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NATURAL BIJECTIONS BETWEEN DIRECTED AND UNDIRECTED SELF-COMPLEMENTARY GRAPHS

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To the memory of J. Bosák

A graph (directed or undirected) is called self-complementary if it is isomorphic to its complement. We shall consider three kinds of graphs. Undirected graphs will be always considered without loops; the complement is taken with respect to the complete undirected graph without loops. In the case of directed graphs without loops the complement is taken with respect to the complete directed graph without loops, i.e. the graph in which any two distinct vertices are joined by a pair of oppositely directed edges and no vertex is joined by an edge with itself. In the case of directed graphs with loops the complement is taken with respect to the complete directed graph with loops, i.e. the directed graph in which the edge from u to v exists for any two vertices u, v (even if $u = v$). All considered graphs will be finite.

A self-complementary undirected graph G [4, 5] with n vertices exists if and only if $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$. If $n \equiv 0 \pmod 4$, then each isomorphism of G onto its complement $\bar{G}$ has all cycles of lengths divisible by 4. If $n \equiv 1 \pmod 4$, then such an isomorphism has exactly one fixed vertex and all its cycles not containing the fixed vertex have again the lengths divisible by 4.

A self-complementary directed graph G without loops [8] with n vertices exists for every positive integer n. If n is even, then each isomorphism of G onto $\bar{G}$ has all cycles of even lengths. If n is odd, then such an isomorphism has exactly one fixed vertex and all its cycles not containing the fixed vertex have again even lengths.

A self-complementary directed graph with loops and with n vertices exists if and only if n is even. Each isomorphism of G onto $\bar{G}$ has all cycles of even lengths.

By s_n (or s_n^*, or s_n^{**}) we denote the number of pairwise non-isomorphic self-complementary undirected (or directed loopless, or directed with loops, respectively) graphs with n vertices. R. C. Read [3] has proved that $s_{2k}^* = s_{4k}$ for any positive integer k; D. Wille [6, 7] has proved that $s_{2k}^{**} = s_{4k+1}$ again for any positive integer k.

J. Bosák [1] has pointed out that, though the above equalities hold, neither a natural bijection between self-complementary directed loopless graphs with $2k$ vertices and self-complementary undirected graphs with $4k$ vertices, nor a natural bijection between self-complementary directed graphs with loops and with $2k$ vertices and self-complementary undirected graphs with $4k + 1$ vertices has been found yet. This is mentioned also in [2]. The aim of this paper is to construct these bijections.

Construction 1. *Natural bijection between self-complementary loopless directed graphs with 2k vertices and self-complementary undirected graphs with 4k vertices.*

Let G be a self-complementary loopless directed graph with $2k$ vertices, let φ be an isomorphism of G onto its complement $\bar{G}$. Then the vertex set $V(G)$ of G is partitioned into cycles of φ; each of these cycles $\mathscr{C}$ has an even length. Therefore the cycle $\mathscr{C}$ consists of vertices $u_0^{\mathscr{C}}, u_1^{\mathscr{C}}, \ldots, u_{h-1}^{\mathscr{C}}$, where h is even and $\varphi(u_i^{\mathscr{C}}) = u_{i+1}^{\mathscr{C}}$ for $i = 0, \ldots, h-2$, $\varphi(u_{h-1}^{\mathscr{C}}) = u_0^{\mathscr{C}}$. We choose the subscripts in such a way that the edge $u_0^{\mathscr{C}} u_1^{\mathscr{C}}$ is in G; this is evidently always possible In order to determine the existence or non-existence of any edge in the subgraph of G induced by $\mathscr{C}$ it is sufficient to determine it for all edges $u_0^{\mathscr{C}} u_i^{\mathscr{C}}$ for $i = 1, \ldots, h-1$. Now let $\mathscr{C}, \mathscr{C}'$ be two distinct cycles of φ of lengths h, h' respectively. Then the vertices of $\mathscr{C}$ are $u_0^{\mathscr{C}}, \ldots, u_{h-1}^{\mathscr{C}}$, the vertices of $\mathscr{C}'$ are $u_0^{\mathscr{C}'}, \ldots, u_{h'-1}^{\mathscr{C}'}$. In order to determine the existence or non-existence of any edge going from a vertex of $\mathscr{C}$ to a vertex of $\mathscr{C}'$ it is sufficient to determine it for all edges $u_0^{\mathscr{C}} u_i^{\mathscr{C}'}$ for $i = 0, \ldots$ $\ldots, h_0 - 1$, where h_0 is the greatest common divisor of h and h'.

Take a set $V(H)$ with $4k$ vertices. Choose a partition of $V(H)$ with the property that there is a one-to-one correspondence between cycles $\mathscr{C}$ of φ in G and classes $M(\mathscr{C})$ of this partition $\mathscr{P}$, while the number of vertices of each class of $\mathscr{P}$ is equal to the length of the corresponding cycle multiplied by 2. Thus if the cycle $\mathscr{C}$ has the length h, the corresponding set $M(\mathscr{C})$ has $2h$ vertices. Choose a permutation ψ of $V(H)$ such that its restriction to any $M(\mathscr{C})$ is a cyclic permutation of $M(\mathscr{C})$. We denote the vertices of $M(\mathscr{C})$ by $v_0^{\mathscr{C}}, \ldots, v_{2h-1}^{\mathscr{C}}$ in such a way that $\psi(v_i^{\mathscr{C}}) = v_{i+1}^{\mathscr{C}}$ for $i = 0, \ldots, 2h-2$ and $\psi(v_{2h-1}^{\mathscr{C}}) = v_0^{\mathscr{C}}$. Now we construct a graph H with the vertex set $V(H)$. In H the undirected edge $v_0^{\mathscr{C}} v_i^{\mathscr{C}}$ for some i such that $1 \leqq i \leqq h-1$ will exist if and only if the directed edge $u_0^{\mathscr{C}} u_i^{\mathscr{C}}$ exists in G. If an edge $u_0^{\mathscr{C}} u_j^{\mathscr{C}}$, where $1 \leqq j \leqq h-1$, does not exist in H, then in H there exists the edge $\psi(u_0^{\mathscr{C}})\, \psi(u_j^{\mathscr{C}})$. Further the edge $v_0^{\mathscr{C}} v_h^{\mathscr{C}}$ will exist in H. If $\mathscr{C}, \mathscr{C}'$ are two distinct cycles of φ in G of the lengths h, h' respectively and h_0 is the greatest common divisor of h and h', then the undirected edge $v_0^{\mathscr{C}} v_i^{\mathscr{C}'}$ for any i such that $0 \leqq i \leqq h_0 - 1$ will exist if and only if the edge $u_0^{\mathscr{C}} u_i^{\mathscr{C}'}$ exists in G. Further the undirected edge $v_0^{\mathscr{C}} v_i^{\mathscr{C}'}$ for any i such that $h_0 \leqq i \leqq 2h_0 - 1$ will exist if and only if the directed edge $u_0^{\mathscr{C}'} u_{i-h_0}^{\mathscr{C}}$ exists in G. If an edge $v_0^{\mathscr{C}} v_i^{\mathscr{C}'}$ for any i such that $0 \leqq i \leqq 2h_0 - 1$ does not exist in H, then the edge $\psi(v_0^{\mathscr{C}})\, \psi(v_i^{\mathscr{C}'})$ will exist in H. Further we add edges $\psi^j(x)\,\psi^j(y)$ for all even j and all pairs $\{x, y\}$ which were joined by an edge in H. Thus we have constructed the undirected graph H which is self-complementary with $4k$ vertices.

Now let a self-complementary undirected graph H with $4k$ vertices be given. Let ψ be an isomorphism of H onto $\bar{H}$. In each cycle $\mathscr{C}$ of ψ of the length $2h$ we denote the vertices by $v_0^{\mathscr{C}}, \ldots, v_{2h-1}^{\mathscr{C}}$ in such a way that $\psi(v_i^{\mathscr{C}}) = v_{i+1}^{\mathscr{C}}$ for $i = 0, \ldots, \ldots, 2h - 1$, $\psi(v_{2h-1}^{\mathscr{C}}) = v_0^{\mathscr{C}}$ and the edges $u_0 u_1$, $u_0 u_h$ are in H (evidently this may be always thus chosen). Then in the converse way to the above described construction we may construct a self-complementary loopless directed graph G. If H has been constructed from a directed graph G by the above described construction, then from H we obtain by the converse construction again G. Thus we have a one-to-one correspondence betweᵉn self-complementary loopless directed graphs G and self-complementary undirected graphs H. Evidently if G_1, G_2 are isomorphic, then the corresponding graphs H_1, H_2 are also isomorphic. Thus from the equality $s_{2k}^* = s_{4k}$ it follows that to non-isomorphic graphs G_1, G_2 there correspond also non-isomorphic graphs H_1, H_2. We have the required natural bijection.

Construction 2. *Natural bijection between self-complementary directed graphs with loops and with $2k$ vertices and self-complementary undirected graphs with $4k + 1$ vertices.*

Let G be a self-complementary directed graph with loops and with $2k$ vertices, let φ be an isomorphism of G onto $\bar{G}$. Let G_0 be obtained from G by deleting all loops. We construct the undirected graph H_0 from G_0 by Construction 1. To H_0 we add a new vertex w. If $u_i^{\mathscr{C}}$ has a loop in G and $\mathscr{C}$ has the length h, then the vertices $v_i^{\mathscr{C}}$, $v_{h+i}^{\mathscr{C}}$ in H will be joined by edges with w. Thus the required natural bijection is constructed.

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НАТУРАЛЬНЫЕ БИЕКЦИИ МЕЖДУ ОРИЕНТИРОВАННЫМИ И НЕОРИЕНТИРОВАННЫМИ САМОДОПОЛНИТЕЛЬНЫМИ ГРАФАМИ

Bohdan Zelinka

Резюме

В работе описанны натуральные биекции между неориентированными самодополнительными графами с $4k$ вершинами и ориентированными без петель с $2k$ вершинами, а также неориентированными с $4k + 1$ вершинами и ориентированными с петлями и с $2k$ вершинами. Это является решением проблемы Ю. Босака.