František Kmeť

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ON RADICALS OF THE SEMIGROUP OF TRIANGULAR MATRICES

FRANTIŠEK KMEŤ

Let S be the multiplicative semigroup of all $m\times m$ upper triangular matrices over a ring T, i. e. matrices of the form

$$A=\begin{bmatrix} a_1 & a_{12} & \dots & a_{1m} \\ 0 & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & a_{mm} \end{bmatrix}$$

where $a_{ik}\in T$, i, $k=1, 2, \dots, m$ and $a_{ik}=0$ for $i>k$.

Let U be a semigroup and J be a two-sided ideal of U. Denote by $R_J(U)$, $M_J(U)$, $L_J(U)$, $R_J^*(U)$, $C_J(U)$ and $N_J(U)$ respectively the radical of Schwarz, McCoy, Ševrin, Clifford, Luh and the set of all nilpotent elements of U with respect to J.

R. Šulka [5, Theorem 7] proved that in a commutative semigroup U we have: $R_J(U) = M_J(U) = R_J^*(U) = N_J(U) = C_J(U)$. The same results were obtained by J. Kuczkowski [2] for C_2-semigroups and by H. Lal [3] for quasi-commutative semigroups.

J. Bosák [1, Theorem 2] proved that for the radicals of an arbitrary semigroup U we have:

$$R_J(U)\subseteq M_J(U)\subseteq L_J(U)\subseteq R_J^*(U)\subseteq N_J(U)\subseteq C_J(U).$$

The purpose of this paper is to prove that in the semigroup S of all $m\times m$ upper triangular matrices over a commutative ring T we have: $R(S) = M(S) = L(S) = R^*(S) = N(S)$.

We introduce some definitions. Let U be a semigroup with a zero O and all ideals considered in the following two-sided.

An element $x\in U$ is called nilpotent (nilpotent with respect to J) if for some positive integer n: $x^n=O (x^n\in J)$.

An ideal (subsemigroup) I of U is called nilpotent (nilpotent with respect to J) if for some positive integer n: $I^n=O$ $(I^n\subseteq J)$.

An ideal P of U is called prime if for any two ideals A and B of U, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

An ideal of U, each element of which is nilpotent (nilpotent with respect to J), is called a nilideal (nilideal with respect to J).

An ideal I of U, each finitely generated subsemigroup of which is nilpotent (with respect to J), is called locally nilpotent.

The set of all nilpotent elements of U (with respect to J) will be denoted by $N(U)$ $(N_J(U))$.

The union $R(U)$ $(R_J(U))$ of all nilpotent ideals of U (with respect to J) is called the Schwarz radical of U (with respect to J).

The union $L(U)$ $(L_J(U))$ of all locally nilpotent ideals of U (with respect to J) is called the Ševrin radical of U (with respect to J).

The intersection $M(U)$ $(M_J(U))$ of all prime ideals of U (which contain J) is called the McCoy radical of U (with respect to J).

The union $R^*(U)$ $(R^*_J(U))$ of all nilideals of U (with respect to J) is called the Clifford radical of U (with respect to J).

Denote by N_0 the set of all matrices of S with a zero diagonal.

Lemma 1. *Let S be the semigroup of all $m \times m$ upper triangular matrices over a ring T. Then N_0 is a nilpotent ideal of S and $N_0^m = O$.*

Proof. Let $A \in N_0$, $B, C \in S$ be arbitrary matrices. Then BA, AC are matrices with the zero diagonal, i. e. $BA \in N_0$, $AC \in N_0$ and hence N_0 is an ideal of S. The set N_0^2 is evidently an ideal of S. If $A, B \in N_0$ are two arbitrary matrices, then in the nth row $(n = 1, 2, \ldots, m-1)$ of the matrix AB the first possible non-zero element is equal to $a'_{n,n+2} = a_{n,n+1}b_{n+1,n+2}$. Therefore the ideal N_0^2 is contained in the set of matrices of the form:

$$\begin{pmatrix} 0 & 0 & a_{13} & \ldots & a_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 0 & a_{m-2,m} \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

Similarly, the ideal N_0^s for $s = 3, \ldots, m-1$ is contained in the set of matrices of the form:

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & a_{1,s+1} & \ldots & a_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & \ldots & \ldots & 0 & a_{m-s,m} \\ 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \end{pmatrix}$$

and so $N_0^m = O$.

Lemma 2. *Let S be the semigroup of al upper triangular $m \times m$ matrices over a commutative ring T and let A be a matrix of S. Then A is a nilpotent matrix if and only if all its diagonal elements $a_{11}, \ldots, a_{mm}$ are nilpotent elements of T.*

Proof. If A is a nilpotent matrix of S and $A^r = O$ for a positive integer r, then we have $a_{11}^r = a_{22}^r = \ldots = a_{mm}^r = 0$. Conversely, if $a_{11}^{r_1} = a_{22}^{r_2} = \ldots = a_{mm}^{r_m} = 0$, then A^r with $r = \max(r_1, r_2, \ldots, r_m)$ is a matrix with a zero diagonal, so $A^r \in N_0$ and hence by Lemma 1 we have $A^{rm} = O$.

Lemma 3. *Let S be the semigroup of all upper triangular $m \times m$ matrices over a commutative ring T. Then the set $N(S)$ is a nilideal of S and $R^*(S) = N(S)$.*

Proof. We shall show that the set $N(S)$ of all nilpotent matrices of S is an ideal of S.

Let $A \in N(S)$, $B, C \in S$ be arbitrary matrices and let $a_{11}^r = a_{22}^r = \ldots = a_{mm}^r = 0$. Then the diagonal elements $(b_{11}a_{11})^r, \ldots, (b_{mm}a_{mm})^r$ of the matrix $(BA)^r$ are equal to zero and so $(BA)^r \in N_0$. But $(BA)^r \in N_0$ implies by Lemma 1 that $(BA)^{rm} = O$ and hence $BA \in N(S)$. Analogously $(AC)^r \in N_0$ implies $AC \in N(S)$.

Then from the definition of the Clifford radical we obtain $N(S) \subseteq R^*(S)$. Conversely $R^*(S) \subseteq N(S)$ is true for any semigroup with zero and therefore $N(S) = R^*(S)$ holds.

Remark. In the semigroup S of all upper triangular $m \times m$ matrices over a non-commutative ring the set $N(S)$ in general does not form an ideal and only $R^*(S) \subset N(S)$ is true.

For example, let K_2 be the ring of all 2×2 matrices over a commutative ring K and let U be the multiplicative semigroup of all $m \times m$ triangular matrices over K_2. Consider the matrices

$$C_1 = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \in K_2, \quad C_2 = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \in K_2,$$

with $c^2 = c \neq 0$, $c \in K$. Then the product AB of two nilpotent matrices

$$A = \begin{bmatrix} C_1 & 0 & \ldots & 0 \\ 0 & C_1 & \ldots & 0 \\ & & & \\ 0 & 0 & \ldots & C_1 \end{bmatrix}, \qquad B = \begin{bmatrix} C_2 & 0 & \ldots & 0 \\ 0 & C_2 & \ldots & 0 \\ & & & \\ 0 & \ldots & 0 & C_2 \end{bmatrix}, \quad (0 \in K_2)$$

of U is equal to the matrix

$$AB = \begin{bmatrix} C & 0 & \ldots & 0 & 0 \\ 0 & C & \ldots & 0 & 0 \\ & & & & \\ 0 & 0 & \ldots & 0 & C \end{bmatrix}, \quad 0 \in K_2, \quad C = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix},$$

and $AB \notin N(U)$. The set $N(U)$ of all nilpotent matrices of the semigroup U cannot be an ideal of U and therefore by [1] we have only $R^*(U) \subset N(U)$.

Lemma 4. *Let S be the semigroup of all upper triangular $m \times m$ matrices over a commutative ring T. Then $R(S) = R^*(S)$.*

Proof. In an arbitrary semigroup with zero we have $R(S) \subseteq R^*(S)$. We shall prove that $R^*(S) \subseteq R(S)$.

It is sufficient to show that for each $A \in R^*(S)$ the principal ideal (A) is a nilpotent ideal of S. If $A \in R^*(S)$ and $A^r = 0$ for a positive integer r, then there exists a least positive integer $s \leqq r$ such that A^s is a matrix with a zero diagonal. Then an arbitrary matrix $C = B_1AB_2A \dots B_sAB_{s+1}$ of the ideal $(A)^s$ (where some of B_i can be empty) is a matrix with a zero diagonal and hence $(A)^s \subseteq N_0$. Since by Lemma 1 we have $(A)^{sm} = 0$, this implies $(A) \subseteq R(S)$ and hence $A \in R(S)$.

From the inclusions between radicals in an arbitrary semigroup: $R(S) \subseteq M(S) \subseteq L(S) \subseteq R^*(S) \subseteq N(S)$, from Lemma 3 and Lemma 4 we obtain

Theorem. *Let S be the semigroup of all upper triangular $m \times m$ matrices over a commutative ring T. Then*

$$R(S) = M(S) = L(S) = R^*(S) = N(S).$$

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Katedra matematiky PEF
Vysokej školy poľnohospodárskej
Mostná 16
949 01 Nitra

РАДИКАЛЫ ПОЛУГРУППЫ ТРЕУГОЛЬНЫХ МАТРИЦ

Франтишек Кметь

Резюме

В мультипликативной полугруппе треугольных матриц над коммутативным кольцом радикалы Шварца, Маккойа, Шеврина, Клиффорда и Луга равны множеству всех нильпотентных элементов полугруппы.