H=[H_n^2\oplus\overline{\Delta L_n^2}]\ominus\{\Theta_T w\oplus\Delta w:\ w\in H_n^2\}$$

by

$$T(f\oplus g)=P(e^{it}f\oplus e^{it}g)\quad\text{for}\quad f\oplus g\in H,$$

where

$$\Delta(t)=(I-\Theta_T(t)^*\Theta_T(t))^{1/2}$$

and P denotes the orthogonal projection onto H, $n=d_T=d_{T^*}$.

There is a one-to-one correspondence between the invariant subspaces of T and the regular factorizations of Θ_T [1, theorem VII.1.1]. Moreover, the invariant subspace K corresponding to the regular factorization $\Theta_T=\Theta_2\Theta_1$ has the representation

$$K=\{\Theta_2 u\oplus Z^{-1}(\Delta_2 u\oplus v):\ u\in H_m^2,\ v\in\overline{\Delta_1 L_n^2}\}\ominus\{\Theta_T w\oplus\Delta w:\ w\in H_n^2\},$$

where $\Delta_j(t)=(I-\Theta_j(t)^*\Theta_j(t))^{1/2}$, $j=1, 2$, m is the dimension of the intermediate space of this factorization and Z denotes the unitary operator from $\overline{\Delta L_n^2}$ onto $\overline{\Delta_2 L_m^2}\oplus\overline{\Delta_1 L_n^2}$ for which $Z(\Delta v)=\Delta_2\Theta_1 v\oplus\Delta_1 v$ for $v\in L_n^2$.

For c.n.u. C_{11} contractions Sz.-Nagy and Foias [1, chap. VII.5] developed a spectral decomposition. Let H_F be the spectral subspace associated with the Borel subset F of C. Note that H_F is the (unique) invariant subspace corresponding to the regular factorization $\Theta_T=\Theta_2\Theta_1$ satisfying:

(*i*) *Θ_1 is outer.*

(*ii*) *$\Theta_1(t)$ is isometric (hence unitary) for $t\in F'$, the complement of F.*

(*iii*) *$\Theta_2(t)$ is isometric for $t\in F$.*

Recall that

$$H_F=H_{F\cap E},\tag{2.1}$$

where $E=\{t:\ \Theta_T(t)$ is not isometric$\}$.

Radu I. Teodorescu [3] showed that hyperlat(T) consists of all H_F. For any Borel subset $F\subset C$ let

$$K_F=\{f\oplus g\in H:\ -\Delta_* f+\Theta_T g=0 \text{ on } F'\},$$

where $\Delta_*=(I-\Theta_T\Theta_T^*)^{1/2}$. We shall show that $K_F=H_F$.

First we shall prove the following additional properties of the factorization corresponding to H_F.

Lemma 2.1. *Let T be a c.n.u. weak C_{11} contraction and let F be a Borel subset of C. Let $\Theta_T=\Theta_2\Theta_1$ be the regular factorization corresponding to H_F. Then*

(*iv*) *For $t\in C$ there exist $\Theta_T(t)^{-1}$, $\Theta_1(t)^{-1}$, $\Theta_2(t)^{-1}$.*

(*v*) *For* $t \in F$ $\Theta_2(t)$ *is a unitary operator.*
(*vi*) *The intermediate space of this factorization is of the dimension* $n = d_T = d_{T*}$.

Proof. As already mentioned Θ_T admits a scalar multiple $\delta(\neq 0)$. In the proof of [1, theorem VII.6.2] it was shown that Θ_1 admits the scalar multiple δ too. According to [1, proposition V.6.4] δ is a scalar multiple of Θ_2 too. Moreover, since Θ_T is outer we may suppose that δ is outer and then Θ_1 and Θ_2 are both outer [1, theorem V.6.2]. Let Ω be the contractive analytic function such that $\Theta_T\Omega = \Omega\Theta_T = \delta I$. Then $\Theta_T(t)^{-1} = \frac{1}{\delta(t)}\,\Omega(t)$. Similarly also $\Theta_1(t)^{-1}$ and $\Theta_2(t)^{-1}$ do exist.

This proves both (*iv*) and (*vi*). Since Θ_2 is outer, $\overline{\Theta_2(t)H_n^2} = H_n^2$. For $t \in F$ $\Theta_2(t)$ is isometry, hence unitary. And so (*v*) is also proved.

Now we shall show that the proof of the equality $H_F = K_F$ in [4, §3], where only C_{11} contractions with finite defect indices were considered, applies to c.n.u. weak contractions (with not necessarily finite defect indices) with only a few changes.

Lemma 2.2. *For any Borel subset* $F \subset C$
(*a*) $K_F \in \mathrm{lat}(T)$
(*b*) $K_{F\cap E} = K_F$
(*c*) *If* $\{F_m\}$ *is a sequence of Borel subsets of* C *and* $F = \bigcap_m F_m$, *then* $K_F = \bigcap_m K_{F_m}$.

Proof. The proof of [4, lemmas 3.1 and 3.2] applies to our case without any change.

Lemma 2.3. *For any Borel subset* $F \subset C$, $H_F \subset K_F$.

Proof. By lemma 2.2(*b*) and by (2.1) we may assume that $F \subset E$. Let $\Theta_T = \Theta_2\Theta_1$ be the regular factorization corresponding to H_F. By lemma 2.1(*vi*) the intermediate space of this factorization has the dimension $n = d_T = d_{T*}$. Hence

$$H_F = \{\Theta_2 u \oplus Z^{-1}(\Delta_2 u \oplus v) \colon\ u \in H_n^2,\ v \in \overline{\Delta_1 L_n^2}\} \ominus \{\Theta_T w \oplus \Delta w \colon\ w \in H_n^2\}.$$

Recall that by (*ii*) for $t \in F'$ $\Theta_1(t)$ is unitary, hence $\Delta_1(t) = 0$. Let

$$\Theta_2 u \oplus Z^{-1}(\Delta_2 u \oplus v) \in H_F.$$

Since on F' $Z^{-1}(\Delta_2 u \oplus v) = Z^{-1}(\Delta_2 u \oplus 0) = \Delta\Theta_1^* u$ and $\Theta_T\Delta = \Delta_*\Theta_T$ we have on F':

$$\begin{aligned} -\Delta_*\Theta_2 u + \Theta_T Z^{-1}(\Delta_2 u \oplus v) &= -\Delta_*\Theta_2 u + \Theta_T\Delta\Theta_1^* u = \\ &= -\Delta_*\Theta_2 u + \Delta_*\Theta_2\Theta_1\Theta_1^* u = 0. \end{aligned}$$

This shows that $\Theta_2 u \oplus Z^{-1}(\Delta_2 u \oplus v) \in K_F$, and so $H_F \subset K_F$.

The proof of the following lemma is also the same as the proof of the corresponding lemma 3.4 of [4].

Lemma 2.4. *Let $O_T - \Theta_2 O_1$ be the regular factorization corresponding to H_F. If there exists an $M>0$ such that (for almost all t) $\|\Theta_2(t)^{-1}\| \leqq M$, then $H_F = K_F$.*

Theorem 2.5. *Let T be a c.n.u. weak contraction. For any Borel subset $F \subset C$ let H_F and K_F be defined as before. Then $H_F = K_F$.*

Proof. Let $O_T = \Theta_2 O_1$ be the regular factorization corresponding to H_F. For each positive integer m let

$$F_m = \{t: \|O_2(t)^{-1}\| > m\} \cup F.$$

Then $\bigcap_m F_m = F$. Accordıng to [1, theorem VII.6.2] $\bigcap_m H_{F_m} = H_F$ and by lemma 2.2(c) we have $\bigcap_m K_{F_m} - K_F$. Thus to complete the proof it suffices to show that $H_{F_m} = K_{F_m}$ for all m.

Let $\Theta_T = O_{2m}\Theta_{1m}$ be the regular factorızation corresponding to H_{F_m}. Since $F \subset F_m$, $H_F \subset H_{F_m}$. Hence there exıst a contractive analytic function Ω_m such that $\Theta_{1m} = \Omega_m O_1$ [1, proposıtion VII.2.4]. Hence $\Theta_T = O_{2m}\Omega_m\Theta_1 = \Theta_2\Theta_1$, since Θ_1 is outer, then $\Theta_2 = O_{2m}\Omega_m$ By lemma 2.1(iv) both $O_2(t)$ and $\Theta_{2m}(t)$ are invertible (for almost all t) We h ve

$$\|O_{2m}(t)^{-1}\| - \|\Omega_m(t)\Omega_2(t)\ ^1\| \leq \|\Theta_2(t)\ ^1\| \leqq m$$

for $t \in F'_m$. By lemma 2.1(v) for $t \in F_m$ $O_{2m}(t)$ is unitary and so $\|\Theta_{2m}(t)\ ^1\| = 1$. Hence (for almost all t) $\|O_{2m}(t)\ ^1\| \leqq m$. Applying lemma 2.4 we have $H_{F_m} = K_{F_m}$ and consequently $H_F - K_F$.

Theorem 2.6. *Let T be a c.n u. weak contraction of the class C_{11} defined on*

$$H = [H_n^2 \oplus \overline{\Delta L_n^2}] \ominus [\Theta_T w \oplus \Delta w:\ w \in H_n^2].$$

Let $K \in \mathscr{S}(H)$. Then the following are equivalent to each other
(1) $K \in \text{hyperlat}(T)$
(2) $K = \ker S$ *for some* $S \in \{T\}''$
(3) $K - \overline{\text{rng}}\, V$ *for some* $V \in \{T\}'$,
hence T has property (L').

We have ju t proved that $H_F = K_F$. Every hyperinvariant sub pace for T is of the form H_F [3, proposition 3]. And so the proof of this theorem in the case of finite defect indices [4, theor m 3.6] applies to our case too.

3. G neral c.n u. weak c ntraction

P.Y. WU showed [5, theorem 8] that every c n.u. weak contraction with fınite defect indices has the property (L′). Using the results of § 2 and of [6] it will now be easy to show that all c n u weak contractıons have the property (L′).

For a c.n.u. weak contraction T on $\mathfrak{H}$ we can consider its $C_0 - C_{11}$ decomposition [1, chap. VIII.2]. Let $\mathfrak{H}_0$, $\mathfrak{H}_1$ be the invariant subspaces for T such that $T_0 = T|\mathfrak{H}_0$ and $T_1 = T|\mathfrak{H}_1$ are the C_0 and the C_{11} parts of T, respectively. $\mathfrak{H}_0$ and $\mathfrak{H}_1$ are even hyperinvariant for T and

$$\mathfrak{H}_0 \vee \mathfrak{H}_1 = \mathfrak{H}, \quad \mathfrak{H}_0 \cap \mathfrak{H}_1 = \{0\}. \tag{3.1}$$

Moreover by [1, proposition VIII.2.4]

$$\mathfrak{H}_0 = \ker m(T), \quad \mathfrak{H}_1 = \overline{\operatorname{rng}}\, m(T), \tag{3.2}$$

where m is the minimal function of T_0. Note that $m(T) \in \{T\}''$. By [5, theorem 1] there exists also $S \in \{T\}''$ such that

$$\mathfrak{H}_0 = \overline{\operatorname{rng}}\, S \quad \mathfrak{H}_1 = \ker S \tag{3.3}$$

Lemma 3.1. *Let $\mathfrak{H}_0, \mathfrak{H}_1 \in \mathscr{S}(\mathfrak{H})$ be such that $T_0 = T|\mathfrak{H}_0$ and $T_1 = T|\mathfrak{H}_1$ are the C_0 and the C_{11} parts of T, respectively, let $S \in \{T\}''$ be such that* (3.3) *holds and let m be the minimal function of T_0.*

If $S_0 \in \{T_0\}''$, $S_1 \in \{T_1\}''$, then $S_0 S \in \{T\}''$, $S_1 m(T) \in \{T\}''$ and

(*i*) $\ker S_0 = \ker S_0 S \cap \overline{\operatorname{rng}}\, S$, $\overline{\operatorname{rng}}\, S_0 = \overline{\operatorname{rng}}\, S_0 S$

(*ii*) $\ker S_1 = \ker S_1 m(T) \cap \overline{\operatorname{rng}}\, m(T)$, $\overline{\operatorname{rng}}\, S_1 = \overline{\operatorname{rng}}\, S_1 m(T)$

Proof. Let $V \in \{T\}'$, since $\mathfrak{H}_0$, $\mathfrak{H}_1$ are from hyperlat(T), $V\mathfrak{H}_0 \subset \mathfrak{H}_0$, $V\mathfrak{H}_1 \subset \mathfrak{H}_1$. Let $V_0 = V|\mathfrak{H}_0$, $V_1 = V|\mathfrak{H}_1$, obviously

$$V_0 T_0 = T_0 V_0, \quad V_1 T_1 = T_1 V_1$$

and so

$$S_0 V_0 = V_0 S_0, \quad S_1 V_1 = V_1 S_1.$$

For $h_0 \in \mathfrak{H}_0$ we have then

$$S_0 S V h_0 = S_0 V S h_0 = S_0 V_0 S h_0 = V_0 S_0 S h_0 = V S_0 S h_0$$

and similarly for $h_1 \in \mathfrak{H}_1$ $S_0 S V h_1 = V S_0 S h_1$. This shows that $S_0 S \in \{T\}''$; $S_1 m(T) \in \{T\}''$ can be shown in the same way.

$S \in \{T\}'' \subset \{T\}'$. It follows that $S|\mathfrak{H}_0 \in \{T_0\}'$, $S|\mathfrak{H}_1 \in \{T_1\}'$. Let $h_0 \in \ker S_0$. Then $S_0 S h_0 = S_0(S|\mathfrak{H}_0) h_0 = (S|\mathfrak{H}_0) S_0 h_0 = 0$, together with (3.3) this shows that $\ker S_0 \subset \ker S_0 S \cap \overline{\operatorname{rng}}\, S$. Let $h_0 \in \ker S_0 S \cap \overline{\operatorname{rng}}\, S$; then $S_0 S h_0 = S S_0 h_0 = 0$ and by (3.1) and (3.3) $S_0 h_0 = 0$.

$\overline{\operatorname{rng}}\, S_0 = \overline{S_0 \mathfrak{H}_0} = \overline{S_0 \overline{S \mathfrak{H}}} = \overline{S_0 S \mathfrak{H}}$ and so (*i*) is proved. Using (3.1) and (3.2) (*ii*) can be proved in the same way.

Theorem 3.2. *Every c.n.u. weak contraction has the property* (L′).

Proof. Let $\mathfrak{H}_0$, $\mathfrak{H}_1$, T_0, T_1, S and m be as in the preceding lemma. Let $K\in\text{hyperlat}(T)$. If z does not belong to the spectrum of T, then $(z-T)^{-1}$ commutes with T, it follows that $(z-T)^{-1}|K=(z-T|K)^{-1}$. This shows that $T|K$ is also a c.n.u. weak contraction and we may consider its C_0 — part $T|K_0$ and its C_{11} — part $T|K_1$. According to [1, proposition VIII.2.2] $K_0=K\cap\mathfrak{H}_0$, $K_1=K\cap\mathfrak{H}_1$. As was shown in the proof of [5, theorem 3] $K_0\in\text{hyperlat}(T_0)$, $K_1\in\text{hyperlat}(T_1)$. It follows by theorem 2.6, by [6, corollary 3.4] and by lemma 3.1 that T has the property (L′).

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РЕШЕТКА ГИПЕРИНВАРИАНТНЫХ ПОДПРОСТРАНСТВ СЛАБЫХ СЖАТИЙ

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Резюме

Рассматриваются решетки гиперинвариантных подпространств для вполне неунитарных слабых сжатий *T*. Показано, что решетка гиперинвариантных подпространств такого оператора порождена замыканиями областей значений и ядрами операторов из бикоммутанта *T*. Это обобщает результат Ву [5] (для сжатий с конечными дефектными индексами).