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RENEWAL THEOREMS FOR RANDOM WALKS IN MULTIDIMENSIONAL TIME

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ABSTRACT. Suppose that $\{X, X_n : n \in K_r\,,\ r \geq 1\}$ is a family of nonnegative, independent and identically distributed random variables where K_r is the set of r-tuples of positive integers. Let $\{S_n : n \in K_r\}$ denote the corresponding random walk in r-dimensional time. The paper deals with the asymptotic properties of its renewal function when either the mean of X is finite or when the distribution function F of X satisfies $1 - F(x) = x^{-\alpha}L(x)$ where L is a slowly varying function.

1. Introduction and main results

For $r \geq 1$, let K_r denote the set of r-tuples $n = (n_1, n_2, \dots, n_r)$ where the components n_j are positive integers. Let $\{X, X_n : n \in K_r\}$ be a family of nonnegative, independent and identically distributed random variables. For $m = (m_j)$ and $n = (n_j)$ in K_r, we say that $m \leq n$ if $m_j \leq n_j$ for all $1 \leq j \leq r$. If $n \in K_r$, define

$$S_n = \sum_{j \leq n} X_j\,.$$

The sequence $\{S_n : n \in K_r\}$ is called a *random walk in r-dimensional time.* Note that when $r = 1$, $\{X_n : n \in K_r\}$ reduces to the sequence $\{X_n : n \geq 1\}$ of independent random variables each distributed as X and S_n becomes the ordinary partial sum, $S_n = \sum\limits_{j=1}^{n} X_j$.

Define the function $U_r(x)$ by

$$U_r(x) = \sum_{n \in K_r} P\{S_n \leq x\}\,. \tag{1}$$

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$U_r(x)$ is called the *renewal function of the random walk* $\{S_n : n \in K_r\}$ *in* r*-dimensional time.*

Note that if for $n \geq 1$, $d_r(n)$ is the number of r-tuples $m = (m_1, m_2, \ldots, m_r)$ in K_r with $m_1 m_2 \ldots m_r = n$, then $U_r(x)$ can be expressed equivalently by

$$U_r(x) = \sum_{n=1}^{\infty} d_r(n) P\{S_n \leq x\} \tag{2}$$

where $S_n = X_1 + X_2 + \cdots + X_n$ is the ordinary partial sum of n independent copies X_j of X.

In the present paper we study the asymptotic properties of $U_r(x)$ as $x \to \infty$ which is one of the fundamental problems in renewal theory in the classical case of one dimensional time. For the case of multidimensional time this question has been studied by several authors for an integer-valued random walk and

$$\Delta U_r(k) = U_r(k) - U_r(k-1) = \sum_{n \in K_r} P\{S_n = k\}.$$

For two dimensional time, Ney and Wainger [8] proved that if X is integer valued, and aperiodic with $E(X^4) < \infty$,

$$\Delta U_r(k) \sim \frac{1}{\mu} \log k \qquad \text{as} \quad k \to \infty \quad \text{where } \mu = E(X).$$

M. Maejima and T. Mori [7] replaced the moment condition by $E|X|^3 < \infty$ and proved that for $r = 2$ or $r = 3$,

$$\Delta U_r(k) \sim \frac{(\log k)^{r-1}}{(r-1)!\,\mu} \qquad \text{as} \quad k \to \infty. \tag{3}$$

An extension of the result of Maejima and Mori, which also gave an error term in (3) was obtained by Galambos and Katai [3] which was also later proved under a weaker assumption by Galambos and Katai [4]. The best result, however, was obtained by Galambos, Indlekofer and Katai [2]. They proved that (3) holds for X integer valued, aperiodic with finite mean μ and finite positive variance and for $r = 2$ or 3.

The main results of this paper are the following theorems which give the asymptotic behavior of $U_r(x)$ as $x \to \infty$ for the cases when the mean of X is finite or infinite.

THEOREM 1. *Let* $X, X_1, X_2, \ldots$ *be nonnegative, independent, identically distributed random variables such that* $0 < \mu = EX < \infty$. *Then, as* $x \to \infty$,

$$U_r(x) \sim \frac{1}{(r-1)!\,\mu} x (\log x)^{r-1} \qquad \textit{for} \quad r = 1, 2, 3 \ldots.$$

Before we state the result of Theorem 2, we give the following:

Definition. A positive measurable function $L(x)$ defined on (A, ∞), $A \geq 0$, is said to be *slowly varying* if

$$\lim_{t \to \infty} \frac{L(tx)}{L(t)} = 1 \qquad \text{for all} \quad x > 0 .$$

Theorem 2. *Let $X, X_1, X_2, \ldots$ be nonnegative, independent, identically distributed random variables. Suppose that the distribution function F of X satisfies $1 - F(x) = x^{-\alpha} L(x)$, as $x \to \infty$, where $0 \leq \alpha \leq 1$ and L is a slowly varying function. Then, as $x \to \infty$,*

(i)

$$U_r(x) \sim \frac{1}{(r-1)!} \frac{\left(\log\left(\frac{1}{L(x)}\right)\right)^{r-1}}{L(x)} \qquad \textit{if} \quad \alpha = 0 ,$$

(ii)

$$U_r(x) \sim \frac{\alpha^{r-1}}{(r-1)!\,\Gamma(1-\alpha)\Gamma(1+\alpha)} \frac{x^{\alpha}(\log x)^{r-1}}{L(x)} \qquad \textit{if} \quad 0 < \alpha < 1 ,$$

(iii)

$$U_r(x) \sim \frac{1}{(r-1)!} \frac{x(\log x)^{r-1}}{m(x)} \qquad \textit{if} \quad \alpha = 1 \quad \textit{where } m(x) = \int_0^x (1 - F(y))\, dy .$$

Remarks.

(i) When $r = 1$, $U_r(x)$ becomes $U(x)$, the renewal function in one dimension and Theorem 1 reduces to the classical elementary renewal theorem.

(ii) In the case when $\alpha = 1$, $\mu = EX$ may or may not be finite. However, when $\alpha = 1$ and $\mu < \infty$, the result of (iii) in Theorem 2 becomes the same as that of Theorem 1.

The following theorem is a generalization of Theorem 1.

Theorem 3. *Let $a_n \geq 0$ such that $A(n) = \sum_{k=1}^{n} a_k \sim n^p L(n)$ as $n \to \infty$ for $p \geq 0$ where L is slowly varying. Suppose $X, X_1, X_2, \ldots$ are nonnegative, independent, identically distributed random variables with $0 < EX = \mu < \infty$. If $S_n = X_1 + X_2 + \cdots + X_n$, and*

$$G(x) = \sum_{n=1}^{\infty} a_n P\{S_n \leq x\} ,$$

then

$$G(x) = EA(N(x)) \sim \frac{1}{\mu^p} x^p L(x) \qquad \textit{as} \quad x \to \infty .$$

2. Proofs

Before we give the proofs of the above theorems, we make the following observations.

In the study of the asymptotic behavior of $\Delta U_r(k)$ or $U_r(x)$, the principal role is played by the divisor function $d_r(n)$. It is known that $d_r(n) = O(n^\varepsilon)$ for any $\varepsilon > 0$. For $r = 2$, see for example Hardy and Wright [6; p. 260, Theorem 315], and the case $r > 2$ follows by induction.

If we put

$$D_r(x) = \sum_{n \le x} d_r(n),$$

then it is known (see Titchmarsh [10; p. 313]) that

$$D_r(x) = xP_r(\log x) + \Delta_r(x) \tag{4}$$

where

$$P_r(u) = \frac{1}{(r-1)!}u^{r-1} + a_{r-2}u^{r-2} + \cdots + a_0,$$

whose coefficients a_j can explicitly be evaluated from the Laurent series of the Riemann zeta function at $s = 1$, and $\Delta_r(x) = O(x^v)$ for some $0 < v < 1$. Thus, from (4) we see that

$$D_r(x) \sim \frac{1}{(r-1)!}x(\log x)^{r-1} \qquad \text{as} \quad x \to \infty. \tag{5}$$

Next, if we define the renewal counting process $\{N(t) : t \ge 0\}$ by

$$N(t) = \max\{n : S_n \le t\},$$

then the following strong law holds.

Lemma 1. *Let $0 < \mu = E(X_1) \le \infty$. Then*

(i) $\frac{N(t)}{t} \to \frac{1}{\mu}$ *a.s. as* $t \to \infty$,

(ii) $\left\{\left(\frac{N(t)}{t}\right)^r : t \ge 1\right\}$ *is uniformly integrable for all* $r > 0$,

(iii) $E\left(\frac{N(t)}{t}\right)^r \to \frac{1}{\mu^r}$ *as* $t \to \infty$ *for all* $r > 0$.

The proof may be found in Gut [5; p 54, Theorem 5.1].

From the above definition of $N(t)$ it follows that $P\{S_n \le t\} = P\{N(t) \ge n\}$ and so we see that:

$$U_r(t) = \sum_{k=1}^{\infty} D_r(k)P\{N(t) = k\} = ED_r(N(t)). \tag{6}$$

We now give a proof of Theorem 1 using the above representation of $U_r(t)$.

Proof of Theorem 1. From (4) we can see that $D_r(x) = f(x) + g(x) + h(x)$ where $f(x) = \frac{1}{(r-1)!}x(\log x)^{r-1}$, $g(x) = a_{r-2}x(\log x)^{r-2} + \cdots + a_1 x\log x + a_0 x$, and $h(x) \le Cx^v$ for some $C > 0$ and $0 < v < 1$.

Hence we see that

$$D_r(N(t)) = f(N(t)) + g(N(t)) + h(N(t)).$$

We now claim that as $t \to \infty$,

(i) $E\left(\frac{f(N(t))}{f(t)}\right) \to \dfrac{1}{\mu}$,

(ii) $E\left(\frac{g(N(t))}{f(t)}\right) \to 0$,

(iii) $E\left(\frac{h(N(t))}{f(t)}\right) \to 0$.

The above together with (6) will then imply that,

$$U_r(t) \sim \frac{1}{\mu} f(t) = \frac{1}{\mu(r-1)!}\, t(\log t)^{r-1} \qquad \text{as} \quad t \to \infty$$

which is what we want to prove.

To prove (i), we observe that

$$\frac{f(N(t))}{f(t)} = \frac{N(t)}{t}\left(\frac{\log N(t)}{\log t}\right)^{r-1}.$$

Since $\frac{N(t)}{t} \to \frac{1}{\mu}$ a.s. as $t \to \infty$, it follows that $\frac{\log N(t)}{\log t} \to 1$ a.s. as $t \to \infty$. Furthermore for $t \ge \mathrm{e}$,

$$\left(\frac{\log N(t)}{\log t}\right)^{r-1} \le \left(1 + \frac{N(t)}{t}\right)^{r-1} \le 2^{r-1}\left(1 + \left(\frac{N(t)}{t}\right)^{r-1}\right).$$

Hence we see that for $t \ge \mathrm{e}$,

$$\frac{f(N(t))}{f(t)} \le 2^{r-1}\left(\frac{N(t)}{t} + \left(\frac{N(t)}{t}\right)^{r}\right).$$

Thus, using Lemma 1(ii) we conclude that $\left\{\frac{f(N(t))}{f(t)} \;:\; t \ge \mathrm{e}\right\}$ is uniformly integrable for all $r > 0$. Since $\frac{f(N(t))}{f(t)} \to \frac{1}{\mu}$ a.s. as $t \to \infty$, we see that (i) holds.

Next we consider,

$$\frac{g(N(t))}{f(t)} = \left(\frac{b_{r-2}}{\log N(t)} + \cdots + \frac{b_0}{(\log N(t))^{r-1}}\right)\frac{f(N(t))}{f(t)} \tag{7}$$

where $b_k = (r-1)!\,a_k$ for $k = 0, 1, \dots, r-2$. However, for $k \geq 1$ and large t

$$\frac{1}{\left(\log N(t)\right)^k} \frac{f\left(N(t)\right)}{f(t)} \leq \frac{f\left(N(t)\right)}{f(t)},$$

and so $\left\{ \frac{f(N(t))}{f(t)} \frac{1}{(\log N(t))^r} \right\}$ is uniformly integrable for all $r > 0$. Now since $\frac{1}{(\log N(t))^k} \frac{f(N(t))}{f(t)} \to 0$ a.s. as $t \to \infty$, for all $k \geq 1$, we see that

$$E\left(\frac{1}{\left(\log N(t)\right)^k} \frac{f\left(N(t)\right)}{f(t)} \right) \to 0 \qquad \text{as} \quad t \to \infty .$$

Hence, from (7) we conclude that (ii) holds.

Finally, to prove (iii) observe that

$$\frac{h\left(N(t)\right)}{f(t)} \leq \frac{C}{\left(N(t)\right)^{1-v} \left(\log N(t)\right)^{r-1}} \frac{f\left(N(t)\right)}{f(t)} \qquad \text{where} \quad C > 0 . \tag{8}$$

Note that the expression on the right side of (8) is uniformly integrable for $r > 0$. Moreover, as $t \to \infty$ this expression converges to 0 a.s. Thus we conclude that (iii) holds and this proves Theorem 1. □

In order to prove Theorem 2 we need the following two lemmas.

Lemma 2.

(i) *If L is a slowly varying function, then* $\lim\limits_{x \to \infty} \frac{\log L(x)}{\log x} = 0$.

(ii) *If L_1 and L_2 are slowly varying and $L_2(x) \to \infty$ as $x \to \infty$, then $L_1\left(L_2(x)\right)$ is slowly varying.*

For the proof see Seneta [9; pp. 18–19].

Lemma 3. *Suppose $1 - F(x) = x^{-\alpha} L(x)$ as $x \to \infty$ where $0 \leq \alpha \leq 1$, L is slowly varying and $m(t) = \int_0^t \left(1 - F(y)\right) \mathrm{d}y$. Then, as $s \to 0^+$*

(i) $\int_0^\infty \mathrm{e}^{-st} \left(1 - F(t)\right) \mathrm{d}t \sim \Gamma(1-\alpha) s^{\alpha - 1} L(1/s)$ *for* $0 \leq \alpha < 1$,

(ii) $\int_0^\infty \mathrm{e}^{-st}\, m(t)\, \mathrm{d}t \sim \frac{1}{s} m(1/s)$ *for* $\alpha = 1$.

P r o o f.

(i) Let $0 \le \alpha < 1$. Then,

$$\int_0^\infty e^{-st}\big(1 - F(t)\big)\, dt = \int_0^\infty e^{-st}\, t^{-\alpha} L(t)\, dt$$
$$= s^{\alpha-1} \int_0^\infty e^{-u}\, u^{-\alpha} L(u/s)\, du\,.$$

Since $\frac{L(u/s)}{L(1/s)} \to 1$ as $s \to 0^+$ for all $u > 0$, by the dominated convergence theorem it follows that

$$\int_0^\infty e^{-u}\, u^{-\alpha} L(u/s)\, du \sim L(1/s) \int_0^\infty e^{-u}\, u^{-\alpha}\, du \qquad \text{as} \quad s \to 0^+\,.$$

Hence,

$$\int_0^\infty e^{-st}\big(1 - F(t)\big)\, dt \sim s^{\alpha-1} L(1/s) \int_0^\infty e^{-u}\, u^{-\alpha}\, du \qquad \text{as} \quad s \to 0^+$$
$$= \Gamma(1-\alpha) s^{\alpha-1} L(1/s)\,.$$

(ii) Suppose $\alpha = 1$. Then, $m(t)$ is slowly varying and the result holds by (i) of the Lemma for $\alpha = 0$. □

We now give a proof of Theorem 2.

P r o o f o f T h e o r e m 2. Let $f(s)$ and $\varphi(s)$ be the Laplace-Stieltjes transforms of $F(x)$ and $U_r(x)$, respectively. That is,

$$f(s) = \int_0^\infty e^{-sx}\, dF(x) \qquad \text{and} \qquad \varphi(s) = \int_0^\infty e^{-sx}\, dU_r(x)\,.$$

If we let,

$$Q(s) = \sum_{k=1}^{\infty} d_r(k) s^k$$

then we observe that $\varphi(s) = Q\big(f(s)\big)$. From (5), and Karamata's Tauberian theorem (see F e l l e r [1; p. 447, Theorem 5] it follows that

$$Q(s) \sim \frac{1}{(r-1)!}\frac{1}{1-s}\left(\log\left(\frac{1}{1-s}\right)\right)^{r-1} \qquad \text{as} \quad s \to 1^-\,.$$

Since $f(s) \to 1^-$ as $s \to 0^+$, the above implies that

$$\varphi(s) \sim \frac{1}{(r-1)!}\frac{1}{1-f(s)}\left(\log\left(\frac{1}{1-f(s)}\right)\right)^{r-1} \qquad \text{as} \quad s \to 0^+ . \tag{9}$$

We now consider asymptotic relations of $1-f(s)$ as $s \to 0^+$. If $0 \le \alpha < 1$, then we see that $1-f(s) = s\int_0^\infty e^{-st}(1-F(t))\,dt$ and so by Lemma 3 we get

$$1-f(s) \sim \Gamma(1-\alpha)s^\alpha L(1/s) \qquad \text{as} \quad s \to 0^+ . \tag{10}$$

On the other hand, for $\alpha = 1$ we have $1-f(s) \sim s^2\int_0^\infty e^{-st}\,m(t)\,dt$. Hence from Lemma 3 we conclude that for $\alpha = 1$,

$$1-f(s) \sim sm(1/s) \qquad \text{as} \quad s \to 0^+ . \tag{11}$$

We now consider the following three cases.

Case (i). Suppose $0 < \alpha < 1$. Then, from (9) and (10) we get

$$\varphi(s) \sim \frac{1}{(r-1)!}\frac{1}{\Gamma(1-\alpha)s^\alpha L(1/s)}\left(\log\left(\frac{1}{\Gamma(1-\alpha)s^\alpha L(1/s)}\right)\right)^{r-1} \quad \text{as} \quad s \to 0^+ . \tag{12}$$

However,

$$\begin{aligned}\log\left(\frac{1}{\Gamma(1-\alpha)s^\alpha L(1/s)}\right) &= \alpha\log(1/s) - \log\bigl(\Gamma(1-\alpha)L(1/s)\bigr) \\ &\sim \alpha\log(1/s) \qquad \text{as} \quad s \to 0^+\end{aligned} \tag{13}$$

since $\frac{\log L(1/s)}{\log(1/s)} \to 0$ as $s \to 0^+$ by Lemma 2(i). Hence, in this case from (12) and (13) we get

$$\varphi(s) \sim s^{-\alpha}\frac{\alpha^{r-1}}{(r-1)!\,\Gamma(1-\alpha)}\frac{(\log(1/s))^{r-1}}{L(1/s)} \qquad \text{as} \quad s \to 0^+ .$$

Thus, by Karamata's Tauberian theorem it follows that

$$U_r(x) \sim \frac{\alpha^{r-1}}{(r-1)!\,\Gamma(1-\alpha)\Gamma(1+\alpha)}\frac{x^\alpha(\log x)^{r-1}}{L(x)} \qquad \text{as} \quad x \to \infty .$$

Case (ii). Suppose $\alpha = 0$. Then, from (9) and (10), we get

$$\varphi(s) \sim \frac{1}{(r-1)!}\frac{\left(\log\left(\frac{1}{L(1/s)}\right)\right)^{r-1}}{L(1/s)} \qquad \text{as} \quad s \to 0^+ .$$

In this case, $\lim\limits_{x\to\infty} L(x) = 0$ and so $\log(\frac{1}{L(x)})$ is slowly varying by Lemma 2(ii). Thus by Karamata's Tauberian theorem, it follows that

$$U_r(x) \sim \frac{1}{(r-1)!}\frac{\left(\log\left(\frac{1}{L(x)}\right)\right)^{r-1}}{L(x)} \qquad \text{as} \quad x\to\infty\,.$$

Case (iii). Suppose $\alpha = 1$. In this case, (9) and (11) imply that

$$\varphi(s) \sim s^{-1}\frac{1}{(r-1)!}\frac{\left(\log\left(\frac{1}{sm(1/s)}\right)\right)^{r-1}}{m(1/s)} \qquad \text{as} \quad s\to 0^+\,. \tag{14}$$

However, since $m(x)$ is slowly varying and $m(x)\to\infty$ as $x\to\infty$, it follows by Lemma 2 that $\frac{\log(m(1/s))}{\log(1/s)} \to 0$ as $s\to 0^+$. Hence $\log(\frac{1}{sm(1/s)}) \sim \log(1/s)$ as $s\to 0^+$, and so from (14), we have

$$\varphi(s) \sim s^{-1}\frac{1}{(r-1)!}\frac{\left(\log(1/s)\right)^{r-1}}{m(1/s)} \qquad \text{as} \quad s\to 0^+\,.$$

Therefore, by Karamata's Tauberian theorem, it follows that

$$U_r(x) \sim \frac{1}{(r-1)!}\frac{x(\log x)^{r-1}}{m(x)} \qquad \text{as} \quad x\to\infty\,.$$

This proves Theorem 2. □

Remarks.

(i) The proof of Theorem 2 can be easily modified to give another proof of Theorem 1.

(ii) We omit the proof of Theorem 3 since its proof is almost identical to that of Theorem 1.

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